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Discrete Fourier Analysis of Multigrid Algorithms[1]

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Contents

Abstract

The main topic of this report is a detailed discussion of the discrete Fourier multilevel analysis of multigrid algorithms. First, a brief overview of multigrid methods is given for discretizations of both linear and nonlinear partial differential equations. Special attention is given to the hp-Multigrid as Smoother algorithm, which is a new algorithm suitable for higher order accurate discontinuous Galerkin discretizations of advection dominated flows. In order to analyze the performance of the multigrid algorithms the error transformation operator for several linear multigrid algorithms are derived. The operator norm and spectral radius of the multigrid error transformation are then computed using discrete Fourier analysis. First, the main operations in the discrete Fourier analysis are defined, including the aliasing of modes. Next, the Fourier symbol of the multigrid operators is computed and used to obtain the Fourier symbol of the multigrid error transformation operator. In the multilevel analysis, two and three level h-multigrid, both for uniformly and semi-coarsened meshes, are considered, and also the analysis of the hp-Multigrid as Smoother algorithm for three polynomial levels and three uniformly and semi-coarsened meshes. The report concludes with a discussion of the multigrid operator norm and spectral radius. In the appendix some useful auxiliary results are summarized.

Chapter 1

Introduction

Multigrid algorithms are very efficient and versatile techniques for the solution of large systems of (non)linear algebraic equations. During the past decades many different multigrid algorithms have been developed and applied to a wide variety of problems. In particular, the solution of the algebraic systems resulting from discretizations of partial differential equations using finite difference, finite volume or finite element methods has been very important. Apart from the development and application of multigrid algorithms also extensive mathematical analysis has been conducted for many multigrid algorithms. This has resulted in detailed knowledge about the design of optimal multigrid algorithms, their performance and efficient implementation. For many problems multigrid algorithms now achieve an excellent computational efficiency and robustness and are widely used in many (commercial) codes. Also, their suitability for use on parallel computers, which is nowadays essential for large scale problems, is very important.

Achieving excellent multigrid performance is, however, nontrivial. In particular, new classes of problems frequently require a detailed analysis and optimization of the multigrid algorithm. The objectives of these notes are to summarize some important mathematical techniques for the analysis of the performance of multigrid algorithms. An important tool is discrete Fourier analysis, which will be used to estimate the convergence rate of both two- and three-level h-multigrid algorithms. The performance estimates obtained with discrete Fourier analysis, in particular the spectral radius and operator norms of the multigrid operator, are very useful in the analysis and optimization of multigrid algorithms.

These notes do not aim at providing a comprehensive survey of multigrid methods. Some basic knowledge of multigrid methods is assumed. There are many introductory text books on multigrid methods with different levels of mathematical sophistication which can be consulted for additional information. See for instance Briggs et al. [2], Hackbusch [3], Hackbusch and Trottenberg [4], Shaidurov [10], Trottenberg et al. [11] and Wesseling [16].

The main components in a multigrid algorithm are an iterative method and coarsened approximations of the algebraic system. In addition, restriction and prolongation operators are necessary to connect the various approximations of the algebraic system. In case of partial differential equations the coarsened algebraic systems can be obtained either by discretizing the equations on meshes with a different number of degrees of freedom, resulting in h-multigrid algorithms, or by using discretizations with different orders of accuracy, which give p-multigrid methods. Of course combinations of both techniques are possible, resulting in hp-multigrid methods.

The design of the iterative method, which is frequently called a smoother since it mainly acts on the high frequency components of the error, and the restriction and prolongation operators are crucial for multigrid performance. Also, the coarsening of the algebraic system, in particular the discretization on the coarse meshes in case of numerical approximations of partial differential equations, can have a significant impact on multigrid performance. If these components in the multigrid algorithm are not chosen properly then a severe degradation of the convergence rate can be observed, and even divergence of the multigrid algorithm is possible.

For linear problems discrete Fourier analysis can provide detailed information on these aspects. This is achieved by analyzing the full two- or three-level multigrid algorithm, which will be discussed in detail in this report. These analysis techniques are rather technical, but they provide a wealth of information about the multigrid algorithm. Due to its complexity, the analysis of multigrid algorithms is frequently restricted to two-level analysis, or even the simpler analysis of the multigrid smoother. For many problems this results in a rather poor prediction of the actual multigrid performance. It is therefore important to consider realistic model problems and extend the analysis to three grid levels, see e.g. Wienands and Oosterlee [18]. This can significantly enhance the accuracy of the analysis and is essential if one aims at optimizing the multigrid algorithm.

For higher order accurate discretizations it is important to use *hp*-multigrid algorithms. These algorithms generally use a V-cycle *p*-multigrid and *h*-multigrid at the coarsest *p*-level. These multigrid algorithms give a significantly improved convergence rate for higher order problems, but are not always sufficiently efficient, e.g. for higher order accurate discontinuous Galerkin discretizations of advection dominated flows. For this purpose we extended the *hp*-multigrid algorithm to the *hp*-Multigrid as Smoother algorithm, which also includes semi-coarsening, see Van der Vegt and Rhebergen [14, 15]. In this report we will also discuss the multilevel Fourier analysis of an *hp-MGS* algorithm with three *p*-levels and three uniformly coarsened and three semi-coarsened *h*-multigrid levels. This analysis then essentially covers all reasonable *hp*-multigrid algorithms.

The multilevel analysis is also important for nonlinear problems. These problems, which are frequently solved with (versions of) a Newton-multigrid method or a Full Approximation Scheme (FAS), are much harder to solve. An important component in many nonlinear algorithms is, however, the solution of linearized equations, but also in case of fully nonlinear algorithms the analysis of linearizations of these algorithms is important.

The outline of these notes is as follows. In Chapter 2 we give an overview of basic multigrid algorithms for linear and nonlinear systems, including the *hp-MGS* algorithm. Next, in Chapter 3 the general formulation of the multigrid error transformation operator for linear problems will be derived. First for standard *h*-multigrid and then for the *hp-MGS* algorithm. In Chapter 4 multilevel Fourier analysis will be discussed. Both, two- and three-level *h*-multigrid and the *hp-MGS* algorithm will be discussed in detail. This analysis provides the spectral radius and operator norms of the multigrid algorithm which be discussed in Section 5.

Chapter 2

Brief Overview of Multigrid Techniques

In these notes we are interested in the analysis of multigrid techniques for the solution of algebraic systems originating from the discretization of partial differential equations with for instance a finite difference, finite volume or finite element method. Since the main analysis tool, viz. discrete Fourier analysis, is primarily limited to linear problems, we will first discuss the standard h-multigrid algorithm for linear systems and its extension to higher order accurate discretizations, viz. the *hp-MGS* algorithm. In addition, several Runge-Kutta type smoothers will be discussed. Since the analysis techniques for linear problems are also applicable to linearizations of nonlinear algorithms, we will also briefly discuss multigrid techniques for nonlinear problems, in particular the Newton multigrid method and the Full Approximation Scheme (FAS).

In order to simplify notation we define the product and division of vectors element-wise. Hence for $a, b \in \mathbb{R}^d$ we have

$$ab := (a_1 b_1, \cdots, a_d b_d) \in \mathbb{R}^d \quad \text{and} \quad a/b := (a_1/b_1, \cdots, a_d/b_d) \in \mathbb{R}^d.$$

2.1 Standard h-Multigrid algorithm for linear systems

In a standard h-multigrid algorithm for the solution of the algebraic system obtained after the discretization of partial differential equations we introduce a finite sequence of increasingly coarser meshes $\mathcal{M}_{nh}$, with $n = (n_1, \cdots, n_d) \in \mathbb{N}^d$ and $h \in (\mathbb{R}^+)^d$. These meshes are used to generate approximations of the discretization on the fine mesh $\mathcal{M}_h$. For simplicity we will only consider in the analysis uniformly and semi-coarsened meshes, but multigrid algorithms can also be applied to discretizations on general unstructured meshes.

In the h-multigrid algorithm we need to connect the different meshes using restriction operators

$$R_{nh}^{mh} : \mathcal{M}_{nh} \to \mathcal{M}_{mh}$$

and prolongation operators

$$P_{mh}^{nh} : \mathcal{M}_{mh} \to \mathcal{M}_{nh},$$

with $n, m \in \mathbb{N}^d$ and $n_i \leq m_i$, $i \in \{1, \cdots, d\}$, where $n_j < m_j$ for some $j \in \{1, \cdots, d\}$. The main goal of the multigrid algorithm is to iteratively solve in an efficient way the system of

Algorithm 1 Standard h-Multigrid Algorithm (H_{nh})

$v_{nh} := H_{nh}(L_{nh}, f_{nh}, v_{nh}, n, \nu_1, \nu_2, \gamma)$
{
if coarsest mesh **then**
    $v_{nh} := L_{nh}^{-1} f_{nh}$;
    return
end if
// pre-smoothing
for $it = 1, \cdots, \nu_1$ **do**
    $v_{nh} := v_{nh} - \mathcal{S}_{nh}(L_{nh} v_{nh} - f_{nh})$;
end for
// coarse grid solution
$r_{nh} := f_{nh} - L_{nh} v_{nh}$;
$f_{2nh} := R_{nh}^{2nh} r_{nh}$;
$v_{2nh} := 0$;
for $ic = 1, \cdots, \gamma$ **do**
    $v_{2nh} := H_{nh}(L_{2nh}, f_{2nh}, v_{2nh}, 2n, \nu_1, \nu_2, \gamma)$;
end for
//coarse grid correction
$v_{nh} := v_{nh} + P_{2nh}^{nh} v_{2nh}$;
// post-smoothing
for $it = 1, \cdots, \nu_2$ **do**
    $v_{nh} := v_{nh} - \mathcal{S}_{nh}(L_{nh} v_{nh} - f_{nh})$;
end for
}

algebraic equations

$$L_h u_h = f_h \qquad \text{on } \mathcal{M}_h, \tag{2.1}$$

with L_h a discretion operator and f_h a given righthand side. In these notes we will assume that L_h is a linear operator and represented by a matrix. The multigrid algorithm also uses a set of auxiliary problems at the grid levels $\mathcal{M}_{nh}$

$$L_{nh} u_{nh} = f_{nh}. \tag{2.2}$$

We assume that each operator L_{nh} is invertible. In the multigrid algorithm the linear systems are solved approximately using an iterative method $\mathcal{S}_{nh}$, which starts from an initial guess. Since, the main effect of the multigrid algorithm should be the damping of high frequency error components, the operator $\mathcal{S}_{nh}$ is also called a smoothing operator. The main steps in a multigrid algorithm for linear problems are summarized in Table 1. Using different sequences of meshes $\mathcal{M}_{nh}$ various multigrid cycles, such as the V, W or F-cycle can be constructed by selecting the proper values of the multigrid parameters ν_1, ν_2 and γ.
The multigrid algorithm discussed in this section is a so-called h-multigrid method, which refers to the use of meshes with different grid resolution. For higher order discretizations one can also use approximations with different order of accuracy, which results in p-multigrid. The p-multigrid algorithm is essentially the same as the h-multigrid method. The only difference are the restriction and prolongation operators. The restriction operator is a projection of the data on a lower order polynomial space, whereas the prolongation interpolates the data to a higher order polynomial space.

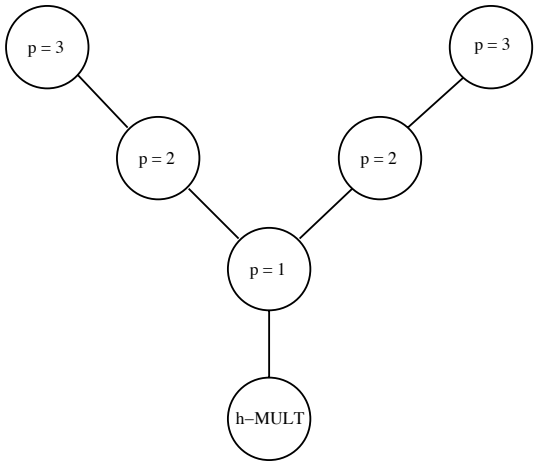

Figure 2.1: *hp*-multigrid algorithm using *h*-multigrid with uniform coarsening at the $p = 1$ polynomial level.

2.2 *hp*-Multigrid as Smoother Algorithm

For higher order accurate discretizations the standard *h*-multigrid algorithm generally is not sufficiently efficient. One option to improve multigrid efficiency is to use an *hp*-multigrid algorithm in which a V-cycle *p*-multigrid algorithm is combined with an *h*-multigrid algorithm at the lowest polynomial level, see Figure 2.1. The *hp*-multigrid algorithm can significantly improve multigrid performance, but in particular for higher order accurate discretizations of partial differential equations with a boundary layer solution a further performance improvement is frequently necessary. This can be accomplished by introducing semi-coarsened meshes which are only coarsened in one (local) coordinate direction. The semi-coarsening multigrid is then used as smoother in the *h*-multigrid, resulting in the *h*-Multigrid as Smoother (*h-MGS*) algorithm. Next, the *h-MGS* algorithm is used as smoother in the *p*-multigrid, which gives the *hp*-Multigrid as Smoother (*hp-MGS*) algorithm. A schematic overview is given in Figures 2.2 - 2.3.

The *hp-MGS* algorithm for the solution of (2.1) is described in Algorithms 2, 3 and 4, with $n = (n_1, n_2) \in \mathbb{N}^2$ and $h = (h_1, h_2) \in (\mathbb{R}^+)^2$. The first part of the *hp-MGS* algorithm is given recursively by Algorithm 2 and consists of the V-cycle *p*-multigrid algorithm $HP_{nh,p}$ with the *h-MGS* smoother $HU_{nh,p}$. In Algorithm 2 the linear system is denoted as $L_{nh,p}$. The linear system originates from a numerical discretization with polynomial order p and mesh sizes h_1 and h_2 in the different local coordinate directions. The mesh coarsening is indicated by the integer $n = (n_1, n_2)$. The unknown coefficients in the linear system are $v_{nh,p}$ and the known righthand as $f_{nh,p}$. The parameters $\gamma_1, \gamma_2, \nu_1, \nu_2, \mu_1, \mu_2$ and μ_3 are used to control the multigrid algorithm, such as the number of pre- and post-relations at each grid level and polynomial order. The $HP_{nh,p}$-multigrid algorithm uses the restriction operators $Q^{p-1}_{nh,p}$ and the prolongation operator $T^{p}_{nh,p-1}$. The restriction operators $Q^{p-1}_{nh,p}$ project data from a discretization with polynomial order p to a discretization with polynomial order $p-1$. The prolongation operators $T^{p}_{nh,p-1}$ interpolate data from a discretization with polynomial order $p-1$ to a discretization with polynomial order p. The *h-MGS*-multigrid algorithm $HU_{nh,p}$ is given by Algorithm 3.

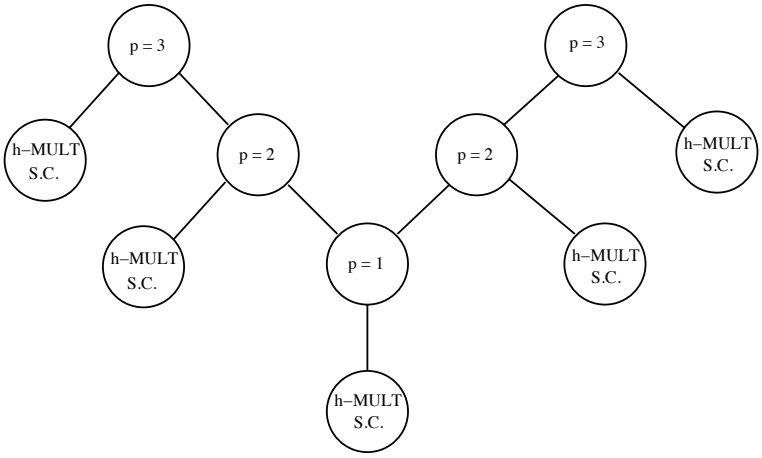

Figure 2.2: *hp-MGS*-algorithm combining p-multigrid and h-multigrid at each polynomial level. The smoother is the h-Multigrid as Smoother algorithm combining semi-coarsening in the local x_1- and x_2-directions and a semi-implicit Runge-Kutta method.

