Introduction to Multigrid Methods

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Multigrid Theory

- In order to obtain accurate estimates of the multigrid performance we need (h-independent) estimates of the spectral radius of the multigrid iteration operator

$$\rho(M_h) \leq \textit{constant} < 1.$$

- In addition, we need h-independent bounds on the error reduction factor $\|M_h\|$, the smoothing factors $\mu(S_h)$, and the two grid convergence factors $\rho(M_h^{2h})$ and $\|M_h^{2h}\|$.

- In general two-grid convergence factors are sufficient to obtain a realistic estimate of multigrid convergence since
    - If h-independent small bounds for $\|M_h^{2h}\|$ are known a simple recursive estimates yields bounds for $\|M_h\|$ using the relation

$$M_0 = 0$$

$$M_k = S_k^{\nu_2}(I_k - I_{k-1}^k(I_{k-1} - (M_{k-1})^\gamma)(L_{k-1})^{-1} I_k^{k-1} L_k)S_k^{\nu_1}, \quad k = 1, \cdots, l$$

    - Using bounds for $\|M_h\|$ we can obtain bounds for $\|u_h - u_h^{FMG}\|$, but this analysis fails for a V-cycle multigrid algorithm.

Multigrid Theory

- Remark

  The difference between the (h_k, h_{k-1}) two-grid operator

$$M_k^{k-1} = S_k^{\nu_2}(I_k - I_{k-1}^k(L_{k-1})^{-1}I_k^{k-1}L_k)S_k^{\nu_1}$$

  and the general multigrid operator

$$M_k = S_k^{\nu_2}(I_k - I_{k-1}^k(I_{k-1} - (M_{k-1})^{\gamma})(L_{k-1})^{-1}I_k^{k-1}L_k)S_k^{\nu_1}$$

  is the replacement of $(L_{k-1})^{-1}$ with $(I_{k-1} - (M_{k-1})^{\gamma})(L_{k-1})^{-1}$.

  Thus the general multigrid algorithm is a perturbed two-grid method.

- Lemma 1. For $k = 1, \cdots, l-1$, the multigrid iteration operator M_k satisfies

$$M_{k+1} = M_{k+1}^{k} + A_{k}^{k+1}(M_k)^{\gamma} A_{k+1}^{k}, \qquad \text{with } M_0 = 0,$$

where

$$A_k^{k+1} := (S_{k+1})^{\nu_2} I_k^{k+1} \; : \; \mathcal{G}(\Omega_k) \to \mathcal{G}(\Omega_{k+1})$$

$$A_{k+1}^{k} := (L_k)^{-1} I_{k+1}^{k} L_{k+1} (S_{k+1})^{\nu_1} \; : \; \mathcal{G}(\Omega_{k+1}) \to \mathcal{G}(\Omega_k)$$

$$M_{k+1}^{k} := (S_{k+1})^{\nu_2} (I_{k+1} - I_k^{k+1} (L_k)^{-1} I_{k+1}^{k} L_{k+1})(S_{k+1})^{\nu_1} \; : \; \mathcal{G}(\Omega_{k+1}) \to \mathcal{G}(\Omega_{k+1})$$

The multigrid algorithm is thus a perturbation of two-grid algorithm M_{k+1}^{k}.

Multigrid Theory

- Assume norms for $\|M_{k+1}^k\|$, $\|A_{k+1}^k\|$ and $\|A_k^{k+1}\|$ are known.
- Theorem 1. Let the following estimates hold uniformly with respect to k $(\leq l-1)$

$$\|M_{k+1}^k\| \leq \sigma^*, \qquad \|A_k^{k+1}\|\|A_{k+1}^k\| \leq C.$$

Then we have $\|M_l\| \leq \eta_l$, where η_l is defined recursively as

$$\eta_1 := \sigma^*, \quad \eta_{k+1} := \sigma^* + C\eta_k^\gamma \qquad (k = 1, \cdots, l-1).$$

- If we additionally assume that

$$4C\sigma^* \leq 1 \quad \text{and} \quad \gamma = 2 \;\; \text{(W-cycle)}$$

we obtain the following estimate for $M_h = M_l$ $(h = h_l)$

$$\|M_h\| \leq \eta := \frac{1 - \sqrt{1 - 4C\sigma^*}}{2C} \leq 2\sigma^* \qquad (l \geq 1)$$

- Proof. From Lemma 1. we have

$$\|M_{k+1}\| \leq \|M^{k}_{k+1}\| + \|A^{k+1}_{k}\| \|M_k\|^{\gamma} \|A^{k}_{k+1}\|.$$

Using $\|M_k\| \leq \eta_k$, obtained from the previous mesh, we obtain the recursion for η_k

$$\eta_{k+1} \leq \sigma^* + C\eta_k^{\gamma} \qquad \text{with } \gamma = 2 \text{ (W-cycle assumption)}$$

Take the limit of the recursion for η then

$$\eta = \sigma^* + C\eta^2 \qquad \text{using the assumption } \gamma = 2.$$

Solving for η then gives

$$\eta_{1,2} = \frac{1 \pm \sqrt{1 - 4\sigma^* C}}{2C}$$

The lowest limit then gives the *h*-independent bound

$$\|M_h\| \leq \eta = \frac{1 - \sqrt{1 - 4\sigma^* C}}{2C}$$

- Example. If $C = 1$ we obtain

$$\eta \leq 0.113 \qquad \text{if } \sigma^* \leq 0.1$$

Thus if $\|M_h^{2h}\| \leq 0.1$ then $\|M_h\| \leq 0.113$.

Note, C will in general not be very large.

This estimate implies that if the two-grid reduction factor is not too large, then the two-level analysis will give a good prediction of M_h.

Theoretical Estimate for Full Multigrid

- Under natural assumptions:
  - FMG provides approximations with discretization accuracy.
  - FMG has cost $O(N)$.

  This implies that FMG is optimal.

Theoretical Estimate for Full Multigrid

- Assume standard coarsening and

1. $\|M_k\| \leq \eta < 1, \qquad (k = 1, 2, \cdots).$
2. The FMG interpolation operator Π^k_{k-1} is uniformly bounded

$$\|\Pi^k_{k-1}\| \leq P, \qquad (k = 1, 2, \cdots).$$

3. The discretization error and the FMG error satisfy

$$\|u - u_h\| \leq K h^{\kappa}$$

$$\|u - \Pi^k_{k-1} u\| \leq \bar{K} h^{\kappa_{FMG}}$$

4. The condition $\kappa_{FMG} > \kappa$ is satisfied.

Note, condition 4 implies that the accuracy of the FMG-interpolation is higher than the discretization accuracy.

Theoretical Estimate for Full Multigrid

- Theorem 2. Let Assumptions 1, 2 and 3 be satisfied. Assume also that

$$\eta^r < \frac{1}{2^\kappa P}, \qquad \text{with } r \text{ the number of MG iterations in FMG.}$$

Then the following estimate holds for any $l \geq 1$

$$\|u_l - u_l^{FMG}\| \leq \delta h^\kappa, \qquad (h = h_l)$$

where

$$\delta = \eta^r \frac{B}{1 - \eta^r A}$$

with $A = 2^\kappa P$ and B such that $K(1 + A) + \bar{K} h^{\kappa_{FMG} - \kappa} \leq B$.

Theoretical Estimate for Full Multigrid

- Proof. By the definition of the FMG method, we have for all $l \geq 1$

$$u_l^{FMG} - u_l = (M_l)^r (u_l^0 - u_l)$$

with $u_l^0 = \Pi_{l-1}^l u_{l-1}^{FMG}$, the interpolated solution from grid level $l-1$.

Next, use the identity

$$u_l^0 - u_l = \Pi_{l-1}^l (u_{l-1}^{FMG} - u_{l-1}) + \Pi_{l-1}^l (u_{l-1} - u) + (\Pi_{l-1}^l u - u) + (u - u_l)$$

Then we obtain the estimate

$$\begin{aligned}
\|u_l^{FMG} - u_l\| &\leq \|(M_l)^r\| \|u_l^0 - u_l\| \\
&\leq \eta^r \big(\|\Pi_{l-1}^l (u_{l-1}^{FMG} - u_{l-1})\| + \|\Pi_{l-1}^l (u_{l-1} - u)\| \\
&\quad + \|\Pi_{l-1}^l u - u\| + \|u - u_l\| \big) \\
&\leq \eta^r \big(P \|u_{l-1}^{FMG} - u_{l-1}\| + P \|u_{l-1} - u\| + \bar{K} h_{l-1}^{\kappa \, FMG} + K h^\kappa \big)
\end{aligned}$$

Theoretical Estimate for Full Multigrid

- Define

$$\delta_k = \frac{\|u_k^{FMG} - u_k\|}{h_k^{\kappa}}$$

Then we obtain

$$h_l^{\kappa}\delta_l \leq \eta^r \left(P\delta_{l-1}h_{l-1}^{\kappa} + PKh_{l-1}^{\kappa} + \bar{K}h_l^{\kappa_{FMG}} + Kh_l^{\kappa}\right))$$

which is equivalent with

$$\begin{aligned}\delta_l &\leq \eta^r \left(P\delta_{l-1}\left(\frac{2h_l}{h_l}\right)^{\kappa} + PK\left(\frac{2h_l}{h_l}\right)^{\kappa} + \bar{K}h_l^{\kappa_{FMG}-\kappa} + K\right)\\ &\leq \eta^r \left(A\delta_{l-1} + KA + \bar{K}h_l^{\kappa_{FMG}-\kappa} + K\right)\end{aligned}$$

with $A = 2^{\kappa}P$.

Theoretical Estimate for Full Multigrid

- Take the limit $\delta_l \to \delta$ then

$$\delta \leq \eta^r A\delta + \eta^r \left(K(1+A) + \bar{K} h_l^{\kappa_{FMG} - \kappa}\right)$$

Hence

$$\delta \leq \eta^r \frac{B}{1-\eta^r A} \qquad \text{with } K(1+A) + \bar{K} h_l^{\kappa_{FMG}-\kappa} \leq B$$

Convergence is obtained if

$$\eta^r A = \eta^r 2^\kappa P < 1,$$

which implies

$$\eta^r < \frac{1}{2^\kappa P}$$

- Note, this estimate puts an upperbound on the necessary multigrid performance since

$$\|M_k\| \leq \eta < 1 \qquad k = 1, 2, \cdots$$

Theoretical Estimate for Full Multigrid

- Note, it is better to have $\kappa_{FMG} > \kappa$, with κ the discretization accuracy.

  If $\kappa_{FMG} = \kappa$ the proof is still valid, but the FMG interpolation contribution can become arbitrary large.

- Note, the orders of accuracy, e.g. κ_{FMG} and κ may depend on the norms used in the analysis.

Two Grid Convergence Factor by Rigorous Fourier Analysis

- The goal of the rigorous Fourier analysis is to obtain sharp bounds for $\rho(M_h^{2h})$ and $\|M_h^{2h}\|$.

  The spectral radius $\rho(M_h^{2h})$ gives the asymptotic convergence rate of the multigrid method.

  The $\|M_h^{2h}\|$ the error reduction factor after one iteration step.

  Rigorous Fourier analysis is only possible for special cases and boundary conditions.

Two Grid Convergence Factor by Rigorous Fourier Analysis

- Define the asymptotic convergence factor ρ^* as

$$\rho^* = \sup_{h \in \mathcal{H}} \rho(M_h^{2h})$$

Two-grid convergence factors ρ^* for second order finite difference discretization of the Laplace equation on a uniform mesh using the Red-Black Gauss-Seidel multigrid method. (Trottenberg, Oosterlee, Schüller, Multigrid, 2001).

	$\nu = 1$	$\nu = 2$	$\nu = 3$	$\nu = 4$
FW	0.250	0.074	0.053	0.041
HW	0.500	0.125	0.033	0.025

Two-grid convergence factors ρ^* for second order finite difference discretization of the Laplace equation on a uniform mesh using the ω-Jacobi multigrid method. (Trottenberg, Oosterlee, Schüller, Multigrid, 2001).

	$\nu = 1$	$\nu = 2$	$\nu = 3$	$\nu = 4$
$\omega = \frac{4}{5}$	0.600	0.360	0.216	0.137
$\omega = \frac{1}{2}$	0.750	0.563	0.422	0.316

Fourier Analysis

- Fourier analysis gives good estimates of local multigrid behaviour and is a very important tool in multigrid analysis.