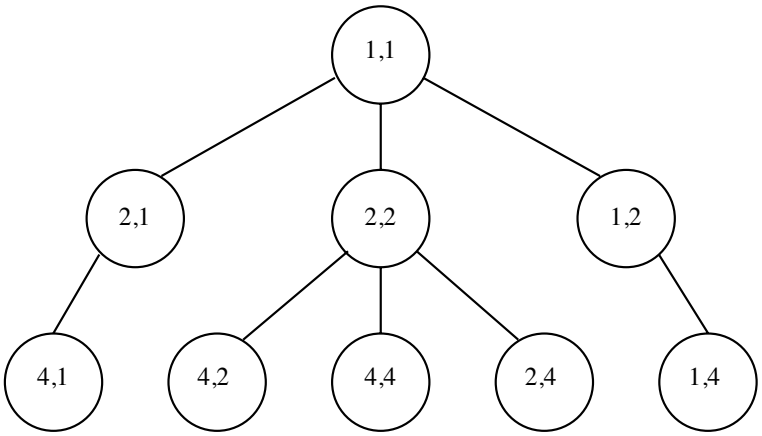

Figure 2.3: h-Multigrid as Smoother algorithm used at each polynomial level in the *hp-MGS* algorithm. The indices refer to grid coarsening. Mesh $(1,1)$ is the fine mesh and e.g. Mesh $(4,1)$ has size $(4h_1, h_2)$.

Algorithm 2 *hp-MGS* Multigrid Algorithm ($HP_{nh,p}$)

$v_{nh,p} := HP_{nh,p}(L_{nh,p}, f_{nh,p}, v_{nh,p}, n, p, \gamma_1, \gamma_2, \nu_1, \nu_2, \mu_1, \mu_2, \mu_3)$
{
if polynomial level $p == 1$ **then**
  $v_{nh,p} := HU_{nh,p}(L_{nh,p}, f_{nh,p}, v_{nh,p}, n, p, \nu_1, \nu_2, \mu_1, \mu_2, \mu_3)$;
  return
end if
// pre-smoothing with *h-MGS* algorithm
for $it = 1, \cdots, \gamma_1$ **do**
  $v_{nh,p} := HU_{nh,p}(L_{nh,p}, f_{nh,p}, v_{nh,p}, n, p, \nu_1, \nu_2, \mu_1, \mu_2, \mu_3)$;
end for
// lower order polynomial solution
$r_{nh,p} := f_{nh,p} - L_{nh,p} v_{nh,p}$;
$f_{nh,p-1} := Q^{p-1}_{nh,p} r_{nh,p}$;
$v_{nh,p-1} := 0$;
$v_{nh,p-1} := HP_{nh,p}(L_{nh,p-1}, f_{nh,p-1}, v_{nh,p-1}, n, p-1, \gamma_1, \gamma_2, \nu_1, \nu_2, \mu_1, \mu_2, \mu_3)$;
// lower order polynomial correction
$v_{nh,p} := v_{nh,p} + T^{p}_{nh,p-1} v_{nh,p-1}$;
// post-smoothing with *h-MGS* algorithm
for $it = 1, \cdots, \gamma_2$ **do**
  $v_{nh,p} := HU_{nh,p}(L_{nh,p}, f_{nh,p}, v_{nh,p}, n, p, \nu_1, \nu_2, \mu_1, \mu_2, \mu_3)$;
end for
}

In the $HU_{nh,p}$-multigrid algorithm semi-coarsening multigrid, indicated with $HS^i_{nh,p}$, $i = 1, 2$, is used as smoother. The restriction of the data from the mesh $\mathcal{M}_{nh}$ to the mesh $\mathcal{M}_{mh}$, with $m_1 \geq n_1$ and $m_2 \geq n_2$, is indicated by the restriction operator $R^{mh}_{nh,p}$. The prolongation of the data from the mesh $\mathcal{M}_{mh}$ to the mesh $\mathcal{M}_{nh}$ is given by the prolongation operator $P^{nh}_{mh,p}$. The semi-coarsening *h*-multigrid smoother $HS^i_{nh,p}$ is defined in Algorithm 4. The smoother in the coordinate direction i is indicated with $S^i_{nh,p}$.

Various multigrid algorithms can be obtained by simplifying the *hp-MGS* algorithm given by Algorithms 2–4. The first simplification is obtained by replacing in the $HP_{nh,p}$ algorithm for polynomial levels $p > 1$ the *h-MGS*-multigrid smoother $HU_{nh,p}$ with the smoothers $S^2_{nh,p} S^1_{nh,p}$ in the pre-smoothing step and $S^1_{nh,p} S^2_{nh,p}$ in the post-smoothing step. We denote this algorithm as the *hp-MGS(1)* algorithm, since the *h-MGS* algorithm is now only used at the $p = 1$ level. The second simplification is to use only uniformly coarsened meshes in the *hp-MGS(1)* algorithm instead of semi-coarsened meshes. In addition, the semi-coarsening smoothers $HS^i_{nh,p}$ in the $HU_{nh,p}$ algorithm are replaced by the smoothers $S^i_{nh,p}$ for $i = 1, 2$. We denote this algorithm as *hp*-multigrid.

2.3 Runge-Kutta type multigrid smoothers

As multigrid smoothers we use in Algorithm 4 a pseudo-time integration method. In a pseudo-time integration method the linear system

$$L_{nh,p} v_{nh,p} = f_{nh,p}, \tag{2.3}$$

Algorithm 3 *h-MGS* Multigrid Algorithm ($HU_{nh,p}$)

$v_{nh,p} := HU_{nh,p}(L_{nh,p}, f_{nh,p}, v_{nh,p}, n, p, \nu_1, \nu_2, \mu_1, \mu_2, \mu_3)$
{
if coarsest uniformly coarsened mesh **then**
    $v_{nh,p} := L^{-1}_{nh,p} f_{nh,p}$;
    return
end if
// pre-smoothing using semi-coarsening multigrid
for $it = 1, \cdots, \nu_1$ **do**
    $v_{nh,p} := HS^1_{nh,p}(L_{nh,p}, f_{nh,p}, v_{nh,p}, 1, n, p, \mu_1, \mu_2, \mu_3)$;
    $v_{nh,p} := HS^2_{nh,p}(L_{nh,p}, f_{nh,p}, v_{nh,p}, 2, n, p, \mu_1, \mu_2, \mu_3)$;
end for
// coarse grid solution
$r_{nh,p} := f_{nh,p} - L_{nh,p} v_{nh,p}$;
$f_{2nh,p} := R^{2nh}_{nh,p} r_{nh,p}$;
$v_{2nh,p} := 0$;
$v_{2nh,p} := HU_{nh,p}(L_{2nh,p}, f_{2nh,p}, v_{2nh,p}, 2n, p, \nu_1, \nu_2, \mu_1, \mu_2, \mu_3)$;
// coarse grid correction
$v_{nh,p} := v_{nh,p} + P^{nh}_{2nh,p} v_{2nh,p}$;
// post-smoothing using semi-coarsening multigrid
for $it = 1, \cdots, \nu_2$ **do**
    $v_{nh,p} := HS^2_{nh,p}(L_{nh,p}, f_{nh,p}, v_{nh,p}, 2, n, p, \mu_1, \mu_2, \mu_3)$;
    $v_{nh,p} := HS^1_{nh,p}(L_{nh,p}, f_{nh,p}, v_{nh,p}, 1, n, p, \mu_1, \mu_2, \mu_3)$;
end for
}

Algorithm 4 Semi-coarsening Multigrid Algorithm ($HS^i_{nh,p}$)

$v_{nh,p} := HS^i_{nh,p}(L_{nh,p}, f_{nh,p}, v_{nh,p}, i, n, p, \mu_1, \mu_2, \mu_3)$
{
if ($i == 1$ **and** coarsest mesh in i_1-direction) **or** ($i == 2$ **and** coarsest mesh in i_2-direction) **then**
  for $it = 1, \cdots, \mu_3$ **do**
    $v_{nh,p} := S^i_{nh,p}(L_{nh,p}, f_{nh,p}, v_{nh,p})$;
  end for
  return
end if
// pre-smoothing
for $it = 1, \cdots, \mu_1$ **do**
  $v_{nh,p} := S^i_{nh,p}(L_{nh,p}, f_{nh,p}, v_{nh,p})$;
end for
// coarse grid solution on semi-coarsened meshes
$r_{nh,p} := f_{nh,p} - L_{nh,p} v_{nh,p}$;
if ($i == 1$) **then**
  // semi-coarsening in i_1-direction
  $f_{(2n_1,n_2)h,p} := R^{(2n_1,n_2)h}_{nh,p} r_{nh,p}$;
  $v_{(2n_1,n_2)h,p} := 0$;
  $v_{(2n_1,n_2)h,p} := HS^1_{nh,p}(L_{(2n_1,n_2)h,p}, f_{(2n_1,n_2)h,p}, v_{(2n_1,n_2)h,p}, i, (2n_1, n_2), p, \mu_1, \mu_2, \mu_3)$;
  $v_{nh,p} := v_{nh,p} + P^{nh}_{(2n_1,n_2)h,p} v_{(2n_1,n_2)h,p}$;
else if ($i == 2$) **then**
  // semi-coarsening in i_2-direction
  $f_{(n_1,2n_2)h,p} := R^{(n_1,2n_2)h}_{nh,p} r_{nh,p}$;
  $v_{(n_1,2n_2)h,p} := 0$;
  $v_{(n_1,2n_2)h,p} := HS^2_{nh,p}(L_{(n_1,2n_2)h,p}, f_{(n_1,2n_2)h,p}, v_{(n_1,2n_2)h,p}, i, (n_1, 2n_2), p, \mu_1, \mu_2, \mu_3)$;
  $v_{nh,p} := v_{nh,p} + P^{nh}_{(n_1,2n_2)h,p} v_{(n_1,2n_2)h,p}$;
end if
// post-smoothing
for $it = 1, \cdots, \mu_2$ **do**
  $v_{nh,p} := S^i_{nh,p}(L_{nh,p}, f_{nh,p}, v_{nh,p})$;
end for
}

is solved by adding a pseudo-time derivative. This results in a system of ordinary differential equations

$$\frac{\partial v^*_{nh,p}}{\partial \sigma} = -\frac{1}{\triangle t}(L_{nh,p} v^*_{nh,p} - f_{nh,p}), \tag{2.4}$$

which is integrated to steady-state in pseudo-time. At steady state, $v_{nh,p} = v^*_{nh,p}$. Note, for nonlinear problems this system is obtained after linearization. The matrix $L_{nh,p}$ is then the Jacobian of the nonlinear algebraic system. The *hp-MGS* algorithm therefore naturally combines with a Newton multigrid method for nonlinear problems.

Since the goal of the pseudo-time integration is to reach steady state as efficiently as possible, time accuracy is not important. This allows the use of low order time integration methods, which can be optimized to improve multigrid convergence to steady state. In [6, 13] optimized explicit pseudo-time Runge-Kutta methods are presented, which are used for the solution of second order accurate space-time DG discretizations of the compressible Euler and Navier-Stokes equations [8, 13]. An important benefit of these explicit pseudo-time smoothers is that they can be directly applied to nonlinear problems without linearization. For higher order accurate DG discretizations, in particular for problems with thin boundary layers, the performance of these smoothers is, however, insufficient. This motivated the development of a semi-implicit Runge-Kutta pseudo-time integration method, which will be discussed in the next section.

Semi-Implicit Runge-Kutta smoother

The system of ordinary differential equations (2.4) can be solved using a five-stage semi-implicit Runge-Kutta method. In the semi-implicit Runge-Kutta method we use the fact that the *hp-MGS* algorithm uses semi-coarsening in the local i_1- and i_2-directions of each element. This makes it a natural choice to use a Runge-Kutta pseudo-time integrator which is implicit in the local directions used for the semi-coarsening. Also, the space-(time) DG discretization uses, next to data on the element itself, only data from elements connected to each of its faces. This results in a linear system with a block matrix structure. It is therefore straightforward to use a Runge-Kutta pseudo-time integrator which is alternating implicit in the local i_1 and i_2-direction. The linear system then consists of uncoupled systems of block tridiagonal matrices, which can be efficiently solved with a direct method. The semi-implicit pseudo-time integration method then can efficiently deal with highly stretched meshes in boundary layers. For this purpose we split the matrix $L_{nh,p}$, when sweeping in the i_1-direction, as

$$L_{nh,p} = L^{i_{11}}_{nh,p} + L^{i_{12}}_{nh,p},$$

and for sweeps in the i_2-direction as

$$L_{nh,p} = L^{i_{21}}_{nh,p} + L^{i_{22}}_{nh,p}.$$

The matrices $L^{i_{11}}_{nh,p}$ and $L^{i_{21}}_{nh,p}$ contain the contribution from the element itself and the elements connected to each face in the i_1-direction, respectively, i_2-direction, which are treated implicitly. The matrices $L^{i_{12}}_{nh,p}$ and $L^{i_{22}}_{nh,p}$ contain the contribution from each face in the i_2-direction, respectively, i_1-direction, which are treated explicitly. Since the DG discretization only uses information from nearest neighboring elements this provides a very natural way to define the lines along which the discretization is implicit. The semi-implicit Runge-Kutta method for sweeps in the i_1-direction then can be defined for the $l+1$ pseudo-

time step as

$$
\begin{aligned}
&v_0 = v^l_{nh,p} \\
&v_k = \big(I_{nh,p} + \beta_k \lambda_\sigma L^{i_{11}}_{nh,p}\big)^{-1}\big(v_0 - \lambda_\sigma \sum_{j=0}^{k-1} \alpha_{kj}(L^{i_{12}}_{nh,p} v_j - f_{nh,p})\big), \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k = 1, \cdots, 5, \\
&v^{l+1}_{nh,p} = S^i_{nh,p} v^l_{nh,p} = v_5,
\end{aligned}
\tag{2.5}
$$

with a similar relation for sweeps in the i_2-direction, where i_{11} is replaced by i_{21} and i_{12} with i_{22}. Here, α_{kj} are the Runge-Kutta coefficients, $\beta_k = \sum_{j=0}^{k-1} \alpha_{kj}$ for $k = 1, \cdots 5$, $\lambda_\sigma = \triangle\sigma/\triangle t$, with $\triangle\sigma$ the pseudo-time step. At steady state of the σ-pseudo-time integration we obtain the solution of the linear system (2.3). The coefficients β_k ensure that the semi-implicit Runge-Kutta operator is the identity operator if $v^l_{nh,p}$ is the exact steady state solution of (2.4). Without this condition the pseudo-time integration method would not converge to a steady state. The only requirement we impose on the Runge-Kutta coefficients α_{kj} is that the algorithm is first order accurate in pseudo-time, which implies the consistency condition

$$\sum_{j=0}^{4} \alpha_{5j} = 1.$$

For each polynomial level all other Runge-Kutta coefficients can be optimized to improve the pseudo-time convergence in combination with the *hp-MGS* algorithm. For the computation of the multigrid error transformation operator we define the semi-implicit Runge-Kutta operator $Q^1_{nh,p}$ recursively for sweeps in the i_1-direction as

$$
\begin{aligned}
&Q_0 = I_{nh,p} \\
&Q_k = \big(I_{nh,p} + \beta_k \lambda_\sigma L^{i_{11}}_{nh,p}\big)^{-1}\big(I_{nh,p} - \lambda_\sigma \sum_{j=0}^{k-1} \alpha_{kj} L^{i_{12}}_{nh,p} Q_j\big), \quad k = 1, \cdots, 5, \\
&Q^1_{nh,p} = Q_5,
\end{aligned}
\tag{2.6}
$$

with a similar expression for $Q^2_{nh,p}$ in the i_2-direction, only with i_{11} and i_{12} replaced by, respectively, i_{21} and i_{22}.