- Assume periodic boundary conditions in $\Omega_h = (0,1)^2 \cap \mathcal{G}_h$.

  Consider $\boldsymbol{\theta} = (\theta_1, \theta_2) = (\frac{2\pi k}{n}, \frac{2\pi l}{n})$, with n the number of mesh points in the x- and y-direction.

  Define

$$T_h = \left\{ \boldsymbol{\theta} = \left(\frac{2\pi k}{n}, \frac{2\pi l}{n} \right) \mid -\frac{n}{2} \leq k, l < \frac{n}{2},\ k, l \in \mathbb{Z} \right\}$$

  The discrete eigenfunctions for the Laplacian with periodic boundary conditions are

$$\phi_h(\boldsymbol{\theta}, \mathbf{x}) = e^{\iota \theta_1 x / h_1} e^{\iota \theta_2 y / h_2}, \qquad \boldsymbol{\theta} \in T_h \subset [-\pi, \pi)^2, \quad \sharp T_h = n^2.$$

- In case of standard coarsening we can split T_h into subsets corresponding to low and high frequencies

$$T_h^{low} := \{\boldsymbol{\theta} = \left(\frac{2\pi k}{n}, \frac{2\pi l}{n}\right) \mid -\frac{n}{4} \leq k, l < \frac{n}{4};\ k, l \in \mathbb{Z}\} \subset \left[-\frac{\pi}{2}, \frac{\pi}{2}\right)^2$$

$$T_h^{high} := T_h \setminus T_h^{low} \subset [-\pi, \pi)^2 \setminus \left[-\frac{\pi}{2}, \frac{\pi}{2}\right)^2$$

Local Fourier Analysis

- Local Fourier Analysis (LFA) is useful to predict the local performance of multigrid, thus without the effect of boundary conditions.

- Computing the smoothing coefficient μ^* is not sufficient to obtain a reasonable estimate of multigrid performance. This requires a two level analysis of the multigrid algorithm.

- LFA is also applicable to local linearizations of nonlinear problems.

- The goal of LFA is to determine $\mu_{loc}(S_h)$, $\rho_{loc}(M_h^{2h})$ and

$$\sigma_{s\,loc}(M_h^{2h}) = \|M_h^{2h}\|.$$

- The grid functions are denoted

$$\phi(\boldsymbol{\theta}, \mathbf{x}) = e^{\iota \boldsymbol{\theta} \cdot \mathbf{x} / \mathbf{h}} := e^{\iota \theta_1 x_1 / h_1} e^{\iota \theta_2 x_2 / h_2}, \quad \mathbf{x} \in \mathbf{G}_h.$$

  We assume that $\boldsymbol{\theta}$ varies continuously in $\mathbb{R}^2$ and consider $\phi(\boldsymbol{\theta}, \mathbf{x})$ on an infinite mesh.

- No effects of boundary conditions are taken into account.

Notation

- The following notation will be used in LFA:

$$\mathbf{x} = (x_1, x_2), \qquad \mathbf{h} = (h_1, h_2)$$

$$\mathbf{G}_h = \left\{ \mathbf{x} = \mathbf{k}\mathbf{h} := (k_1 h_1, k_2 h_2), \;\; \mathbf{k} \in \mathbb{Z}^2 \right\}.$$

On $\mathbf{G}_h$ we have the discrete operator

$$L_h w_h(\mathbf{x}) = \sum_{\boldsymbol{\kappa} \in V} s_{\boldsymbol{\kappa}} w_h(\mathbf{x} + \boldsymbol{\kappa}\mathbf{h}),$$

with $\boldsymbol{\kappa} = (\kappa_1, \kappa_2)$ and V a finite index set.

Grid Functions

- The grid functions are denoted

$$\phi(\boldsymbol{\theta}, \mathbf{x}) = e^{\iota \boldsymbol{\theta} \cdot \mathbf{x}/\mathbf{h}} := e^{\iota \theta_1 x/h_1} e^{\iota \theta_2 y/h_2}, \quad \mathbf{x} \in \mathbf{G}_h.$$

  We assume that $\boldsymbol{\theta}$ varies continuously in $\mathbb{R}^2$.

Grid Functions

- The grid functions satisfy

$$\phi(\boldsymbol{\theta}, \mathbf{x}) = \phi(\boldsymbol{\theta}', \mathbf{x}) \qquad \text{for } \mathbf{x} \in \mathbf{G}_h$$

if and only if $\theta_1 = \theta_1' \pmod{2\pi}$ and $\theta_2 = \theta_2' \pmod{2\pi}$.

Proof: Take $x_1 = jh_1$ and $x_2 = mh_2$ then

$$\begin{aligned}
\phi(\boldsymbol{\theta}, \mathbf{x}) &= e^{\iota(\theta_1' + 2\pi p)j} e^{\iota(\theta_2' + 2\pi q)m}, \qquad p, q \in \mathbb{Z} \\
&= e^{2\pi\iota pj} e^{\iota\theta_1' j} e^{2\pi\iota qm} e^{\iota\theta_1' m} \\
&= e^{\iota\theta_1' j} e^{\iota\theta_1' m} \\
&= e^{\iota\theta_1' x_1/h_1} e^{\iota\theta_2' x_2/h_2} \\
&= \phi(\boldsymbol{\theta}', \mathbf{x}), \qquad \mathbf{x} \in \mathbf{G}_h.
\end{aligned}$$

- The grid functions ϕ therefore only need to consider

$$\phi(\boldsymbol{\theta}, \mathbf{x}), \qquad \text{with } \boldsymbol{\theta} \in [-\pi, \pi)^2, \quad \mathbf{x} \in \mathbf{G}_h.$$

Fourier Symbols

- Lemma. For $\boldsymbol{\theta} \in [-\pi, \pi)^2$ all grid functions $\phi(\boldsymbol{\theta}, \mathbf{x})$ are eigenfunctions of any discrete operator L_h that can be represented on a grid stencil.