Point-Implicit Runge-Kutta smoother

A second approach to solve the system of ordinary differential equations (2.4) is provided by a five-stage Point-Implicit Runge-Kutta (PIRK) method.

$$
\begin{aligned}
&v_0 = v^l_{nh,p} \\
&v_k = \Big(v_0 - \lambda_\sigma \sum_{j=0}^{k-1} \beta_{kj} v_j - \lambda_\sigma \sum_{j=0}^{k-1} \alpha_{kj}\big(L_{nh,p} v_j - f_{nh,p}\big)\Big)/(1 + \lambda_\sigma \beta_{kk}), \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k = 1, \cdots, 5, \\
&v^{l+1}_{nh,p} = v_5,
\end{aligned}
\tag{2.7}
$$

with Runge-Kutta coefficients α_{kj} and β_{kj}, $\lambda_\sigma = \triangle\sigma/\triangle t$, and $\triangle\sigma$ the pseudo-time step. At steady state of the pseudo-time integration we obtain the solution of the linear system (2.3).

The coefficients β_{kj} must satisfy the conditions $\sum_{j=0}^{k} \beta_{kj} = 0$ and $\beta_{kk} > 0$ for $k = 1, \cdots, 5$. The only requirement we impose on the Runge-Kutta coefficients α_{kj} is that the algorithm is first order accurate in pseudo-time, which implies the consistency condition

$$\sum_{j=0}^{4} \alpha_{5j} = 1.$$

All other Runge-Kutta coefficients can be optimized to improve the pseudo-time convergence in combination with the multigrid algorithm. For the computation of the multigrid error transformation operator discussed in Chapter 3, we also define the point-implicit Runge-Kutta operator $P_{nh,p}$ recursively as

$$\begin{aligned} &P_0 = I_{nh,p} \\ &P_k = \Big(I_{nh,p} - \lambda_\sigma \sum_{j=0}^{k-1} \big(\beta_{kj} + \alpha_{kj} L_{nh,p}\big) P_j \Big) / (1 + \beta_{kk}\lambda_\sigma), \qquad k = 1, \cdots, 5, \\ &P_{nh,p} = P_5. \end{aligned} \tag{2.8}$$

2.4 Multigrid algorithms for nonlinear systems

For nonlinear problems we can not directly use the algorithms discussed in Sections 2.1 and 2.2. Two main approaches exist to deal with nonlinear algebraic equations in a multigrid context, viz. the Newton multigrid method and the Full Approximation Scheme (FAS). In the next two sections we will summarize both algorithms.

Consider the nonlinear system of algebraic equations, obtained for instance by discretizing a system of nonlinear partial differential equations on the mesh $\mathcal{M}_h$,

$$N_h v_h = f_h \tag{2.9}$$

with N_h the nonlinear operator and f_h a given righthand side. Assume that w_h is an approximation to the exact solution v_h. We define then the error $e_h = v_h - w_h$ and the residual $r_h = f_h - N_h w_h$. Subtracting $N_h w_h$ from both sides of (2.9) yields

$$N_h v_h - N_h w_h = r_h \tag{2.10}$$

Note, since N_h is nonlinear we have $N_h(v_h - w_h) \neq r_h$. Hence we can not determine the error from linear equations using various meshes in a multigrid algorithm. In order to solve (2.9) we can first (approximately) linearize the equations using a Newton method or use a Picard iteration. The resulting linear algebraic equations then can be solved with a linear multigrid method. In the FAS method, discussed in Section 2.4.2, (2.10) is used as starting point for the derivation of the multigrid algorithm.

2.4.1 Newton multigrid method

The Newton multigrid method is based on Newton's method. Consider the scalar equation $F(x) = 0$. Newton's method is obtained by expanding $F(x)$ in a Taylor series around the point y and truncating at the quadratic term results in

$$F(x) = F(y) + (x - y)F'(y) + \frac{1}{2}(x - y)^2 F''(\xi)$$

for some ξ in between x and y. Newton's method results then in the following iteration method. Given an initial guess x_0, the solution x_j in iteration j is then obtained through

$$x_{j+1} = x_j - \frac{F(x_j)}{F'(x_j)} \qquad \text{with } j \in \mathbb{N}.$$

The extension of the Taylor expansion to a system of n nonlinear equations is given by

$$N_h(w_h + e_h) = N_h w_h + J_h(w_h)e_h + \text{higher order terms}$$

with $e_h = v_h - w_h$ and the Jacobian matrix defined as

$$J_h(w_h) = \begin{pmatrix} \frac{\partial N_{h1}}{\partial w_{h1}} & \cdots & \frac{\partial N_{h1}}{\partial w_{hn}} \\ \vdots & \ddots & \vdots \\ \frac{\partial N_{hn}}{\partial w_{h1}} & \cdots & \frac{\partial N_{hn}}{\partial w_{hn}} \end{pmatrix}$$

and $w_h = (w_{h1}, \cdots, w_{hn})$ and $N_h = (N_{h1}, \cdots, N_{hn})$. Neglecting quadratic terms we obtain, using (2.9) and the definition of e_h,

$$\begin{aligned} J_h(w_h)e_h &= N_h(w_h + e_h) - N_h w_h \\ &= N_h v_h - N_h w_h \\ &= f_h - N_h w_h. \end{aligned}$$

Newton's method for nonlinear systems is then defined through the following iteration process:
Given an initial guess w_h^0, the iterates w_h^j are then obtained from

$$w_h^{j+1} = w_h^j + J_h^{-1}(w_h^j)(f_h - N_h w_h^j).$$

The Newton-multigrid algorithm is now obtained by combining the Newton method in an outer iteration with the solution of the resulting linear system with a multigrid method for linear problems. There are various modifications possible to this algorithm. In many cases it is difficult to compute the Jacobian matrix J_h exactly using analytic methods or it is computationally too expensive to compute an accurate Jacobian matrix either through automatic differentiation or numerical approximation. Then it is more practical to approximate the Jacobian matrix, e.g. by neglecting certain contributions. This results in an approximate Newton method which generally converges slower but can be computationally more efficient. The process of computing the Jacobian matrix can also be combined with the iterative solution of the linear system using a Krylov method. Multigrid then can be used as a preconditioner for the Krylov method. This results in the "Jacobian free" method which requires significantly less memory since the Jacobian matrix is not stored, only the vector $J_h w_h$. The success of this technique, however, strongly depends on the preconditioner for the Krylov method, which is generally a nontrivial task.

2.4.2 Full approximation scheme

The Full Approximation Scheme (FAS) directly considers the nonlinear algebraic equations. We first consider the FAS algorithm for two mesh levels. Given a numerical approximation w_h^j of (2.9) on the fine mesh $\mathcal{M}_h$. This solution satisfies the nonlinear equation

$$N_h(w_h^j + e_h^j) = f_h, \tag{2.11}$$

with $e_h^j = v_h - w_h^j$. Restrict (2.11) now to the next coarser mesh $\mathcal{M}_{2h}$ and use (2.10) to obtain

$$N_{2h}(w_{2h}^j + e_{2h}^j) - N_{2h}w_{2h}^j = r_{2h}^j.$$

The coarse grid residual is obtained by restricting the fine grid residual r_h^j to $\mathcal{M}_{2h}$, resulting in

$$r_{2h}^j = \bar{R}_h^{2h} r_h^j = \bar{R}_h^{2h}(f_h - N_h w_h^j).$$

Also, the coarse grid solution w_{2h}^j is obtained by restriction of the fine grid solution

$$w_{2h}^j = R_h^{2h} w_h^j.$$

Note, the restriction operators R_h^{2h} and $\bar{R}_h^{2h}$ do not necessarily have to be the same operators. The coarse grid equation can now be expressed as

$$N_{2h}(\underbrace{R_h^{2h}w_h^j + e_{2h}^j}_{v_{2h}^j}) = \underbrace{N_{2h}(R_h^{2h}w_h^j) + \bar{R}_h^{2h}(f_h - N_h w_h^j)}_{f_{2h}^j},$$

where the right hand side is known. Assume we can obtain an accurate (approximate) solution to the equation

$$N_{2h}v_{2h}^j = f_{2h}^j$$

e.g. with a Newton method, then we can define the error at mesh $\mathcal{M}_{2h}$ as

$$e_{2h}^j = v_{2h}^j - R_h^{2h}w_h^j.$$

The error e_{2h}^j can now be interpolated to the mesh $\mathcal{M}_h$ using the prolongation operator P_{2h}^h and used to correct the numerical solution on the mesh $\mathcal{M}_h$

$$\begin{aligned} w_h^{j+1} &= w_h^j + P_{2h}^h e_{2h}^j \\ &= w_h^j + P_{2h}^h(v_{2h}^j - R_h^{2h}w_h^j) \end{aligned}$$

If N_h is a linear operator then the algorithm reduces to the multigrid algorithm discussed in Section 2.1.

The multilevel FAS algorithm is defined in Algorithm 5. As smoothers in the function $\mathcal{SM}_{nh}$ in Algorithm 5 one can for instance use a nonlinear Gauss-Seidel relaxation method or the point implicit Runge-Kutta time integration method discussed in Section 2.3.

2.5 Full multigrid method

Both the Newton and FAS multigrid methods require an initial condition to start the algorithm. If this solution is not sufficiently close to the exact solution then the multigrid algorithm can diverge. In addition, a solution which is closer to the exact solution makes the assumptions in the Newton and FAS algorithm more realistic and can significantly improve the efficiency of the solver. The Full Multigrid Method (FMG) provides a good initial solution by starting the Newton and FAS multigrid algorithms on the coarsest mesh and in case of a higher order accurate discretization also for the lowest possible discretization order. After a reasonable reduction of the initial residual is obtained then the solution is interpolated to the next mesh level

$$w_{nh} = \tilde{P}_{mh}^{nh} w_{mh}, \qquad \text{with } 1 \le n < m \le N_c$$

Algorithm 5 Standard h-Multigrid FAS Algorithm (FAS_{nh})

$v_{nh} := FAS_{nh}(N_{nh}, f_{nh}, v_{nh}, n, \nu_1, \nu_2, \gamma)$
{
if coarsest mesh **then**
    $v_{nh} := N_{nh}^{-1} f_{nh}$;
    return
end if
// pre-smoothing with nonlinear smoother $\mathcal{SM}_{nh}$
for $it = 1, \cdots, \nu_1$ **do**
    $v_{nh} := \mathcal{SM}_{nh}(v_{nh}, f_{nh})$;
end for
// coarse grid solution
$r_{nh} := f_{nh} - N_{nh} v_{nh}$;
$v_{2nh} := R_{nh}^{2nh} v_{nh}$;
$f_{2nh} := N_{2nh} v_{2nh} + \bar{R}_{nh}^{2nh} r_{nh}$;
for $ic = 1, \cdots, \gamma$ **do**
    $v_{2nh} := FAS_{nh}(N_{2nh}, f_{2nh}, v_{2nh}, 2n, \nu_1, \nu_2, \gamma)$;
end for
//coarse grid correction
$v_{nh} := v_{nh} + P_{2nh}^{nh}(v_{2nh} - R_h^{2nh} v_{nh})$;
// post-smoothing with nonlinear smoother $\mathcal{SM}_{nh}$
for $it = 1, \cdots, \nu_2$ **do**
    $v_{nh} := \mathcal{SM}_{nh}(v_{nh}, f_{nh})$;
end for
}

The interpolation operator $\tilde{P}_{mh}^{nh}$ should be of sufficiently high order and not generate unnecessary high frequency errors. If $m < N_c$ then at each level the FMG procedure can of course be combined with a Newton-multigrid or FAS multigrid method on the mesh levels which already have an initial solution.

Chapter 3

Multigrid Error Transformation Operators

3.1 h-Multigrid error transformation operator

In order to understand the performance of the h-multigrid algorithm, defined in Algorithm 1, we need the multigrid error transformation operator. This operator shows how much the error in the iterative solution of the algebraic system (2.1) is reduced by one full multigrid cycle. Given an initial guess v_{nh}^0 of the linear system (2.1) at grid level n then the initial error is equal to

$$e_{nh}^0 = u_{nh} - v_{nh}^0,$$

with u_{nh} the exact solution of (2.2). The multigrid error after application of the h-multigrid algorithm is then equal to

$$e_{nh}^1 = u_{nh} - v_{nh}^1,$$

with $v_{nh}^1 = H_{nh}(L_{nh}, f_{nh}, v_{nh}^0, n, \nu_1, \nu_2, \gamma)$. The initial and multigrid error at grid level n are related through the multigrid error transformation operator

$$e_{nh}^1 = M_{nh} e_{nh}^0.$$

We will now derive a recursive expression for the multigrid error transformation operator M_{nh}.

1. At the coarsest mesh $\mathcal{M}_{Nh}$ we solve (2.2) exactly, hence the error at this level is zero and M_{Nh} is the null operator.

2. At grid level n the error after l pre-smoothing iterations is defined as

   $$\sigma_{nh}^l = u_{nh} - v_{nh}^l, \qquad l = 0, 1, \cdots, \nu_1,$$

   with $\sigma_{nh}^0 = e_{nh}^0$. In the pre-smoothing step the numerical solution is updated as

   $$v_{nh}^{l+1} = v_{nh}^l - \mathcal{S}_{nh}(L_{nh} v_{nh}^l - f_{nh}), \qquad l = 0, 1, \cdots, \nu_1 - 1. \tag{3.1}$$

   Subtracting (3.1) from u_{nh} and using $L_{nh} u_{nh} = f_{nh}$ we obtain

   $$\sigma_{nh}^{l+1} = \sigma_{nh}^l - \mathcal{S}_{nh} L_{nh} \sigma_{nh}^l, \qquad l = 0, 1, \cdots, \nu_1 - 1.$$

After ν_1 pre-smoothing steps the error then is equal to

$$\sigma_{nh}^{\nu_1} = S_{nh}^{\nu_1} e_{nh}^0, \tag{3.2}$$

with $S_{nh} = I_{nh} - \mathcal{S}_{nh} L_{nh}$ and I_{nh} the identity operator at grid level n.

3. For the coarse grid solution at grid level $m > n$, we first compute the restriction of the residual

$$\begin{aligned} r_{mh} &= R_{nh}^{mh}(f_{nh} - L_{nh} v_{nh}^{\nu_1}) \\ &= R_{nh}^{mh}(L_{nh} u_{nh} - L_{nh} v_{nh}^{\nu_1}) \\ &= R_{nh}^{mh} L_{nh} \sigma_{nh}^{\nu_1}. \end{aligned}$$

At the grid level m we need to solve now

$$L_{mh} z_{mh}^* = r_{mh}, \tag{3.3}$$

which has the exact solution

$$\begin{aligned} z_{mh}^* &= L_{mh}^{-1} r_{mh} \\ &= L_{mh}^{-1} R_{nh}^{mh} L_{nh}\, \sigma_{nh}^{\nu_1}. \end{aligned} \tag{3.4}$$

4. The linear system (3.3) at the coarse grid level m is also solved iteratively using the multigrid algorithm. We start with the initial guess $z_{mh}^0 = 0$, hence the initial error is $\delta_{mh}^0 = z_{mh}^* - z_{mh}^0 = z_{mh}^*$. After γ applications of the h-multigrid algorithm the error in the multigrid solution z_{mh}^γ is reduced to

$$\delta_{mh}^\gamma = M_{mh}^\gamma z_{mh}^*,$$

hence

$$\begin{aligned} z_{mh}^\gamma &= z_{mh}^* - \delta_{mh}^\gamma \\ &= (I_{mh} - M_{mh}^\gamma)\, z_{mh}^*, \end{aligned} \tag{3.5}$$

with I_{mh} the identity operator at grid level m.

5. The solution after the coarse grid correction is denoted as

$$y_{nh}^0 = v_{nh}^{\nu_1} + P_{mh}^{nh} z_{mh}^\gamma. \tag{3.6}$$

After l post-smoothing steps this solution is updated to y_{nh}^l, $l = 0, 1, \cdots, \nu_2$, and the multigrid error is equal to $\rho_{nh}^l = u_{nh} - y_{nh}^l$. Then using subsequently (3.6), (3.5), (3.4) and (3.2) we obtain

$$\begin{aligned} \rho_{nh}^0 &= u_{nh} - y_{nh}^0 \\ &= u_{nh} - v_{nh}^{\nu_1} - P_{mh}^{nh} z_{mh}^\gamma \\ &= \sigma_{nh}^{\nu_1} - P_{mh}^{nh} (I_{mh} - M_{mh}^\gamma)\, z_{mh}^* \\ &= \left(I_{nh} - P_{mh}^{nh} (I_{mh} - M_{mh}^\gamma)\, L_{mh}^{-1} R_{nh}^{mh} L_{nh}\right) \sigma_{nh}^{\nu_1} \\ &= \left(I_{nh} - P_{mh}^{nh} (I_{mh} - M_{mh}^\gamma)\, L_{mh}^{-1} R_{nh}^{mh} L_{nh}\right) S_{nh}^{\nu_1} e_{nh}^0. \end{aligned}$$

6. Finally, due to the post-smoothing step the error ρ_{nh}^0 is modified using (3.2) into

$$\rho_{nh}^{\nu_2} = S_{nh}^{\nu_2} \rho_{nh}^0.$$

Combining all steps we obtain a recursive expression for the multigrid error transformation operator at grid level n

$$M_{nh} = S_{nh}^{\nu_2} \left(I_{nh} - P_{mh}^{nh} \left(I_{mh} - M_{mh}^{\gamma} \right) L_{mh}^{-1} R_{nh}^{mh} L_{nh} \right) S_{nh}^{\nu_1}.$$

For two grid levels ($n = 1, m = 2$) with uniform mesh coarsening the multigrid error transformation operator is equal to

$$M_h^{2g} = S_h^{\nu_2} \left(I_h - P_{2h}^h L_{2h}^{-1} R_h^{2h} L_h \right) S_h^{\nu_1} \tag{3.7}$$

and for three grid levels ($n = 1, m = 2$) and ($n = 2, m = 4$) with uniform mesh coarsening we obtain

$$M_h^{3g} = S_h^{\nu_2} \left(I_h - P_{2h}^h \left(I_{2h} - M_{2h}^{\gamma} \right) L_{2h}^{-1} R_h^{2h} L_h \right) S_h^{\nu_1} \tag{3.8}$$

with

$$M_{2h} = S_{2h}^{\nu_2} \left(I_{2h} - P_{4h}^{2h} L_{4h}^{-1} R_{2h}^{4h} L_{2h} \right) S_{2h}^{\nu_1}. \tag{3.9}$$

The two-level and three-level multigrid error transformation operators will be studied in detail in Chapter 4 using discrete Fourier analysis.

3.2 *hp-MGS* Multigrid error transformation operator

In this section we analyze the error after one application of the *hp-MGS* multigrid algorithm. We assume that the linear system (2.1) is obtained from a finite element discretization of a partial differential equation using polynomial basis functions of order p. We define the initial error in the solution of the algebraic system on the grid $\mathcal{M}_{nh}$ as

$$e_{nh,p}^0 = u_{nh,p} - v_{nh,p}^0.$$

Here, $u_{nh,p}$ is the exact solution of the algebraic system

$$L_{nh,p} u_{nh,p} = f_{nh,p},$$

and $v_{nh,p}^0$ the initial guess used in the multigrid algorithm. Similarly, the error after one application of the multigrid algorithm is defined as

$$e_{nh,p}^1 = u_{nh,p} - v_{nh,p}^1,$$

with $v_{nh,p}^1 = HP_{nh,p} v_{nh,p}^0$. The operator $HP_{nh,p}$ denotes the action of the *hp*-multigrid algorithm defined in Algorithm 2. The initial and multigrid error are related through the *hp-MGS* error transformation operator

$$e_{nh,p}^1 = M_{nh,p} e_{nh,p}^0.$$

The error transformation operator of the *hp-MGS* multigrid algorithm is obtained by computing the error transformation operators of Algorithms 2-4 defined in Section 2.2 and the pseudo-time smoothers defined in Section 2.3.

1. *p-multigrid step.* At the lowest polynomial order, which we set equal to $p = 1$, the multigrid solution is equal to

$$v^1_{nh,1} = HU_{nh,1} v^0_{nh,1}.$$

The h-multigrid operator $HU_{nh,p}$, defined in Algorithm 3, must satisfy the consistency condition

$$u_{nh,p} = HU_{nh,p} u_{nh,p},$$

hence the multigrid error at the lowest polynomial level is equal to

$$\begin{aligned} e^1_{nh,1} &= u_{nh,1} - HU_{nh,1} v^0_{nh,1} \\ &= HU_{nh,1}(u_{nh,1} - v^0_{nh,1}) \\ &= HU_{nh,1} e^0_{nh,1}. \end{aligned}$$

For $p = 1$ the multigrid error transformation operator then is equal

$$M_{nh,1} = HU_{nh,1}.$$

For polynomial orders $p > 1$, the *hp*-multigrid algorithm starts with γ_1 pre-smoothing steps using the $HU_{nh,p}$ algorithm. The error after l pre-smoothing steps is defined as

$$\sigma^l_{nh,p} = u_{nh,p} - v^l_{nh,p}, \qquad l = 0, 1, \cdots, \gamma_1,$$

with $\sigma^0_{nh,p} = e^0_{nh,p}$. During the pre-smoothing step the multigrid solution $v^0_{nh,p}$ is updated as

$$v^l_{nh,p} = (HU_{nh,p})^l v^0_{nh,p}, \qquad l = 0, 1, \cdots, \gamma_1 - 1.$$

After $l + 1$ pre-smoothing steps the error then is equal to

$$\begin{aligned} \sigma^{l+1}_{nh,p} &= u_{nh,p} - v^{l+1}_{nh,p} \\ &= HU_{nh,p} u_{nh,p} - HU_{nh,p} v^l_{nh,p} \\ &= HU_{nh,p} \sigma^l_{nh,p} \\ &= (HU_{nh,p})^{l+1} e^0_{nh,p}, \end{aligned}$$

hence $\sigma^{\gamma_1}_{nh,p} = (HU_{nh,p})^{\gamma_1} e^0_{nh,p}$. For the correction from the lower order polynomial discretization we first compute the residual and project this to the lower order polynomial space

$$\begin{aligned} f_{nh,p-1} &= Q^{p-1}_{nh,p}(f_{nh,p} - L_{nh,p} v^{\gamma_1}_{nh,p}) \\ &= Q^{p-1}_{nh,p}(L_{nh,p} u_{nh,p} - L_{nh,p} v^{\gamma_1}_{nh,p}) \\ &= Q^{p-1}_{nh,p} L_{nh,p} \sigma^{\gamma_1}_{nh,p}. \end{aligned}$$

At the polynomial level $p - 1$ we need to solve now

$$L_{nh,p-1} z^*_{nh,p-1} = f_{nh,p-1} \tag{3.10}$$

which has the exact solution

$$\begin{aligned} z^*_{nh,p-1} &= (L_{nh,p-1})^{-1} f_{nh,p-1} \\ &= (L_{nh,p-1})^{-1} Q^{p-1}_{nh,p} L_{nh,p} \sigma^{\gamma_1}_{nh,p}. \end{aligned}$$

We use the p-multigrid algorithm with $p-1$ polynomials to solve the system (3.10). Set $z^0_{nh,p-1} = 0$. The initial error at polynomial level $p-1$ is then

$$\delta^0_{nh,p-1} = z^*_{nh,p-1} - z^0_{nh,p-1} = z^*_{nh,p-1}.$$

After one step of the $HP_{nh,p}$-multigrid algorithm at the polynomial level $p-1$ the error is reduced to

$$\begin{aligned} \delta^1_{nh,p-1} &= M_{nh,p-1} \delta^0_{nh,p-1} \\ &= M_{nh,p-1} z^*_{nh,p-1} \end{aligned}$$

We also have $\delta^1_{nh,p-1} = z^*_{nh,p-1} - z^1_{nh,p-1}$, hence

$$\begin{aligned} z^1_{nh,p-1} &= z^*_{nh,p-1} - \delta^1_{nh,p-1} \\ &= z^*_{nh,p-1} - M_{nh,p-1} z^*_{nh,p-1} \\ &= (I_{nh,p-1} - M_{nh,p-1}) z^*_{nh,p-1}, \end{aligned}$$

with $I_{nh,p-1}$ the identity operator for polynomial level $p-1$. The solution after the correction with the lower order polynomial solution is equal to

$$y^0_{nh,p} = v^{\gamma_1}_{nh,p} + T^p_{nh,p-1} z^1_{nh,p-1}.$$

After l post-smoothing iterations with the $HU_{nh,p}$ algorithm this solution is updated to $y^l_{nh,p}$, and the multigrid error is equal to

$$\rho^l_{nh,p} = u_{nh,p} - y^l_{nh,p}, \qquad l = 0, 1, \cdots, \gamma_2.$$

This error can be further evaluated into

$$\begin{aligned} \rho^0_{nh,p} &= u_{nh,p} - y^0_{nh,p} \\ &= u_{nh,p} - v^{\gamma_1}_{nh,p} - T^p_{nh,p-1} z^1_{nh,p-1} \\ &= \sigma^{\gamma_1}_{nh,p} - T^p_{nh,p-1} (I_{nh,p-1} - M_{nh,p-1}) z^*_{nh,p-1} \\ &= \sigma^{\gamma_1}_{nh,p} - T^p_{nh,p-1} (I_{nh,p-1} - M_{nh,p-1}) (L_{nh,p-1})^{-1} Q^{p-1}_{nh,p} L_{nh,p} \sigma^{\gamma_1}_{nh,p}. \end{aligned}$$

The post-processing error is analogous to the pre-processing error

$$\rho^{\gamma_2}_{nh,p} = (HU_{nh,p})^{\gamma_2} \rho^0_{nh,p}.$$

Combining all terms we obtain that the error after one step on the mesh $\mathcal{M}_{nh}$ with the $HP_{nh,p}$-multigrid algorithm is equal to

$$\begin{aligned} e^1_{nh,p} &= \sigma^{\gamma_2}_{nh,p} (I_{nh,p} - T^p_{nh,p-1} (I_{nh,p-1} - M_{nh,p-1}) (L_{nh,p-1})^{-1} Q^{p-1}_{nh,p} L_{nh,p}) \sigma^{\gamma_1}_{h,p} \\ &= (HU_{nh,p})^{\gamma_2} (I_{nh,p} - T^p_{nh,p-1} (I_{nh,p-1} - M_{nh,p-1}) (L_{nh,p-1})^{-1} Q^{p-1}_{nh,p} L_{nh,p}) \\ &\quad (HU_{nh,p})^{\gamma_1} e^0_{nh,p}. \end{aligned}$$

The *hp-MGS* multigrid error transformation operator $M_{nh,p}$ on the mesh $\mathcal{M}_{nh}$ is defined recursively as

$$\begin{aligned} M_{nh,p} = &\big(HU_{nh,p}\big)^{\gamma_2}\big(I_{nh,p} - T^p_{nh,p-1}(I_{nh,p-1} - M_{nh,p-1})(L_{nh,p-1})^{-1} \\ &Q^{p-1}_{nh,p}L_{nh,p}\big)\big(HU_{nh,p}\big)^{\gamma_1} && \text{if } p > 1, \quad (3.11) \\ = &\, HU_{nh,1} && \text{if } p = 1. \end{aligned}$$

2. *h-multigrid step.* In the h-multigrid step we first compute the error reduction using the $HU_{nh,p}$-multigrid algorithm. Given the initial solution $v^0_{nh,p}$ the initial error is equal to

$$e^0_{nh,p} = u_{nh,p} - v^0_{nh,p}.$$

The error after ν_1 $HU_{nh,p}$-multigrid steps at the grid level n then is equal to

$$\sigma^{\nu_1}_{nh,p} = (HU_{nh,p})^{\nu_1} e^0_{nh,p}.$$

At the coarsest mesh with $n = N$ we use an exact solver and

$$v_{N,p} = (L_{Nh,p})^{-1} f_{Nh,p}.$$

The error is then equal to

$$e^1_{Nh,p} = u_{Nh,p} - (L_{Nh,p})^{-1} f_{Nh,p} = 0,$$

and we obtain that $HU_{Nh,p} = 0$.