Consider

$$L_h w_h(\mathbf{x}) = \sum_{\boldsymbol{\kappa} \in V} s_{\boldsymbol{\kappa}} w_h(\mathbf{x} + \boldsymbol{\kappa}\mathbf{h}), \qquad \mathbf{x} \in \mathbf{G}_h,$$

then

$$L_h \phi(\boldsymbol{\theta}, \mathbf{x}) = \tilde{L}_h(\boldsymbol{\theta})\, \phi(\boldsymbol{\theta}, \mathbf{x}),$$

with

$$\tilde{L}_h(\boldsymbol{\theta}) = \sum_{\boldsymbol{\kappa} \in V} s_{\boldsymbol{\kappa}} e^{\iota \boldsymbol{\theta} \cdot \boldsymbol{\kappa}}$$

We call $\tilde{L}_h(\boldsymbol{\theta})$ the formal eigenvalue or symbol of L_h.

Fourier Symbols

- Proof

$$
\begin{aligned}
L_h\phi(\boldsymbol{\theta},\mathbf{x}) &= \sum_{\boldsymbol{\kappa}\in V} s_{\boldsymbol{\kappa}}\phi(\boldsymbol{\theta},\mathbf{x}+\boldsymbol{\kappa}\mathbf{h}) \\
&= \sum_{\boldsymbol{\kappa}\in V} s_{\boldsymbol{\kappa}} e^{\iota\boldsymbol{\theta}\cdot(\mathbf{x}+\boldsymbol{\kappa}\mathbf{h})/\mathbf{h}} \\
&= \sum_{\boldsymbol{\kappa}\in V} s_{\boldsymbol{\kappa}} e^{\iota\boldsymbol{\theta}\cdot\boldsymbol{\kappa}} e^{\iota\boldsymbol{\theta}\cdot\mathbf{x}/\mathbf{h}} \\
&= \sum_{\boldsymbol{\kappa}\in V} s_{\boldsymbol{\kappa}} e^{\iota\boldsymbol{\theta}\cdot\boldsymbol{\kappa}}\phi(\boldsymbol{\theta},\mathbf{x}) \\
&= \tilde{L}_h(\boldsymbol{\theta})\phi(\boldsymbol{\theta},\mathbf{x})
\end{aligned}
$$

Fourier Symbols

- Example: Laplace operator

$$\begin{aligned} L_h w_h &= \frac{1}{h^2}(4w_{i,j} - w_{i-1,j} - w_{i+1,j} - w_{i,j-1} - w_{i,j+1}) \\ &= \frac{1}{h^2}\begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}_h w_h(\mathbf{x}) \\ &= \sum_{\boldsymbol{\kappa}\in V} s_{\boldsymbol{\kappa}} w_h(\mathbf{x} + \boldsymbol{\kappa}\mathbf{h}) \end{aligned}$$

The Fourier symbol of L_h is then

$$\begin{aligned} \tilde{L}_h(\boldsymbol{\theta}) &= \frac{1}{h^2}\left(4 - \left(e^{-\iota\theta_1} + e^{\iota\theta_1} + e^{-\iota\theta_2} + e^{\iota\theta_2}\right)\right) \\ &= \frac{2}{h^2}\left(2 - (\cos\theta_1 + \cos\theta_2)\right) \end{aligned}$$

Coarse Grid

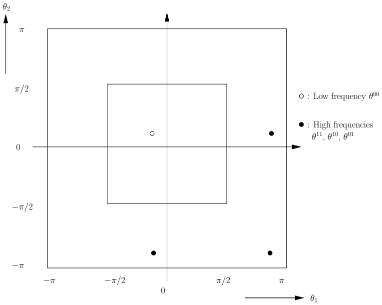
- The coarse grid is given by

$$\mathbf{G_H} = \left\{ \mathbf{x} = \boldsymbol{\kappa}\mathbf{H} \mid \boldsymbol{\kappa} \in \mathbb{Z}^2 \right\}, \qquad \mathbf{H} = 2\mathbf{h}.$$

- Due to aliasing only modes $\phi(\boldsymbol{\theta}, \cdot)$ with $\boldsymbol{\theta} \in [-\frac{\pi}{2}, \frac{\pi}{2})^2$ are distinguishable on the coarse grid

$$\text{Low frequency} \qquad \boldsymbol{\theta} \in T^{low} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right)^2,$$

$$\text{High frequency} \qquad \boldsymbol{\theta} \in T^{high} = [-\pi, \pi)^2 \setminus \left[-\frac{\pi}{2}, \frac{\pi}{2}\right)^2.$$



Low and high frequencies Fourier modes in two-level multigrid.

- Four fine grid modes alias to one coarse grid mode.

- Consider $L_h = -\Delta_h$ with ($h_1 = h_2 = h$) and use the Gauss-Seidel method with Lexicographic ordering as smoother.

  We can write the discretization

  $$L_h u_h = f_h$$

  with the GS-LEX smoother as

  $$L_h^+ \bar{w}_h + L_h^- w_h = f_h$$

Smoothing Analysis

- The stencil for the GS-LEX smoother is

$$L_h^+ = \frac{1}{h^2}\begin{bmatrix} & 0 & \\ -1 & 4 & 0 \\ & -1 & \end{bmatrix}_h, \qquad \text{updated nodes}$$

$$L_h^- = \frac{1}{h^2}\begin{bmatrix} & -1 & \\ 0 & 0 & -1 \\ & 0 & \end{bmatrix}_h, \qquad \text{old nodes}$$

- Similarly, the stencil for the ω-Jacobi smoother is

$$L_h^+ = \frac{1}{h^2}\begin{bmatrix} & 0 & \\ 0 & \frac{4}{\omega} & 0 \\ & 0 & \end{bmatrix}_h, \qquad \text{updated nodes}$$

$$L_h^- = \frac{1}{h^2}\begin{bmatrix} & -1 & \\ -1 & 4(1-\frac{1}{\omega}) & -1 \\ & -1 & \end{bmatrix}_h, \qquad \text{old nodes}$$

Smoothing Analysis

- Consider now

$$L_h u_h = f_h$$

$$L_h^+ \bar{w}_h + L_h^- w_h = f_h$$

Subtracting both equations then gives

$$L_h^+ \bar{v}_h + L_h^- v_h = 0$$

with $\bar{v}_h = u_h - \bar{w}_h$ and $v_h = u_h - w_h$. We obtain then

$$\bar{v}_h = S_h v_h \qquad \text{with } S_h = -(L_h^+)^{-1} L_h^- .$$

- For $L_h^{\pm}$ we obtain then relation

$$L_h^{\pm} e^{\iota\boldsymbol{\theta}\cdot\mathbf{x}/\mathbf{h}} = \tilde{L}_h^{\pm}(\boldsymbol{\theta}) e^{\iota\boldsymbol{\theta}\cdot\mathbf{x}/\mathbf{h}}$$

with

$$\tilde{L}_h^{\pm}(\boldsymbol{\theta}) = \sum_{\boldsymbol{\kappa}\in V} s_{\boldsymbol{\kappa}}^{\pm} e^{\iota\boldsymbol{\theta}\cdot\boldsymbol{\kappa}}$$

and $s_{\boldsymbol{\kappa}}^{\pm}$ the stencil of $L_h^{\pm}$.