At the finer meshes with $n \leq N$, with $n_i < N_i$ for some i, we set $w^0_{nh,p} = v^0_{nh,p}$. Define the error after l semi-coarsening smoother steps, in respectively the local i_1 and i_2-direction, as

$$\tau^l_{nh,p} = u_{nh,p} - w^l_{nh,p}, \qquad l = 0, 1, \cdots, \nu_1,$$

with $\tau^0_{nh,p} = e^0_{nh,p}$. After one semi-coarsening smoothing step in, respectively, the local i_1- and i_2-direction, we obtain the multigrid solution

$$w^1_{nh,p} = HS^2_{nh,p} HS^1_{nh,p} w^0_{nh,p}.$$

If we apply the semi-coarsening smoothers now ν_1-times then the initial solution $w^0_{nh,p}$ becomes equal to

$$w^{\nu_1}_{nh,p} = (HS^2_{nh,p} HS^1_{nh,p})^{\nu_1} w^0_{nh,p}.$$

Using the consistency of the semi-coarsening smoothers $HS^i_{nh,p}$, with $i = 1, 2$, we can now express the error after $l + 1$ smoother steps as

$$\begin{aligned} \tau^{l+1}_{nh,p} &= u_{nh,p} - HS^2_{nh,p} HS^1_{nh,p} w^l_{nh,p} \\ &= HS^2_{nh,p} HS^1_{nh,p} u_{nh,p} - HS^2_{nh,p} HS^1_{nh,p} w^l_{nh,p} \\ &= HS^2_{nh,p} HS^1_{nh,p} \tau^l_{nh,p} \\ &= (HS^2_{nh,p} HS^1_{nh,p})^{l+1} \tau^0_{nh,p}, \end{aligned}$$

hence $\tau_{nh,p}^{\nu_1} = (HS_{nh,p}^2 HS_{nh,p}^1)^{\nu_1} e_{nh,p}^0$. The correction from the coarse mesh with level $2n$ is obtained by first restricting the residual to this level

$$\begin{aligned} f_{2nh,p} &= R_{nh,p}^{2nh}(f_{nh,p} - L_{nh,p} w_{nh,p}^{\nu_1}) \\ &= R_{nh,p}^{2nh}(L_{nh,p} u_{nh,p} - L_{nh,p} w_{nh,p}^{\nu_1}) \\ &= R_{nh,p}^{2nh} L_{nh,p}\, \tau_{nh,p}^{\nu_1}. \end{aligned}$$

At the coarse mesh with level $2n$ we need to solve

$$L_{2nh,p} x_{2nh,p}^* = f_{2nh,p} \tag{3.12}$$

which results in

$$\begin{aligned} x_{2nh,p}^* &= (L_{2nh,p})^{-1} f_{2nh,p} \\ &= (L_{2nh,p})^{-1} R_{nh,p}^{2nh} L_{nh,p}\, \tau_{nh,p}^{\nu_1}. \end{aligned}$$

We use the h-multigrid algorithm with initial solution $x_{2nh,p}^0 = 0$ to solve the linear system (3.12). The initial error is then $\mu_{2nh,p}^0 = x_{2nh,p}^* - x_{2nh,p}^0 = x_{2nh,p}^*$. After one step of the h-multigrid algorithm we obtain

$$\begin{aligned} \mu_{2nh,p}^1 &= HU_{2nh,p} \mu_{2nh,p}^0 \\ &= HU_{2nh,p} x_{2nh,p}^*. \end{aligned}$$

We also have $\mu_{2nh,p}^1 = x_{2nh,p}^* - x_{2nh,p}^1$, thus

$$\begin{aligned} x_{2nh,p}^1 &= x_{2nh,p}^* - \mu_{2nh,p}^1 \\ &= x_{2nh,p}^* - HU_{2nh,p} x_{2nh,p}^* \\ &= (I_{2nh,p} - HU_{2nh,p}) x_{2nh,p}^*. \end{aligned}$$

The solution after the coarse grid correction is now equal to

$$t_{nh,p}^0 = w_{nh,p}^{\nu_1} + P_{2nh,p}^{nh,p} x_{2nh,p}^1$$

After ν_2 post-smoothing iterations the solution is updated to $t_{nh,p}^l$, $l = 0, 1, \cdots, \nu_2 - 1$, and the multigrid error is equal to

$$\beta_{nh,p}^l = u_{nh,p} - t_{nh,p}^l, \qquad l = 0, 1, \cdots, \nu_2.$$

The multigrid error can now be expressed as

$$\begin{aligned} \beta_{nh,p}^0 &= u_{nh,p} - t_{nh,p}^0 \\ &= u_{nh,p} - w_{nh,p}^{\nu_1} - P_{2nh,p}^{nh,p} x_{2nh,p}^1 \\ &= \tau_{nh,p}^{\nu_1} - P_{2nh,p}^{nh,p} (I_{2nh,p} - HU_{2nh,p}) x_{2nh,p}^* \\ &= \tau_{nh,p}^{\nu_1} - P_{2nh,p}^{nh,p} (I_{2nh,p} - HU_{2nh,p}) (L_{2nh,p})^{-1} R_{nh,p}^{2nh,p} L_{nh,p} \tau_{nh,p}^{\nu_1}. \end{aligned}$$

The post-smoothing step is analogous to the pre-smoothing step. Combining now the various contributions we obtain

$$\begin{aligned} e_{nh,p}^1 &= HU_{nh,p} e_{nh,p}^0 \\ &= (HS_{nh,p}^1 HS_{nh,p}^2)^{\nu_2} (I_{nh,p} - P_{2nh,p}^{nh,p} (I_{2nh,p} - HU_{2nh,p}) (L_{2nh,p})^{-1} \\ &\quad R_{nh,p}^{2nh,p} L_{nh,p}) (HS_{nh,p}^2 HS_{nh,p}^1)^{\nu_1} e_{nh,p}^0. \end{aligned}$$

The *h-MGS* error transformation operator $HU_{nh,p}$ can now be defined as

$$\begin{aligned} HU_{nh,p} = &\left(HS^1_{nh,p}HS^2_{nh,p}\right)^{\nu_2}\left(I_{nh,p} - P^{nh}_{2nh,p}(I_{2nh,p} - HU_{2nh,p})\right. \\ &\left.(L_{2nh,p})^{-1}R^{2nh}_{nh,p}L_{nh,p}\right)(HS^2_{nh,p}HS^1_{nh,p})^{\nu_1}, \qquad \text{if } n < m, \qquad (3.13) \\ = &\, 0, \qquad \text{if } n = m. \qquad (3.14) \end{aligned}$$

The $HU_{nh,p}$ error transformation operator given by (3.13)-(3.14) can also be used to obtain the semi-coarsening error transformation operators $HS^1_{nh,p}$ and $HS^2_{nh,p}$, which are equal to

$$\begin{aligned} HS^1_{nh,p} = &\left(S^1_{nh,p}\right)^{\mu_2}\left(I_{nh,p} - P^{nh}_{(2n_1,n_2)h,p}(I_{(2n_1,n_2)h,p} - HS^1_{(2n_1,n_2)h,p})\right. \\ &\left.(L_{(2n_1,n_2)h,p})^{-1}R^{(2n_1,n_2)h}_{nh,p}L_{nh,p}\right)\left(S^1_{nh,p}\right)^{\mu_1}, \qquad \text{if } n < m, \\ = &\, I_{nh,p} - \left(S^1_{nh,p}\right)^{\mu_3}, \qquad \text{if } n = m, \\ HS^2_{nh,p} = &\left(S^2_{nh,p}\right)^{\mu_2}\left(I_{nh,p} - P^{nh}_{(n_1,2n_2)h,p}(I_{(n_1,2n_2)h,p} - HS^2_{(n_1,2n_2)h,p})\right. \\ &\left.(L_{(n_1,2n_2)h,p})^{-1}R^{(n_1,2n_2)h}_{nh,p}L_{nh,p}\right)\left(S^2_{nh,p}\right)^{\mu_1}, \qquad \text{if } n < m, \\ = &\, I_{nh,p} - \left(S^2_{nh,p}\right)^{\mu_3}, \qquad \text{if } n = m, \end{aligned}$$

where we used that at the coarsest level μ_3 smoother iterations are performed.

3. *Multigrid smoothers.* The pseudo-time integrators solve the linear system

$$L_{nh,p}w_{nh,p} = f_{nh,p}. \qquad (3.15)$$

We define the error after the lth and $l+1$st pseudo-time integration step as

$$\begin{aligned} e^0_{nh,p} &= w_{nh,p} - w^l_{nh,p} \\ e^1_{nh,p} &= w_{nh,p} - w^{l+1}_{nh,p}. \end{aligned}$$

We also define the error in each Runge-Kutta stage

$$\bar{e}_i = w_{nh,p} - w_i,$$

with $\bar{e}_0 = e^0_{nh,p}$.

(a) Semi-Implicit Runge-Kutta pseudo-time integrator. This pseudo-time integrator solves at steady state the linear system (3.15). Using (3.15) the semi-implicit Runge-Kutta method (2.5) for the local i_1 direction can be transformed into

$$(I_{nh,p} + \beta_k\lambda_\sigma L^{i_{11}}_{nh,p})w_k = w_0 - \lambda_\sigma\sum_{j=0}^{k-1}\alpha_{kj}(L^{i_{12}}_{nh,p}w_j - L_{nh,p}w_{nh,p}),$$
$$k = 1, \cdots, 5,$$

with $w_0 = w^l_{nh,p}$. This relation can be further evaluated into

$$\begin{aligned} w_{nh,p} - w_k &= w_{nh,p} - w_0 + \beta_k \lambda_\sigma L^{i_{11}}_{nh,p} w_k \\ &\quad + \lambda_\sigma \sum_{j=0}^{k-1} \alpha_{kj} (L^{i_{12}}_{nh,p} w_j - L_{nh,p} w_{nh,p}) \\ &= w_{nh,p} - w_0 - \lambda_\sigma \sum_{j=0}^{k-1} \alpha_{kj} L^{i_{11}}_{nh,p} (w_{nh,p} - w_k) \\ &\quad - \lambda_\sigma \sum_{j=0}^{k-1} \alpha_{kj} L^{i_{12}}_{nh,p} (w_{nh,p} - w_j), \qquad k = 1, \cdots, 5. \end{aligned}$$

The error after one semi-implicit Runge-Kutta step can now be defined recursively as

$$\begin{aligned} &\bar{e}_0 = e^0_{nh,p} \\ &\bar{e}_k = (I_{nh,p} + \lambda_\sigma \beta_k L^{i_{11}}_{nh,p})^{-1} \Big(\bar{e}_0 - \lambda_\sigma \sum_{j=0}^{k-1} \alpha_{kj} L^{i_{12}}_{nh,p} \bar{e}_j \Big), \;\; k = 1, \cdots, 5, \\ &e^1_{nh,p} = \bar{e}_5 = Q_{nh,p} e^0_{nh,p}. \end{aligned}$$

(b) Point-Implicit Runge-Kutta pseudo-time integrator. Using (3.15) we can transform the point-implicit Runge-Kutta method (2.7) into

$$(1 + \lambda_\sigma \beta_{kk}) w_k = w_0 - \lambda_\sigma \sum_{j=0}^{k-1} \big(\beta_{kj} w_j + \alpha_{kj} L_{nh,p} (w_j - w_{nh,p}) \big), \qquad k = 1, \cdots, 5,$$

with $w_0 = w^l_{nh,p}$. This relation can be further evaluated into

$$\begin{aligned} w_{nh,p} - w_k &= w_{nh,p} - w_0 + \lambda_\sigma \sum_{j=0}^{k} \beta_{kj} w_j \\ &\quad + \lambda_\sigma \sum_{j=0}^{k-1} \alpha_{kj} L_{nh,p} (w_j - w_{nh,p}) \\ &= w_{nh,p} - w_0 + \lambda_\sigma \sum_{j=0}^{k} \beta_{kj} (w_j - w_{nh,p}) \\ &\quad + \lambda_\sigma \sum_{j=0}^{k-1} \alpha_{kj} L_{nh,p} (w_j - w_{nh,p}), \qquad i = 1, \cdots, 5, \end{aligned}$$

where we used in the second step that $\sum_{j=0}^{k} \beta_{kj} = 0$, $k = 1, \cdots, 5$. The error after one point-implicit Runge-Kutta step can now be defined recursively as

$$\begin{aligned} &\bar{e}_0 = e^0_{nh,p} \\ &\bar{e}_k = \Big(\bar{e}_0 - \lambda_\sigma \sum_{j=0}^{k-1} \big(\beta_{kj} \bar{e}_j + \alpha_{kj} L_{nh,p} \bar{e}_j \Big) / (1 + \lambda_\sigma \beta_{kk}), \qquad k = 1, \cdots, 5, \\ &e^1_{nh,p} = \bar{e}_5 = P_{nh,p} e^0_{nh,p}. \end{aligned}$$

Chapter 4

Fourier Analysis of Discrete Operators

4.1 Introduction

In this chapter we will analyze in detail the two- and three-level error transformation operators derived in Section 3.1. We will also analyze the error transformation operator of the *hp-MGS* algorithm, which was derived in Section 3.2. The analysis closely follows Brandt [1] and Wienands and Joppich [17], see also Hackbusch [3], Hackbusch and Trottenberg [4], Trottenberg et al. [11] and Wesseling [16]. The analysis will be general and includes both uniformly coarsened and semi-coarsened meshes. For the analysis of the two- and three-level error transformation operators we will use discrete Fourier analysis. In this section we will introduce some important definitions which will be used throughout this report.

Assume a finite mesh $G_{nh}^N \subset \mathbb{R}^d$, with $n, N \in \mathbb{N}^d$ and $h \in (\mathbb{R}^+)^d$, which is defined in $\mathbb{R}^d$ as

$$G_{nh}^N := \big\{x = (x_1, \cdots, x_d) = (k_1 n_1 h_1, \cdots, k_d n_d h_d) \mid k \in \mathcal{G}_n^N\big\},$$

with the index set $\mathcal{G}_n^N$ given by

$$\mathcal{G}_n^N = \{k \in \mathbb{Z}^d \mid -N_i/n_i \leq k_i \leq (N_i/n_i) - 1,\ N_i/n_i \in \mathbb{N},\ i = 1, \cdots, d\}. \tag{4.1}$$

On G_{nh}^N we define for $v_{nh}, w_{nh} : G_{nh}^N \to \mathbb{C}$ the scaled Euclidian inner product

$$(v_{nh}, w_{nh})_{G_{nh}^N} := \Big(\prod_{i=1}^{d} \frac{n_i}{2N_i}\Big) \sum_{x \in G_{nh}^N} v_{nh}(x)\overline{w_{nh}(x)} \tag{4.2}$$

and norm

$$\|v_{nh}\|_{G_{nh}^N} := (v_{nh}, v_{nh})_{G_{nh}^N}^{\frac{1}{2}}.$$

Here an overbar denotes the complex conjugate. We will also consider an infinite mesh $G_{nh} \subset \mathbb{R}^d$, which is defined as

$$G_{nh} := \big\{x = (x_1, \cdots, x_d) = (k_1 n_1 h_1, \cdots, k_d n_d h_d) \mid k \in \mathbb{Z}^d\big\}.$$

Similarly, on G_{nh} we define for $v_{nh}, w_{nh} : G_{nh} \to \mathbb{C}$ the scaled Euclidian inner product as

$$(v_{nh}, w_{nh})_{G_{nh}} := \lim_{N \to \infty} \frac{1}{(2N)^d} \left(\Pi_{i=1}^d n_i\right) \sum_{x \in G_{nh}^N} v_{nh}(x)\overline{w_{nh}(x)}, \tag{4.3}$$

and the norm

$$\|v_{nh}\|_{G_{nh}} := (v_{nh}, v_{nh})_{G_{nh}}^{\frac{1}{2}}.$$

In $\mathbb{R}^2$ a uniform mesh with mesh sizes $h = (h_1, h_2)$ can now be represented as $G_h = G_{(h_1,h_2)}$ and a uniformly coarsened mesh as $G_{2h} = G_{(2h_1,2h_2)}$. A mesh with semi-coarsening in the x_1-, respectively, x_2-direction is represented as $G_{(2h_1,h_2)}$ and $G_{(h_1,2h_2)}$.

In the analysis we will also need the discrete ℓ^2 inner product and norm on G_{nh}, which are defined for $v_{nh}, w_{nh} : G_{nh} \to \mathbb{C}$, respectively, as

$$(v_{nh}, w_{nh})_{\ell^2(G_{nh})} := \sum_{x \in G_{nh}} v_{nh}(x)\overline{w_{nh}(x)} \tag{4.4}$$

$$\|v_{nh}\|_{\ell^2(G_{nh})} := (v_{nh}, v_{nh})_{\ell^2(G_{nh})}^{\frac{1}{2}}.$$

We consider now on each of the meshes G_{nh} the following linear system

$$L_{nh}v_{nh}(x) = f_{nh}(x), \qquad x \in G_{nh},$$

with L_{nh} the matrix resulting e.g. from a numerical discretization on the mesh G_{nh} of a (system of) linear partial differential equations with constant coefficients and f_{nh} the right hand side. The linear system on the mesh G_{nh} is described using stencil notation

$$L_{nh}v_{nh}(x) = \sum_{k \in J_n} l_{n,k} v_{nh}(x + knh), \qquad x \in G_{nh}, \tag{4.5}$$

with stencil coefficients $l_{n,k} \in \mathbb{R}^{m_k \times m_k}$ and finite index sets $J_n \subset \mathbb{Z}^d$ describing the stencil. For instance, in two dimensions frequently a 9-point stencil is used with

$$J_n := \{k = (k_1, k_2) \,|\, k_1, k_2 \in \{-1, 0, 1\}\}.$$

The stencil of L_{nh} is then given by

$$[L_{nh}] = \begin{pmatrix} l_{n,-1,-1} & l_{n,-1,0} & l_{n,-1,1} \\ l_{n,0,-1} & l_{n,0,0} & l_{n,0,1} \\ l_{n,1,-1} & l_{n,1,0} & l_{n,1,1} \end{pmatrix}.$$

In general the stencil coefficients $l_{n,k}$ are $m_k \times m_k$ matrices, with $m_k \geq 1$.

On the infinite mesh $G_{nh} \subset \mathbb{R}^d$, we define for $x \in G_{nh}$ the continuous Fourier modes with frequency $\theta = (\theta_1, \cdots, \theta_d) \in \Pi_n$, with $\Pi_n = [-\frac{\pi}{n_1}, \frac{\pi}{n_1}) \times \cdots \times [-\frac{\pi}{n_d}, \frac{\pi}{n_d})$, as

$$\phi_{nh}(n\theta, x) := e^{\imath n\theta \cdot x/(nh)}, \tag{4.6}$$

where $n\theta \cdot x/(nh) = \theta_1 x_1/h_1 + \cdots + \theta_d x_d/h_d$, $h \in (\mathbb{R}^+)^d$ and $\imath = \sqrt{-1}$. Note, the Fourier modes are orthonormal with respect to the scaled Euclidian inner product on G_{nh}. For a proof see Appendix A.

We define the space of bounded grid functions on the infinite mesh G_{nh}, with $n \in \mathbb{N}^d$, as

$$\mathcal{F}(G_{nh}) := \{v_{nh} \,|\, v_{nh} : G_{nh} \to \mathbb{C} \text{ with } \|v_{nh}\|_{G_{nh}} < \infty\}.$$

For each $v_{nh} \in \mathcal{F}(G_{nh})$, there exists a Fourier transformation, which is defined as

$$\widehat{v_{nh}}(n\theta) = \Big(\prod_{k=1}^{d} \frac{n_k}{2\pi}\Big) \sum_{x \in G_{nh}} v_{nh}(x) e^{-\imath n\theta \cdot x/(nh)} \qquad \theta \in \Pi_n. \tag{4.7}$$

The inverse Fourier transformation is given by

$$v_{nh}(x) = \int_{\theta \in \Pi_n} \widehat{v_{nh}}(n\theta) e^{\imath n\theta \cdot x/(nh)} d\theta, \qquad x \in G_{nh}. \tag{4.8}$$

Hence v_{nh} can be written as a linear combination of Fourier components, see e.g. Brandt [1]. For more details see also Appendix A.

Due to aliasing, Fourier components with $|\hat{\theta}| := \max\{n_1|\theta_1|, \cdots, n_d|\theta_d|\} \geq \pi$ are not visible on G_{nh}. For more details see Appendix A. These modes therefore coincide with $e^{\imath n\theta \cdot x/(nh)}$, where $\theta = \hat{\theta} \,(\mathrm{mod}\, 2\pi/n)$. Hence, the Fourier space

$$\mathcal{F}_n(G_{nh}) := \mathrm{span}\,\{\phi_h(\theta, x) \mid \theta \in \Pi_n, x \in G_{nh}\}$$

contains any bounded infinite grid function on G_{nh}. The norms of the fields v_{nh} and $\widehat{v_{nh}}$ are related through the Parseval identity

$$\int_{\theta \in \Pi_n} |\widehat{v_{nh}}(n\theta)|^2 d\theta = \Big(\Pi_{l=1}^{d} \frac{n_l}{2\pi}\Big) \|v_{nh}\|^2_{\ell^2(G_{nh})}, \tag{4.9}$$

for a proof, see Appendix A.

On a finite domain with mesh G^N_{nh}, where at the domain boundaries periodic boundary conditions are imposed, only a finite number of frequencies can be represented. Hence, for every $v_{nh} \in \mathcal{F}_n(G^N_{nh})$ the discrete Fourier transformation is defined as

$$\widehat{v_{nh}}(n\theta_k) = \left(\prod_{l=1}^{d} \frac{n_l}{2N_l}\right) \sum_{x \in G^N_{nh}} v_{nh}(x) e^{-\imath n\theta_k \cdot x/(nh)},$$

with $\theta_k = (\theta_{k_1}, \cdots, \theta_{k_d})$, $\theta_{k_l} = \pi k_l/N_l$, $k_l \in \mathcal{G}^{N_l}_{n_l}$. The inverse discrete Fourier transformation is given by

$$v_{nh}(x) = \sum_{k \in \mathcal{G}^N_n} \widehat{v_{nh}}(n\theta_k) e^{\imath n\theta_k \cdot x/(nh)}, \qquad x \in G^N_{nh}.$$

The results of the discrete Fourier analysis on the infinite mesh G_{nh} and the finite mesh G^N_{nh} are equal for a periodic field at the frequencies $\theta = \theta_k$, with $\theta_k = \pi k/N$, $k \in \mathcal{G}^N_n$. This equivalence will be used to find approximate results for the discrete Fourier analysis on the infinite mesh G_{nh}, which generally results in eigenvalue problems which can not be solved analytically.

4.2 Fourier symbols of grid operators

In this section we will derive the Fourier symbols of the basic multigrid operators, namely the fine and coarse grid operators, and the restriction, prolongation and smoothing operators. We will first consider the more general case of three level analysis, which relations can be simplified if only two grid levels are used in the analysis. In order to simplify notation we limit the discussion of the Fourier analysis to two dimensions.

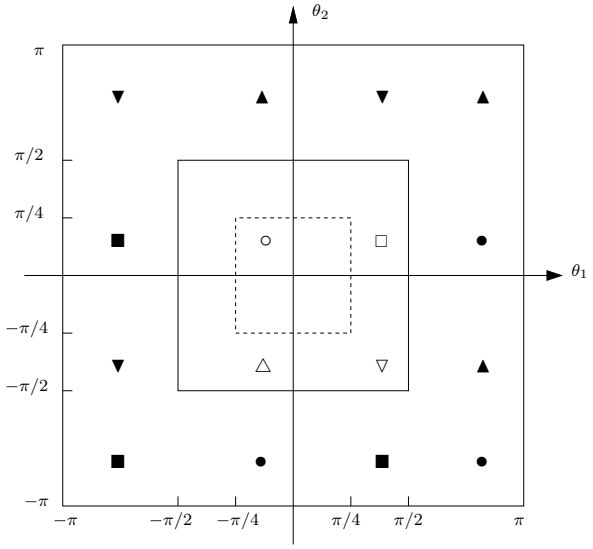

Figure 4.1: Aliasing of Fourier modes for uniform-coarsening. Modes with a black symbol alias on the mesh G_{2h} to the mode with equivalent open symbol in the domain $[-\pi/2, \pi/2)^2$. Modes in the domain $[-\pi/2, \pi/2)^2 \setminus [-\pi/4, \pi/4)^2$ alias on the mesh G_{4h} to the mode in $[-\pi/4, \pi/4)^2$.

4.2.1 Aliasing of Fourier modes

In three-level analysis with uniform mesh coarsening 16 modes on the fine mesh $G_{(h_1,h_2)}$ alias to four independent modes on the mesh $G_{(2h_1,2h_2)}$ and to one mode on the coarsest mesh $G_{(4h_1,4h_2)}$, see Figure 4.1. We therefore introduce the Fourier harmonics $\mathcal{F}_h^3(\theta)$, with $\theta \in \Pi_{(4,4)}$, as

$$\mathcal{F}_h^3(\theta) := \text{span}\big\{\phi_h(\theta_\beta^\alpha, x) \mid \alpha \in \alpha_2, \beta \in \beta_2\big\},$$

with

$$\begin{aligned}
&\theta = \theta_{00}^{00} \in \Pi_{(4,4)} := [-\pi/4, \pi/4)^2,\\
&\theta_\beta^{00} = \theta_{00}^{00} - (\bar\beta_1 \,\text{sign}\,(\theta_1), \bar\beta_2 \,\text{sign}\,(\theta_2))\pi,\\
&\theta_\beta^\alpha := \theta_\beta^{00} - (\bar\alpha_1 \text{sign}\,((\theta_\beta^{00})_1), \bar\alpha_2 \,\text{sign}\,((\theta_\beta^{00})_2))\pi,\\
&\alpha_2 = \{(\bar\alpha_1, \bar\alpha_2) \mid \bar\alpha_i \in \{0, 1\},\, i = 1, 2\},\\
&\beta_2 = \{(\bar\beta_1, \bar\beta_2) \mid \bar\beta_i \in \{0, \frac{1}{2}\},\, i = 1, 2\}.
\end{aligned} \tag{4.10}$$

Next to uniform coarsening, the *hp-MGS* algorithm also uses semi-coarsening multigrid. In this case the grid is coarsened in only one direction, which implies that four modes on the fine mesh alias to two modes on the medium mesh, and to one mode on the coarsest mesh, see Figures 4.2 and 4.3.

The aliasing relations for the Fourier modes on the different coarse meshes can be straightforwardly computed using the representation of the modes θ_β^α given by (4.10). For more details, see Appendix A.5. First, assume the following mesh coarsenings $G_h \to G_{nh}$, with $n \in \{(2,2), (2,1), (1,2)\}$, which includes both uniform and semi-coarsening. For $x \in G_{nh}$

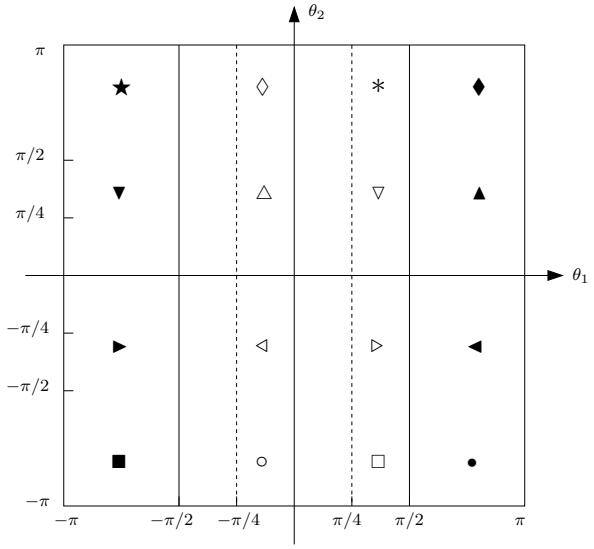

Figure 4.2: Aliasing of Fourier modes for semi-coarsening in the x_1-direction. Modes with a black symbol alias on the mesh $G_{(2h_1,h_2)}$ to the mode with an equivalent open symbol in the domain $[-\pi/2,\pi/2)\times[-\pi,\pi)$. Modes in the domain $\theta \in ([-\pi/2,-\pi/4)\cup[\pi/4,\pi/2))\times[-\pi,\pi)$ alias on the mesh $G_{(4h_1,h_2)}$ to the mode in $[-\pi/4,\pi/4)\times[-\pi,\pi)$ with the same value of θ_2.

Fourier modes with frequency $\theta_\beta^\alpha \in \Pi_{(1,1)}$, with $\alpha \in \alpha_2$, $\beta \in \beta_2$, alias on the mesh G_{nh} to modes with frequency $\theta_\beta^{\alpha'} \in \Pi_n$ with

$$\begin{aligned}\phi_h(\theta_\beta^\alpha, x) &= \phi_h(\theta_\beta^{\alpha'}, x) \\ &= \phi_{nh}(n\theta_\beta^{\alpha'}, x), \qquad \theta_\beta^{\alpha'} \in \Pi_n,\ x \in G_{nh},\end{aligned}$$

and

$$\alpha' = \begin{cases}(0,0) & \text{if } n=(2,2),\\ (0,\bar{\alpha}_2) & \text{if } n=(2,1),\\ (\bar{\alpha}_1,0) & \text{if } n=(1,2).\end{cases} \tag{4.11}$$

Analogously, for the mesh coarsening $G_{nh} \to G_{mh}$, with $m \in \{(4,4),(4,1),(1,4)\}$, modes with frequency $\theta_\beta^{\alpha'} \in \Pi_n$ alias on the mesh G_{mh} to modes with frequency $\theta_{\beta'}^{\alpha'} \in \Pi_m$ as

$$\begin{aligned}\phi_{nh}(n\theta_\beta^{\alpha'}, x) &= \phi_h(\theta_{\beta'}^{\alpha'}, x) \\ &= \phi_{mh}(m\theta_{\beta'}^{\alpha'}, x), \quad \theta_{\beta'}^{\alpha'} \in \Pi_m,\ x \in G_{mh},\end{aligned}$$

with α' and β' given by

$$\begin{aligned}&\alpha' = (0,0), && \beta' = (0,0), && \text{if } m=(4,4),\\ &\alpha' = (0,\bar{\alpha}_2), && \beta' = (0,\bar{\beta}_2) && \text{if } m=(4,1),\\ &\alpha' = (\bar{\alpha}_1,0), && \beta' = (\bar{\beta}_1,0) && \text{if } m=(1,4).\end{aligned}$$

In order to unify the analysis of uniform and semi-coarsening multigrid we also use in the semi-coarsening analysis the sixteen modes θ_β^α defined in (4.10) for uniform coarsening.

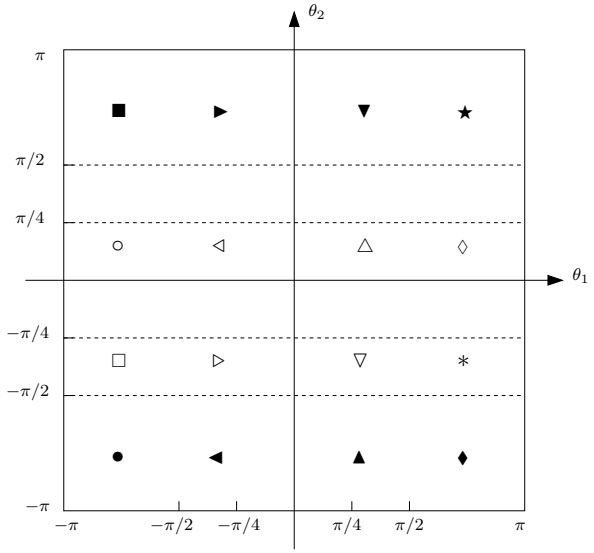

Figure 4.3: Aliasing of Fourier modes for semi-coarsening in the x_2-direction. Modes with a black symbol alias on the mesh $G_{(h_1,2h_2)}$ to the mode with an equivalent open symbol in the domain $[-\pi,\pi)\times[-\pi/2,\pi/2)$. Modes in the domain $\theta\in[-\pi,\pi)\times([-\pi/2,-\pi/4)\cup[\pi/4,\pi/2))$ alias on the mesh $G_{(h_1,4h_2)}$ to the mode in $[-\pi,\pi)\times[-\pi/4,\pi/4)$ with the same value of θ_1.

These modes are, however, subdivided into four independent groups. On the coarser meshes there is no aliasing between modes in different groups, only between modes in the same group.