For all $\phi(\boldsymbol{\theta}), \cdot)$, with $\tilde{L}_h^{+}(\boldsymbol{\theta}) \neq 0$, we have then for $S_h = -(L_h^{+})^{-1} L_h^{-}$ the relation

$$S_h \phi(\boldsymbol{\theta}, \mathbf{x}) = \tilde{S}_h(\boldsymbol{\theta}) \phi(\boldsymbol{\theta}, \mathbf{x})$$

with Fourier symbol

$$\tilde{S}_h(\boldsymbol{\theta}) = -\frac{\tilde{L}_h^{-}(\boldsymbol{\theta})}{\tilde{L}_h^{+}(\boldsymbol{\theta})}.$$

Smoothing Analysis

- Example: Fourier symbol of Gauss-Seidel smoother with Lexicographic ordering

$$L_h^+ e^{\iota \boldsymbol{\theta} \cdot \mathbf{x}/\mathbf{h}} = \frac{1}{h^2} \begin{bmatrix} & 0 & \\ -1 & 4 & 0 \\ & -1 & \end{bmatrix} e^{\iota \boldsymbol{\theta} \cdot \mathbf{x}/\mathbf{h}}$$

$$= \frac{1}{h^2}(4 - e^{-\iota\theta_1} - e^{-\iota\theta_2}) e^{\iota \boldsymbol{\theta} \cdot \mathbf{x}/\mathbf{h}}$$

$$L_h^- e^{\iota \boldsymbol{\theta} \cdot \mathbf{x}/\mathbf{h}} = \frac{1}{h^2} \begin{bmatrix} & -1 & \\ 0 & 0 & -1 \\ & 0 & \end{bmatrix} e^{\iota \boldsymbol{\theta} \cdot \mathbf{x}/\mathbf{h}}$$

$$= \frac{-1}{h^2}(e^{\iota\theta_1} + e^{\iota\theta_2}) e^{\iota \boldsymbol{\theta} \cdot \mathbf{x}/\mathbf{h}}$$

$$\tilde{S}_h(\boldsymbol{\theta}) = -\frac{\tilde{L}_h^-(\boldsymbol{\theta})}{\tilde{L}_h^+(\boldsymbol{\theta})} = \frac{e^{\iota\theta_1} + e^{\iota\theta_2}}{4 - e^{-\iota\theta_1} - e^{-\iota\theta_2}}$$

Smoothing Analysis

- The smoothing factor $\mu_{loc}(S_h)$ is defined as

$$\mu_{loc} = \mu_{loc}(S_h) := \sup\left\{|\tilde{S}_h(\boldsymbol{\theta})| \mid \boldsymbol{\theta} \in T^{high}\right\}$$

- For the Gauss-Seidel relaxation the smoothing factor is

$$\begin{aligned}\mu_{loc}(S_h) &= \sup\left\{\left|\frac{e^{\iota\theta_1} + e^{\iota\theta_2}}{4 - e^{-\iota\theta_1} - e^{-\iota\theta_2}}\right| \mid \boldsymbol{\theta} \in T^{high}\right\} \\ &= 0.5\end{aligned}$$

Smoothing Analysis

- For the ω-Jacobi method the Fourier symbol of the smoother is

$$\tilde{S}_h(\omega, \boldsymbol{\theta}) = 1 - \frac{\omega}{2}(2 - \cos\theta_1 - \cos\theta_2)$$

with smoothing factor

$$\mu_{loc}(S_h(\omega)) = \max\left\{\left|1 - \frac{\omega}{2}\right|, |1 - 2\omega|\right\}$$

Two-Grid Analysis

- Apply the Local Fourier Analysis (LFA) to the two-grid iteration operator

$$M_h^{2h} = S_h^{\nu_2} K_h^{2h} S_h^{\nu_1}, \qquad \text{with } K_h^{2h} = I_h - I_{2h}^h (L_{2h})^{-1} I_h^{2h} L_h.$$

- We need to investigate how the operators L_h, I_h^{2h}, L_{2h}, I_{2h}^h and S_h act on the Fourier components $\phi(\boldsymbol{\theta}, \cdot)$.

- Use the aliasing of four fine grid modes to one coarse grid mode.

Take any $\boldsymbol{\theta} = (\theta_1, \theta_2) \in T^{low} = [-\pi/2, \pi/2)^2$ and consider

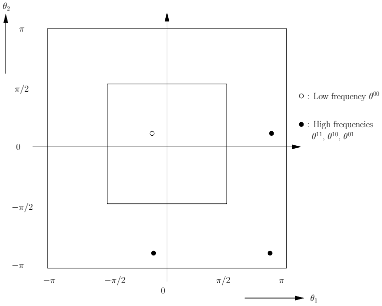
$$\boldsymbol{\theta}^{(0,0)} = (\theta_1, \theta_2), \qquad \boldsymbol{\theta}^{(1,1)} = (\bar{\theta}_1, \bar{\theta}_2)$$

$$\boldsymbol{\theta}^{(1,0)} = (\bar{\theta}_1, \theta_2), \qquad \boldsymbol{\theta}^{(0,1)} = (\theta_1, \bar{\theta}_2)$$

where

$$\bar{\theta}_i = \begin{cases} \theta_i + \pi & \text{if } \theta_i < 0 \\ \theta_i - \pi & \text{if } \theta_i \geq 0 \end{cases}$$

Two-Grid Analysis



Low and high frequencies Fourier modes in two-level multigrid.

- Lemma.
  1. For any low frequency $\boldsymbol{\theta}^{(0,0)} \in T^{low}$, we have

$$\phi(\boldsymbol{\theta}^{(0,0)}, \mathbf{x}) = \phi(\boldsymbol{\theta}^{(1,1)}, \mathbf{x}) = \phi(\boldsymbol{\theta}^{(1,0)}, \mathbf{x}) = \phi(\boldsymbol{\theta}^{(0,1)}, \mathbf{x}), \qquad \mathbf{x} \in \mathbf{G}_{2h}$$

  2. Each of the four Fourier components $\phi(\boldsymbol{\theta}^{\boldsymbol{\alpha}}, \cdot)$, $\boldsymbol{\alpha} \in \{(0,0),(1,1),(1,0),(0,1)\}$ coincides on $\mathbf{G}_{2h}$ with the grid function $\phi_{2h}(2\boldsymbol{\theta}^{(0,0)}, \cdot)$

$$\phi_h(\boldsymbol{\theta}^{\boldsymbol{\alpha}}, \mathbf{x}) = \phi_{2h}(2\boldsymbol{\theta}^{(0,0)}, \mathbf{x}), \qquad \mathbf{x} \in \mathbf{G}_{2h}$$

Two-Grid Analysis

- Proof. For $\boldsymbol{\theta}^{(1,1,)}$ we have

$$
\begin{aligned}
\phi_h(\boldsymbol{\theta}^{(1,1)}, \mathbf{x}) &= e^{\iota\bar{\theta}_1 x_1/h_1} e^{\iota\bar{\theta}_2 x_2/h_2} \\
&= e^{\iota(\theta_1 \pm \pi)x_1/h_1} e^{\iota(\theta_2 \pm \pi)x_2/h_2} \\
&= e^{\iota\theta_1 x_1/h_1} e^{\pm 2\pi\iota k_1} e^{\iota\theta_2 x_2/h_2} e^{\pm 2\pi\iota k_2} \quad \text{since } x_i = 2k_i h_i \\
&= e^{\iota\theta_1 (2x_1)/(2h_1)} e^{\iota\theta_2 (2x_2)/(2h_2)} \\
&= \phi_{2h}(2\boldsymbol{\theta}^{(0,0)}, \mathbf{x}), \qquad \mathbf{x} \in \mathbf{G}_{2h}.
\end{aligned}
$$

Two-Grid Analysis

- For a given $\boldsymbol{\theta} \in T^{low}$ we define the four-dimensional space of space harmonics by

$$E_h^{\theta} := \mathrm{span}\left\{\phi(\boldsymbol{\theta}^{\boldsymbol{\alpha}}, \cdot) \;:\; \boldsymbol{\alpha} \in \{(0,0),(1,1),(1,0),(0,1)\}\right\}$$

- The space E_h^{θ} is invariant under the two-grid operator M_h^{2h}.

All $\phi_h(\boldsymbol{\theta}, \cdot)$ are eigenfunctions of L_h and S_h (for some smoothers).

The operator K_h^{2h} intermixes Fourier components with each other, which originates from using two different grids, $\mathbf{G}_h$ and $\mathbf{G}_{2h}$.

Given an arbitrary $\psi \in E_h^{\theta}$, we can represent ψ in the form

$$\psi = A^{(0,0)}\phi(\boldsymbol{\theta}^{(0,0)}) + A^{(1,1)}\phi(\boldsymbol{\theta}^{(1,1)}) + A^{(1,0)}\phi(\boldsymbol{\theta}^{(1,0)}) + A^{(0,1)}\phi(\boldsymbol{\theta}^{(0,1)}).$$

- Theorem. The coarse grid correction operator K_h^{2h} is represented on E_h^θ by the (4×4) matrix $\hat{K}_h^{2h}(\boldsymbol{\theta})$

$$\hat{K}_h^{2h}(\boldsymbol{\theta}) = \hat{I}_h - \hat{I}_{2h}^h(\boldsymbol{\theta})(\hat{L}_{2h}(2\boldsymbol{\theta}))^{-1}\,\hat{I}_h^{2h}(\boldsymbol{\theta})\hat{L}_h(\boldsymbol{\theta}) \qquad \text{for each } \boldsymbol{\theta} \in T^{low}.$$

Here

$$\hat{I}_h, \hat{L}_h(\boldsymbol{\theta}) \in \mathbb{R}^{4\times 4},$$

$$\hat{I}_h^{2h}(\boldsymbol{\theta}) \in \mathbb{R}^{1\times 4},$$

$$\left(\hat{L}_{2h}(2\boldsymbol{\theta})\right)^{-1} \in \mathbb{R}^{1\times 1},$$

$$\hat{I}_{2h}^h(\boldsymbol{\theta}) \in \mathbb{R}^{4\times 1}.$$

Two-Grid Analysis

- If we apply K_h^{2h} to any $\psi \in E_h^\theta$ the coefficients of ϕ in terms of the Fourier modes transfer as

$$\begin{pmatrix} \bar{A}^{(0,0)} \\ \bar{A}^{(1,1)} \\ \bar{A}^{(1,0)} \\ \bar{A}^{(0,1)} \end{pmatrix} \Leftarrow \hat{K}_h^{2h}(\boldsymbol{\theta}) \begin{pmatrix} A^{(0,0)} \\ A^{(1,1)} \\ A^{(1,0)} \\ A^{(0,1)} \end{pmatrix}$$

- If the space E_h^θ is invariant under the smoothing operator S_h,

$$S_h : E_h^\theta \to E_h^\theta, \qquad \forall \boldsymbol{\theta} \in T^{low},$$

we can represent M_h^{2h} on E_h^θ by a (4×4) matrix $\hat{M}^{2h}(\boldsymbol{\theta})$, with

$$\hat{M}^{2h}(\boldsymbol{\theta}) = \hat{S}_h(\boldsymbol{\theta})^{\nu_2} \hat{K}_h^{2h}(\boldsymbol{\theta}) \hat{S}_h(\boldsymbol{\theta})^{\nu_1}$$

Two-Grid Analysis

- Since in the LFA we need to exclude modes with $\tilde{L}_h(\boldsymbol{\theta}) = 0$ or $\tilde{L}_{2h}(2\boldsymbol{\theta}) = 0$ that are related to the periodic boundary conditions we define the set

$$\Lambda = \left\{ \boldsymbol{\theta} \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right)^2 \mid \tilde{L}_h(\boldsymbol{\theta}) = 0 \text{ or } \tilde{L}_{2h}(\boldsymbol{\theta}) = 0 \right\}.$$

- Asymptotic convergence factor

$$\rho_{loc}(M_h^{2h}) = \sup\{\rho(\hat{M}_h^{2h}(\boldsymbol{\theta})) \mid \boldsymbol{\theta} \in T^{low}, \boldsymbol{\theta} \notin \Lambda\}.$$

- Error reduction factor

$$\sigma_{loc,S}(M_h^{2h}) = \sup\{\|\hat{M}_h^{2h}(\boldsymbol{\theta})\| \mid \boldsymbol{\theta} \in T^{low}, \boldsymbol{\theta} \notin \Lambda\}.$$

with $\|\cdot\|$ the spectral norm.