For the three-level Fourier analysis of semi-coarsening in the x_1-direction we subdivide the Fourier harmonics with frequencies θ_β^α, $\alpha\in\alpha_2,\beta\in\beta_2$, on the mesh $G_{(h_1,h_2)}$ into the groups

$$\begin{aligned}
\alpha^1_{(2,1)} &= \{(0,0),(1,0)\} &\rightarrow\quad \gamma^1_{(2,1)} &= (0,0),\\
\alpha^2_{(2,1)} &= \{(1,1),(0,1)\} &\rightarrow\quad \gamma^2_{(2,1)} &= (0,1),\\
\beta^1_{(2,1)} &= \{(0,0),(\frac{1}{2},0)\} &\rightarrow\quad \delta^1_{(2,1)} &= (0,0),\\
\beta^2_{(2,1)} &= \{(\frac{1}{2},\frac{1}{2}),(0,\frac{1}{2})\} &\rightarrow\quad \delta^2_{(2,1)} &= (0,\frac{1}{2}),
\end{aligned}$$

where also the index of the mode to which each group of modes aliases on the next coarser mesh level is indicated with an arrow, see also Figure 4.2. For example the modes with index in the group $\alpha^1_{(2,1)}$, viz. $(0,0)$ and $(1,0)$, both alias to the mode $\gamma^1_{(2,1)}=(0,0)$. Analogously, for three-level Fourier analysis of semi-coarsening in the x_2-direction we define the groups

$$\begin{aligned}
\alpha^1_{(1,2)} &= \{(0,0),(0,1)\} &\rightarrow\quad \gamma^1_{(1,2)} &= (0,0),\\
\alpha^2_{(1,2)} &= \{(1,1),(1,0)\} &\rightarrow\quad \gamma^2_{(1,2)} &= (1,0),\\
\beta^1_{(1,2)} &= \{(0,0),(0,\frac{1}{2})\} &\rightarrow\quad \delta^1_{(1,2)} &= (0,0),\\
\beta^2_{(1,2)} &= \{(\frac{1}{2},\frac{1}{2}),(\frac{1}{2},0)\} &\rightarrow\quad \delta^2_{(1,2)} &= (\frac{1}{2},0),
\end{aligned}$$

see Figure 4.3. Finally, for uniform mesh coarsening the modes in the three-level Fourier analysis are ordered as

$$\begin{aligned}\alpha^1_{(2,2)} &= \{(0,0),(1,1),(1,0),(0,1)\} \quad &\rightarrow \quad \gamma^1_{(2,2)} &= (0,0),\\ \beta^1_{(2,2)} &= \{(0,0),(\frac{1}{2},\frac{1}{2}),(\frac{1}{2},0),(0,\frac{1}{2})\} \quad &\rightarrow \quad \delta^1_{(2,2)} &= (0,0),\end{aligned}$$

see Figure 4.1. In principle the ordering of the modes in the different groups can be changed, but it is important that the same ordering is used in all steps of the multilevel analysis.

In two-level analysis with uniform mesh coarsening 4 modes on the fine mesh $G_{(h_1,h_2)}$ alias to four independent modes on the mesh $G_{(2h_1,2h_2)}$. We therefore introduce the Fourier harmonics $\mathcal{F}^2_h(\theta)$, with $\theta \in \Pi_{(2,2)}$, as

$$\mathcal{F}^2_h(\theta) := \text{span}\big\{\phi_h(\theta^\alpha, x) \mid \alpha \in \alpha_2\big\},$$

and

$$\begin{aligned}&\theta = \theta^{00} \in \Pi_{(2,2)} := [-\pi/2, \pi/2)^2,\\ &\theta^\alpha := \theta^{00} - (\bar\alpha_1 \text{sign}\,((\theta^{00})_1), \bar\alpha_2\,\text{sign}\,((\theta^{00})_2))\pi,\\ &\alpha_2 = \{(\bar\alpha_1,\bar\alpha_2) \mid \bar\alpha_i \in \{0,1\},\, i = 1,2\},.\end{aligned} \tag{4.12}$$

Analogous to the three-level analysis, the mode subdivision into different groups is also used in the two-level analysis for semi-coarsened meshes.

4.2.2 Discrete operator and smoothing operator

Define for $x \in G_{nh}$ the discrete operator $L_{nh} : \mathcal{F}(G_{nh}) \rightarrow \mathcal{F}(G_{nh})$ as

$$(L_{nh}v_{nh})(x) = \sum_{k \in J_{L_{nh}}} l_{k,nh} v_{nh}(x + knh), \qquad x \in G_{nh}, x + knh \in G_{nh}$$

with $J_{L_{nh}}$ the stencil of the fine grid operator. On the mesh G_{nh} we can express the discrete operator L_{nh} in terms of its discrete Fourier transform $\widehat{L_{nh}v_{nh}}(n\theta)$ through the relation

$$(L_{nh}v_{nh})(x) = \int_{\theta \in \Pi_n} \widehat{L_{nh}v_{nh}}(n\theta) e^{\imath n\theta\cdot x/(nh)} d\theta. \tag{4.13}$$

The discrete Fourier transform can be further evaluated for $\theta \in \Pi_n$ into:

$$\begin{aligned}\widehat{L_{nh}v_{nh}}(n\theta) &= \Big(\prod_{k=1}^d \frac{n_k}{2\pi}\Big) \sum_{x \in G_{nh}} (L_{nh}v_{nh})(x) e^{-\imath n\theta\cdot x/(nh)}\\ &= \Big(\prod_{k=1}^d \frac{n_k}{2\pi}\Big) \sum_{x \in G_{nh}} \sum_{k \in J_{L_{nh}}} l_{k,nh} v_{nh}(x + knh) e^{-\imath n\theta\cdot x/(nh)}\\ &= \sum_{k \in J_{L_{nh}}} l_{k,nh} e^{\imath n\theta\cdot k} \Big(\prod_{k=1}^d \frac{n_k}{2\pi}\Big) \sum_{x \in G_{nh}} v_{nh}(x + knh) e^{-\imath n\theta\cdot(x+knh)/(nh)}\\ &= \widehat{L_{nh}}(n\theta)\widehat{v_{nh}}(n\theta),\end{aligned} \tag{4.14}$$

with

$$\widehat{L_{nh}}(n\theta) = \sum_{k \in J_{L_{nh}}} l_{k,nh} e^{\imath n\theta \cdot k}.$$

The Fourier modes $e^{\imath n\theta \cdot x/(nh)}$ are the eigenfunctions and $\widehat{L_{nh}}(n\theta)$ the eigenvalues of the operator L_{nh}, since

$$L_{nh} e^{\imath n\theta \cdot x/(nh)} = \widehat{L_{nh}}(n\theta) e^{\imath n\theta \cdot x/(nh)},$$

which follows directly from a substitution of (4.6) into (4.5).

Similarly, we obtain for the smoothing operator $S_{nh} : \mathcal{F}(G_{nh}) \rightarrow \mathcal{F}(G_{nh})$, which is defined as

$$(S_{nh} v_{nh})(x) = \sum_{k \in J_{S_{nh}}} s_{k,nh} v_{nh}(x + knh), \qquad x \in G_{nh},\ x + knh \in G_{nh},$$

with $J_{S_{nh}}$ the stencil of the smoothing operator, the relation

$$(S_{nh} v_{nh})(x) = \int_{\theta \in \Pi_n} \widehat{S_{nh} v_{nh}}(n\theta) e^{\imath n\theta \cdot x/(nh)} d\theta, \tag{4.15}$$

where the discrete Fourier transform can be further evaluated into:

$$\widehat{S_{nh} v_{nh}}(n\theta) = \widehat{S_{nh}}(n\theta) \widehat{v_{nh}}(n\theta), \tag{4.16}$$

with

$$\widehat{S_{nh}}(n\theta) = \sum_{k \in J_{S_{nh}}} s_{k,nh} e^{\imath n\theta \cdot k}.$$

4.2.3 Discrete Fourier transform of pseudo-time smoothers

Using the techniques discussed in the previous section it is straightforward to compute the discrete Fourier transform of the pseudo-time integration smoothers discussed in Section 2.3. For the semi-implicit pseudo-time Runge-Kutta operator Q_h^l, with $l = 1, 2$, on the mesh G_h, which is defined in (2.6), the discrete Fourier transform is equal to

$$\begin{aligned}
&\widehat{Q_0}(\theta_\beta^\alpha) = I^{m_q}, \\
&\widehat{Q_k}(\theta_\beta^\alpha) = \big(I^{m_q} + \beta_k \lambda_\sigma \widehat{L_h^{l,1}}(\theta_\beta^\alpha)\big)^{-1} \big(I^{m_q} - \lambda_\sigma \sum_{j=0}^{k-1} \alpha_{kj} \widehat{L_h^{l,2}}(\theta_\beta^\alpha) \widehat{Q_j}(\theta_\beta^\alpha)\big), \\
&\qquad\qquad \forall \alpha \in \alpha_2,\ \forall \beta \in \beta_2,\ k = 1, \cdots, 5, \\
&\widehat{Q_h^l}(\theta_\beta^\alpha) = \widehat{Q_5}(\theta_\beta^\alpha),
\end{aligned}$$

On the coarse mesh G_{nh} the discrete Fourier transform of the semi-implicit pseudo-time Runge-Kutta operator Q_{nh}^l is equal to

$$\begin{aligned}
&\widehat{Q_0}(n\theta_\beta^{\gamma_n^r}) = I^{m_q}, \\
&\widehat{Q_k}(n\theta_\beta^{\gamma_n^r}) = \big(I^{m_q} + \beta_k \lambda_\sigma \widehat{L_{nh}^{l,1}}(n\theta_\beta^{\gamma_n^r}))\big)^{-1} \big(I^{m_q} - \lambda_\sigma \sum_{j=0}^{k-1} \alpha_{kj} \widehat{L_{nh}^{l,2}}(n\theta_\beta^{\gamma_n^r}) \widehat{Q_j}(n\theta_\beta^{\gamma_n^r})\big), \\
&\qquad\qquad \forall \beta \in \beta_n^s,\ r, s \in s_n, k = 1, \cdots, 5, \\
&\widehat{Q_{nh}^l}(n\theta_\beta^{\gamma_n^r}) = \widehat{Q_5}(n\theta_\beta^{\gamma_n^r}).
\end{aligned}$$

The set s_n is defined as $s_n = \{1, 2\}$ if $n = (2, 1)$ or $(1, 2)$ and $s_n = \{1\}$ if $n = (2, 2)$. For the point implicit Runge-Kutta pseudo-time integration operator P_h on the mesh G_h, given by (2.8), the discrete Fourier transform is equal to

$$\begin{aligned}
&\widehat{P_0}(\theta_\beta^\alpha) = I^{m_q},\\
&\widehat{P_k}(\theta_\beta^\alpha) = \Big(I^{m_q} - \lambda_\sigma \sum_{j=0}^{k-1} \big(\beta_{kj}\widehat{P_j}(\theta_\beta^\alpha) + \alpha_{kj}\widehat{L}_{h,p}(\theta_\beta^\alpha)\widehat{P_j}(\theta_\beta^\alpha)\big)\Big)/\\
&\qquad (1 + \lambda_\sigma\beta_{kk}), && k = 1, \cdots, 5,\\
&\widehat{P_h}(\theta_\beta^\alpha) = \widehat{P_5}(\theta_\beta^\alpha), && \forall \alpha \in \alpha_2, \forall \beta \in \beta_2.
\end{aligned}$$

Analogously, on the mesh G_{nh} the discrete Fourier transform of the point-implicit Runge-Kutta method is

$$\begin{aligned}
&\widehat{P_0}(n\theta_\beta^{\gamma_r^n}) = I^{m_q},\\
&\widehat{P_k}(n\theta_\beta^{\gamma_r^n}) = \Big(I^{m_q} - \lambda_\sigma \sum_{j=0}^{i-1} \big(\beta_{kj}\widehat{P_j}(n\theta_\beta^{\gamma_r^n}) + \alpha_{kj}\widehat{L_{nh}}(n\theta_\beta^{\gamma_r^n})\widehat{P_j}(n\theta_\beta^{\gamma_r^n})\big)\Big)/\\
&\qquad (1 + \lambda_\sigma\beta_{kk}), && k = 1, \cdots, 5,\\
&\widehat{P_{nh}}(n\theta_\beta^{\gamma_r^n}) = \widehat{P_5}(n\theta_\beta^{\gamma_r^n}), && \forall \beta \in \beta_n^s,\ r, s \in s_n.
\end{aligned}$$

Depending on the type of pseudo-time integrator the discrete Fourier transforms $\widehat{P_h}(\theta_\beta^\alpha)$ and $\widehat{Q_h^l}(\theta_\beta^\alpha)$ provide the discrete Fourier transform of the smoother $\widehat{S_h}(\theta_\beta^\alpha)$. Analogously, the discrete Fourier transforms $\widehat{P_{nh}}(n\theta_\beta^{\gamma_r^n})$ and $\widehat{Q_{nh}^l}(\theta_\beta^{\gamma_r^n})$ provide the discrete Fourier transform of the smoother $\widehat{S_{nh}}(n\theta_\beta^{\gamma_r^n})$.

4.2.4 h-Multigrid restriction operators

Define the restriction operator $R_h^{nh} : \mathcal{F}(G_h) \to \mathcal{F}(G_{nh})$, with $n \in \{(2, 2), (2, 1), (1, 2)\}$, as

$$(R_h^{nh} v_h)(x) = \sum_{k \in J_{R_h^{nh}}} r_{k,nh} v_h(x + kh), \qquad x \in G_{nh},\ x + kh \in G_h,$$

with $J_{R_h^{nh}}$ the stencil of the restriction operator. On the mesh G_{nh} the restriction operator can be related to its discrete Fourier transform through the relation

$$\begin{aligned}
(R_h^{nh} v_h)(x) &= \int_{\theta \in \Pi_n} \widehat{R_h^{nh} v_h}(n\theta) e^{\imath n\theta \cdot x/(nh)} d\theta\\
&= \sum_{i \in s_n} \sum_{j \in s_n} \sum_{\beta \in \beta_n^j} \int_{\theta \in \Pi_{(4,4)}} \widehat{R_h^{nh} v_h}(n\theta_\beta^{\gamma_n^i}(\theta)) e^{\imath n\theta_\beta^{\gamma_n^i}(\theta) \cdot x/(nh)} d\theta,
\end{aligned} \tag{4.17}$$

with $\theta_\beta^\gamma(\theta)$ given by (4.10). Note, in (4.17) we used the subdivision of modes with frequency $\theta \in \Pi_n$ into different groups as discussed in Section 4.2.1. The discrete Fourier transform

$\widehat{R_h^{nh} v_h}(n\theta)$, with $\theta \in \Pi_n$, is defined as

$$\begin{aligned}\widehat{R_h^{nh} v_h}(n\theta) &= \frac{n_1 n_2}{4\pi^2} \sum_{x \in G_{nh}} (R_h^{nh} v_h)(x) e^{-\imath n\theta \cdot x/(nh)} \\ &= \frac{n_1 n_2}{4\pi^2} \sum_{x \in G_{nh}} \sum_{k \in J_{R_h^{nh}}} r_{k,nh} v_h(x + kh) e^{-\imath n\theta \cdot x/(nh)}.\end{aligned}$$

For $x \in G_{nh}$ we have the aliasing relation

$$\phi_h(\theta_\beta^\alpha, x) = \phi_{nh}(n\theta_\beta^{\gamma_n^i}, x), \tag{4.18}$$

with $\alpha \in \alpha_n^i$, $i \in s_n$ and $\forall \beta \in \beta_n^j$, $j \in s_n$, see Section 4.2.1.