Two-Grid Analysis

- Proof. The operator L_h gives for each mode

$$\begin{pmatrix} A^{(0,0)} \\ A^{(1,1)} \\ A^{(1,0)} \\ A^{(0,1)} \end{pmatrix} \Leftarrow \begin{pmatrix} \tilde{L}(\boldsymbol{\theta}^{(0,0)}) & & & \\ & \tilde{L}(\boldsymbol{\theta}^{(1,1)}) & & \\ & & \tilde{L}(\boldsymbol{\theta}^{(1,0)}) & \\ & & & \tilde{L}(\boldsymbol{\theta}^{(0,1)}) \end{pmatrix} \begin{pmatrix} A^{(0,0)} \\ A^{(1,1)} \\ A^{(1,0)} \\ A^{(0,1)} \end{pmatrix}$$

with a similar relation for I_h and the smoother S_h if it has a stencil representation.

Two-Grid Analysis

- Restriction operator

  Assume

$$I_h^{2h} w_h(\mathbf{x}) = \sum_{\boldsymbol{\kappa}\in V} \hat{t}_{\boldsymbol{\kappa}} w_h(\mathbf{x}+\boldsymbol{\kappa}\mathbf{h}), \qquad \mathbf{x}\in \mathbf{G}_{2h}$$

  then

$$\begin{aligned} I_h^{2h}\phi_h(\boldsymbol{\theta}^{\boldsymbol{\alpha}},\mathbf{x}) &= \sum_{\boldsymbol{\kappa}\in V} \hat{t}_{\boldsymbol{\kappa}}\phi_h(\boldsymbol{\theta}^{\boldsymbol{\alpha}},\mathbf{x}+\boldsymbol{\kappa}\mathbf{h}) \\ &= \sum_{\boldsymbol{\kappa}\in V} \hat{t}_{\boldsymbol{\kappa}} e^{\iota\boldsymbol{\kappa}\cdot\boldsymbol{\theta}^{\boldsymbol{\alpha}}}\phi_h(\boldsymbol{\theta}^{\boldsymbol{\alpha}},\mathbf{x}) \\ &= \sum_{\boldsymbol{\kappa}\in V} \hat{t}_{\boldsymbol{\kappa}} e^{\iota\boldsymbol{\kappa}\cdot\boldsymbol{\theta}^{\boldsymbol{\alpha}}}\phi_{2h}(2\boldsymbol{\theta}^{(0,0)},\mathbf{x}) \qquad \text{since } \mathbf{x}\in\mathbf{G}_{2h} \end{aligned}$$

Thus the Fourier symbol of the restriction operator is

$$\tilde{I}_h^{2h}(\boldsymbol{\theta}) = \sum_{\boldsymbol{\kappa}\in V} \hat{t}_{\boldsymbol{\kappa}} e^{\iota \boldsymbol{\kappa}\cdot\boldsymbol{\theta}^{\boldsymbol{\alpha}}}.$$

Then

$$A_{2h} = \tilde{I}_h^{2h}(\boldsymbol{\theta}) \begin{pmatrix} A^{(0,0)} \\ A^{(1,1)} \\ A^{(1,0)} \\ A^{(0,1)} \end{pmatrix}$$

with A_{2h} the coefficient of the $2h$-Fourier component $\phi_{2h}(2\boldsymbol{\theta}^{(0,0)}$ and $\tilde{I}_h^{2h}(\boldsymbol{\theta}) \in \mathbb{R}^{1\times 4}$ given by

$$\tilde{I}_h^{2h}(\boldsymbol{\theta}) = (\tilde{I}_h^{2h}(\boldsymbol{\theta}^{(0,0)}), \tilde{I}_h^{2h}(\boldsymbol{\theta}^{(1,1)}), \tilde{I}_h^{2h}(\boldsymbol{\theta}^{(1,0)}), \tilde{I}_h^{2h}(\boldsymbol{\theta}^{(0,1)}))$$

- Example. Full weighting restriction operator

$$I_h^{2h} = \frac{1}{16}\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}_h^{2h}$$

$$\begin{aligned}
\tilde{I}_h^{2h}(\boldsymbol{\theta}^{(0,0)}) &= \sum_{\boldsymbol{\kappa}\in V} \hat{t}_{\boldsymbol{\kappa}} e^{\iota \boldsymbol{\kappa}\cdot\boldsymbol{\theta}^{(0,0)}} \\
&= \frac{1}{16}\Big(e^{-\iota\theta_1-\iota\theta_2} + e^{\iota\theta_1-\iota\theta_2} + e^{-\iota\theta_1+\iota\theta_2} + e^{\iota\theta_1+\iota\theta_2} \\
&\quad + 4 + 2e^{-\iota\theta_1} + 2e^{-\iota\theta_2} + 2e^{\iota\theta_1} + 2e^{\iota\theta_2}\Big) \\
&= \frac{1}{4}(1+\cos\theta_1)(1+\cos\theta_2)
\end{aligned}$$

and in general $\tilde{I}_h^{2h}(\boldsymbol{\theta}^{\boldsymbol{\theta}}) = \frac{1}{4}(1+\cos\bar{\theta}_1)(1+\cos\bar{\theta}_2)$.

Two-Grid Analysis

- Solution on $2h$ grid.

The coarse grid operator L_{2h} is given by the stencil $L_{2h} = \left[s_{\boldsymbol{\kappa},2h}\right]_{2h}$.

For $\boldsymbol{\theta}^{(0,0)} \in T^{low}$ we have

$$L_{2h}\,\phi_{2h}(2\boldsymbol{\theta},\cdot) = \tilde{L}_{2h}(2\boldsymbol{\theta})\phi_{2h}(2\boldsymbol{\theta},\cdot)$$

with

$$\tilde{L}_{2h}(2\boldsymbol{\theta}) = \sum_{\boldsymbol{\kappa}\in V} s_{\boldsymbol{\kappa},2h} e^{2\iota\boldsymbol{\theta}\cdot\boldsymbol{\kappa}}$$

and $s_{\boldsymbol{\kappa},2h}$ the stencil of L_{2h} on the mesh $\mathbf{G}_{2h}$.

The solution on the $2h$-grid (for $2\boldsymbol{\theta} \notin \Lambda$) will give

$$A_{2h} \Leftarrow \frac{1}{\tilde{L}_{2h}(2\boldsymbol{\theta})} A_{2h}$$

- Prolongation operator

The prolongation operator I_{2h}^h can be represented as

$$I_{2h}^h \, \phi_{2h}(2\boldsymbol{\theta}^{(0,0)}, \cdot) = \sum_{\boldsymbol{\alpha}} \tilde{I}_{2h}^h(\boldsymbol{\theta}^{\boldsymbol{\alpha}}) \phi(\boldsymbol{\theta}^{\boldsymbol{\alpha}}, \cdot)$$

with

$$\tilde{I}_{2h}^h(\boldsymbol{\theta}^{\boldsymbol{\alpha}}) = \frac{1}{4} \sum_{\boldsymbol{\kappa} \in V} t_{\boldsymbol{\kappa}} e^{\iota \boldsymbol{\theta}^{\boldsymbol{\alpha}} \cdot \boldsymbol{\kappa}}.$$

The key point in the derivation of this relation is to determine an expression of $I_{2h}^h \, \phi_{2h}(2\boldsymbol{\theta}^{(0,0)}, \cdot)$ in terms of $\phi_h(\boldsymbol{\theta}^{(0,0)}, \cdot)$

Two-Grid Analysis

- The coefficients $A^{\boldsymbol{\alpha}}$ and A_{2h} are now related as

$$\begin{pmatrix} A^{(0,0)} \\ A^{(1,1)} \\ A^{(1,0)} \\ A^{(0,1)} \end{pmatrix} \Leftarrow \hat{I}_{2h}^{h}(\boldsymbol{\theta}^{(0,0)})A_{2h} \qquad \text{with } \hat{I}_{2h}^{h}(\boldsymbol{\theta}^{(0,0)}) = \begin{pmatrix} \hat{I}_{2h}^{h}(\boldsymbol{\theta}^{(0,0)}) \\ \hat{I}_{2h}^{h}(\boldsymbol{\theta}^{(1,1)}) \\ \hat{I}_{2h}^{h}(\boldsymbol{\theta}^{(1,0)}) \\ \hat{I}_{2h}^{h}(\boldsymbol{\theta}^{(0,1)}) \end{pmatrix}$$

- Linear interpolation

$$I_{2h}^{h} = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}_{2h}^{h}$$

with

$$\tilde{I}_{2h}^{h}(\boldsymbol{\theta}^{\boldsymbol{\alpha}}) = \frac{1}{4}\sum_{\boldsymbol{\kappa}\in V} t_{\boldsymbol{\kappa}} e^{\iota\boldsymbol{\theta}^{\boldsymbol{\alpha}}\cdot\boldsymbol{\kappa}}$$

Then

$$\tilde{I}_{2h}^{h}(\boldsymbol{\theta}^{(0,0)}) = \frac{1}{4}(1+\cos\theta_1)(1+\cos\theta_2).$$

Two-Grid Analysis

$$
\begin{aligned}
\tilde{I}_{2h}^{h}(\boldsymbol{\theta}^{(1,1)}) &= \frac{1}{16}\big(e^{-\iota(\theta_1\pm\pi)-\iota(\theta_2\pm\pi)} + e^{\iota(\theta_1\pm\pi)-\iota(\theta_2\pm\pi)} \\
&\quad + e^{-\iota(\theta_1\pm\pi)+\iota(\theta_2\pm\pi)} + e^{\iota(\theta_1\pm\pi)+\iota(\theta_2\pm\pi)} \\
&\quad + 2e^{-\iota(\theta_2\pm\pi)} + 2e^{-\iota(\theta_1\pm\pi)} + 2e^{\iota(\theta_2\pm\pi)} + 2e^{\iota(\theta_1\pm\pi)} + 4\big) \\
&= \frac{1}{4}(1-\cos\theta_1)(1-\cos\theta_2)
\end{aligned}
$$

with similar relations for $\boldsymbol{\theta}^{(1,0)}$ and $\boldsymbol{\theta}^{(0,1)}$.

- The Fourier symbol of the prolongation operator is now

$$\begin{pmatrix} \tilde{I}_{2h}^{h}(\boldsymbol{\theta}^{(0,0)}) \\ \tilde{I}_{2h}^{h}(\boldsymbol{\theta}^{(1,1)}) \\ \tilde{I}_{2h}^{h}(\boldsymbol{\theta}^{(1,0)}) \\ \tilde{I}_{2h}^{h}(\boldsymbol{\theta}^{(0,1)}) \end{pmatrix} = \frac{1}{4} \begin{pmatrix} (1+\cos\theta_1)(1+\cos\theta_2) \\ (1-\cos\theta_1)(1-\cos\theta_2) \\ (1+\cos\theta_1)(1-\cos\theta_2) \\ (1-\cos\theta_1)(1+\cos\theta_2) \end{pmatrix}$$

Two-Grid Analysis

- The spectral radius $\rho_{loc}(M_h^{2h})$ and the operator norm $\sigma_{loc,S}(M_h^{2h})$ can now be obtained by computing for each value of $\boldsymbol{\theta} \in T^{low}, \boldsymbol{\theta} \notin \Lambda$ the eigenvalues of the Fourier symbol of the multigrid iteration operator $\hat{M}_h^{2h} \in \mathbb{R}^{4\times 4}$.

  In practice this is done numerically for a sufficiently large number of discrete value of $\boldsymbol{\theta}$.