We can use the aliasing relation (4.18) to express the modes in the different groups of Fourier modes on G_{nh} as the average of aliasing modes on the mesh G_h

$$e^{\imath n\theta_\beta^{\gamma_n^i} \cdot x/(nh)} = \frac{1}{n_1 n_2} \sum_{\alpha \in \alpha_n^i} e^{\imath \theta_\beta^\alpha \cdot x/h}, \qquad \forall \beta \in \beta_n^j,\ i, j \in s_n. \tag{4.19}$$

The discrete Fourier transform $\widehat{R_h^{nh} v_h}(n\theta_\beta^{\gamma^i})$, $\forall \beta \in \beta_n^j$, $i, j \in s_n$, can be further evaluated into

$$\begin{aligned}\widehat{R_h^{nh} v_h}(n\theta_\beta^{\gamma_n^i}) &= \frac{1}{4\pi^2} \sum_{x \in G_{nh}} \sum_{k \in J_{R_h^{nh}}} r_{k,nh} v_h(x + kh) \sum_{\alpha \in \alpha_n^i} e^{-\imath \theta_\beta^\alpha \cdot x/h} \\ &= \frac{1}{4\pi^2} \sum_{\alpha \in \alpha_n^i} \sum_{k \in J_{R_h^{nh}}} r_{k,nh} e^{\imath \theta_\beta^\alpha \cdot k} \sum_{x \in G_{nh}} v_h(x + kh) e^{-\imath \theta_\beta^\alpha \cdot (x+kh)/h} \\ &= \frac{1}{4\pi^2} \sum_{\alpha \in \alpha_n^i} \sum_{k \in J_{R_h^{nh}}} r_{k,nh} e^{\imath \theta_\beta^\alpha \cdot k} \sum_{x \in G_{nh}} \Big(v_h(x + kh) e^{-\imath \theta_\beta^\alpha \cdot (x+kh)/h} \\ &+ \sum_{l \in l_n} v_h(x + kh + lh) e^{-\imath \theta_\beta^\alpha \cdot (x+kh+lh)/h} \\ &- \sum_{l \in l_n} v_h(x + kh + lh) e^{-\imath \theta_\beta^\alpha \cdot (x+kh+lh)/h} \Big) \\ &= \frac{1}{4\pi^2} \sum_{\alpha \in \alpha_n^i} \sum_{k \in J_{R_h^{nh}}} r_{k,nh} e^{\imath \theta_\beta^\alpha \cdot k} \sum_{x \in G_h} v_h(x) e^{-\imath \theta_\beta^\alpha \cdot x/h} \\ &- \frac{1}{4\pi^2} \sum_{\alpha \in \alpha_n^i} \sum_{k \in J_{R_h^{nh}}} r_{k,nh} \sum_{x \in G_{nh}} \sum_{l \in l_n} v_h(x + kh + lh) e^{-\imath \theta_\beta^\alpha \cdot (x+lh)/h} \\ &= \sum_{\alpha \in \alpha_n^i} \sum_{k \in J_{R_h^{nh}}} r_{k,nh} e^{\imath \theta_\beta^\alpha \cdot k} \frac{1}{4\pi^2} \sum_{x \in G_h} v_h(x) e^{-\imath \theta_\beta^\alpha \cdot x/h} \\ &- \frac{1}{4\pi^2} \sum_{k \in J_{R_h^{nh}}} r_{k,nh} \sum_{x \in G_{nh}} \sum_{l \in l_n} v_h(x + kh + lh) \sum_{\alpha \in \alpha_n^i} e^{-\imath \theta_\beta^\alpha \cdot (x+lh)/h}\end{aligned} \tag{4.20}$$

with $l_n := \alpha_n^1 \backslash \gamma_n^1$. Note, in the fourth step we used that for points $x \in G_{nh}$ and $l \in l_n$ the points $x + lh \in G_h \setminus G_{nh}$, hence a summation over both sets is equal to a summation over G_h.

Using (4.10) we obtain for $x \in G_{nh}$, hence $x = jnh, j \in \mathbb{Z}^2$, that

$$\begin{aligned}\sum_{\alpha\in\alpha_n^i} e^{-\imath\theta_\beta^\alpha\cdot(x+lh)/h} &= \sum_{\alpha\in\alpha_n^i} e^{-\imath\theta_\beta^\alpha\cdot(jnh+lh)/h}\\ &= \sum_{\alpha\in\alpha_n^i} e^{-\imath(\theta_\beta^{00}-(\bar\alpha_1\,\mathrm{sign}((\theta_\beta^{00})_1),\bar\alpha_2\,\mathrm{sign}((\theta_\beta^{00})_2))\pi)\cdot(jn+l)}\\ &= e^{-\imath\theta_\beta^{00}\cdot(jn+l)}\sum_{\alpha\in\alpha_n^i} e^{\imath\pi(\bar\alpha_1\,\mathrm{sign}((\theta_\beta^{00})_1),\bar\alpha_2\,\mathrm{sign}((\theta_\beta^{00})_2))\cdot(jn+l)}. \qquad (4.21)\end{aligned}$$

The last summation in (4.21) can be further evaluated as

- For uniform coarsening we have $\alpha^1_{(2,2)} = \{(0,0),(1,1),(1,0),(0,1)\}$ and $l_{(2,2)} = \{(1,1),(1,0),(0,1)\}$ and we obtain

$$\begin{aligned}&\sum_{\alpha\in\alpha^1_{(2,2)}} e^{\imath\pi(\bar\alpha_1\mathrm{sign}((\theta_\beta^{00})_1),\bar\alpha_2\mathrm{sign}((\theta_\beta^{00})_2))\cdot(2j_1+l_1,2j_2+l_2)}\\ &\qquad= \sum_{\alpha\in\alpha^1_{(2,2)}} e^{\imath\pi(\bar\alpha_1\mathrm{sign}((\theta_\beta^{00})_1),\bar\alpha_2\mathrm{sign}((\theta_\beta^{00})_2))\cdot(l_1,l_2)}\\ &\qquad= 1 + e^{\imath\pi\mathrm{sign}((\theta_\beta^{00})_1)} + e^{\imath\pi\mathrm{sign}((\theta_\beta^{00})_1)} + 1 = 0, && \text{if } l=(1,0),\\ &\qquad= 1 + e^{\imath\pi\mathrm{sign}((\theta_\beta^{00})_2)} + 1 + e^{\imath\pi\mathrm{sign}((\theta_\beta^{00})_2)} = 0, && \text{if } l=(0,1),\\ &\qquad= 1 + e^{\imath\pi\mathrm{sign}((\theta_\beta^{00})_1)}e^{\imath\pi\mathrm{sign}((\theta_\beta^{00})_2)} + e^{\imath\pi\mathrm{sign}((\theta_\beta^{00})_1)}\\ &\qquad+ e^{\imath\pi\mathrm{sign}((\theta_\beta^{00})_2)} = 0, && \text{if } l=(1,1).\end{aligned}$$

- For semi-coarsening in the x_1-direction we have two cases: $\alpha^1_{(2,1)} = \{(0,0),(1,0)\}$ and $\alpha^2_{(2,1)} = \{(1,1),(0,1)\}$, with $l_{(2,1)} = \{(1,0)\}$, which can be further evaluated as

$$\begin{aligned}\sum_{\alpha\in\alpha^1_{(2,1)}} e^{\imath\pi(\bar\alpha_1\mathrm{sign}((\theta_\beta^{00})_1),\bar\alpha_2\mathrm{sign}((\theta_\beta^{00})_2))\cdot(2j_1+1,j_2)} &= 1 + e^{\imath\pi(\mathrm{sign}((\theta_\beta^{00})_1),0))\cdot(2j_1+1,j_2)}\\ &= 1 + e^{\imath\pi\mathrm{sign}((\theta_\beta^{00})_1)}\\ &= 0.\\ \sum_{\alpha\in\alpha^2_{(2,1)}} e^{\imath\pi(\bar\alpha_1\mathrm{sign}((\theta_\beta^{00})_1),\bar\alpha_2\mathrm{sign}((\theta_\beta^{00})_2))\cdot(2j_1+1,j_2)} &= e^{\imath\pi(\mathrm{sign}((\theta_\beta^{00})_1),\mathrm{sign}((\theta_\beta^{00})_2))\cdot(2j_1+1,j_2)}\\ &\quad+ e^{\imath\pi(0,\mathrm{sign}((\theta_\beta^{00})_2))\cdot(2j_1+1,j_2)}\\ &= (e^{\imath\pi\mathrm{sign}((\theta_\beta^{00})_1)} + 1)e^{\imath\pi j_2\mathrm{sign}((\theta_\beta^{00})_2)}\\ &= 0.\end{aligned}$$

- For semi-coarsening in the x_2-direction we have two cases: $\alpha^1_{(1,2)} = \{(0,0),(0,1)\}$ and $\alpha^2_{(1,2)} = \{(1,1),(1,0)\}$, with $l_{(1,2)} = \{(0,1)\}$, which can be further evaluated as

$$\begin{aligned}\sum_{\alpha\in\alpha^1_{(1,2)}} e^{\imath\pi(\bar\alpha_1\mathrm{sign}((\theta_\beta^{00})_1),\bar\alpha_2\mathrm{sign}((\theta_\beta^{00})_2))\cdot(j_1,2j_2+1)} &= 1 + e^{\imath\pi(0,\mathrm{sign}((\theta_\beta^{00})_2))\cdot(j_1,2j_2+1)}\\ &= 1 + e^{\imath\pi\mathrm{sign}((\theta_\beta^{00})_2)}\\ &= 0.\end{aligned}$$

$$\sum_{\alpha\in\alpha^2_{(1,2)}} e^{\imath\pi(\bar\alpha_1\mathrm{sign}((\theta^{00}_\beta)_1),\bar\alpha_2\mathrm{sign}((\theta^{00}_\beta)_2))\cdot(j_1,2j_2+1)} = e^{\imath\pi(\mathrm{sign}((\theta^{00}_\beta)_1),\mathrm{sign}((\theta^{00}_\beta)_2))\cdot(j_1,2j_2+1)}$$
$$+ e^{\imath\pi(\mathrm{sign}((\theta^{00}_\beta)_1),0)\cdot(j_1,2j_2+1)}$$
$$= e^{\imath\pi j_1\mathrm{sign}((\theta^{00}_\beta)_1)}(e^{\imath\pi\mathrm{sign}((\theta^{00}_\beta)_2)}+1)$$
$$= 0.$$

Combining all contributions in (4.21) we obtain that

$$\sum_{\alpha\in\alpha^i_n} e^{-\imath\theta^\alpha_\beta\cdot(x+lh)/h} = 0, \qquad \forall\beta\in\beta^j_n,\ i,j\in s_n.$$

The relation for the discrete Fourier transform of the restriction operator, given by (4.20), therefore can be further evaluated into

$$\widehat{R^{nh}_h v_h}(n\theta^{\gamma^i_n}_\beta) = \sum_{\alpha\in\alpha^i_n} \widehat{R^{nh}_h}(\theta^\alpha_\beta)\widehat{v_h}(\theta^\alpha_\beta), \quad \forall\beta\in\beta^j_n,\ i,j\in s_n, \tag{4.22}$$

with the Fourier symbol $\widehat{R^{nh}_h}(\theta^\alpha_\beta)$ defined as

$$\widehat{R^{nh}_h}(\theta^\alpha_\beta) = \sum_{k\in J_{R^{nh}_h}} r_{k,nh}e^{\imath\theta^\alpha_\beta\cdot k}.$$

Relation (4.22) shows that the restriction operator couples the grid modes θ^α_β on the grid G_h to the coarse grid modes $\theta^{\gamma^i_n}_\beta$ on the grid G_{nh}. Using relation (4.22) we can transform (4.17) into

$$(R^{nh}_h v_h)(x) = \sum_{i\in s_n}\sum_{j\in s_n}\sum_{\beta\in\beta^j_n}\int_{\theta\in\Pi_{(4,4)}}\left(\sum_{\alpha\in\alpha^i_n}\widehat{R^{nh}_h}(\theta^\alpha_\beta)\widehat{v_h}(\theta^\alpha_\beta)\right)e^{\imath n\theta^{\gamma^i_n}_\beta\cdot x/(nh)}d\theta,$$

with $\theta^\alpha_\beta = \theta^\alpha_\beta(\theta)$ given by (4.10).

Next, we define the restriction operator $R^{mh}_{nh}:\mathcal{F}(G_{nh})\to\mathcal{F}(G_{mh})$ as

$$(R^{mh}_{nh}v_{nh})(x) = \sum_{k\in J_{R^{mh}_{nh}}} r_{k,mh}v_{nh}(x+knh), \qquad x\in G_{mh},\ x+knh\in G_{nh},$$

with $n\in\{(2,2),(2,1),(1,2)\}, m\in\{(4,4),(4,1),(1,4)\}$ and $J_{R^{mh}_{nh}}$ the stencil of the restriction operator. On the mesh G_{mh} the restriction operator can be related to its discrete Fourier transform through the relation

$$(R^{mh}_{nh}v_{nh})(x) = \int_{\theta\in\Pi_m}\widehat{R^{mh}_{nh}v_{nh}}(m\theta)e^{\imath m\theta\cdot x/(mh)}d\theta$$
$$= \sum_{i\in s_n}\sum_{j\in s_n}\int_{\theta\in\Pi_{(4,4)}}\widehat{R^{mh}_{nh}v_{nh}}(m\theta^{\gamma^i_n}_{\delta^j_n})e^{\imath m\theta^{\gamma^i_n}_{\delta^j_n}\cdot x/(mh)}d\theta, \tag{4.23}$$

where the discrete Fourier transform $\widehat{R_{nh}^{mh} v_{nh}}(m\theta)$, with $\theta \in \Pi_{(4,4)}$, is defined as

$$\begin{aligned}\widehat{R_{nh}^{mh} v_{nh}}(m\theta) &= \frac{m_1 m_2}{4\pi^2} \sum_{x \in G_{mh}} (R_{nh}^{mh} v_{nh})(x) e^{-\imath m\theta \cdot x/(mh)} \\ &= \frac{m_1 m_2}{4\pi^2} \sum_{x \in G_{mh}} \sum_{k \in J_{R_{nh}^{mh}}} r_{k,mh} v_{nh}(x + knh) e^{-\imath m\theta \cdot x/(mh)}.\end{aligned}$$

For $x \in G_{mh}$ we have the aliasing relation

$$\phi_{nh}(n\theta_{\beta}^{\gamma_n^i}, x) = \phi_{mh}(m\theta_{\delta_n^j}^{\gamma_n^i}, x), \tag{4.24}$$

with $\beta \in \beta_n^j$, $i, j \in s_n$, see Section 4.2.1.

We can use the aliasing relation (4.24) to express the modes in the different groups on G_{mh} as the average of the aliasing modes on the mesh G_{nh}

$$e^{\imath m\theta_{\delta_n^j}^{\gamma_n^i} \cdot x/(mh)} = \frac{1}{n_1 n_2} \sum_{\beta \in \beta_n^j} e^{\imath n\theta_{\beta}^{\gamma_n^i} \cdot x/(nh)}, \qquad i, j \in s_n. \tag{4.25}$$

The discrete Fourier transform $\widehat{R_{nh}^{mh} v_{nh}}(m\theta_{\delta_n^j}^{\gamma_n^i})$, $i, j \in s_n$, can be further evaluated into

$$\begin{aligned}\widehat{R_{nh}^{mh} v_{nh}}(m\theta_{\delta_n^j}^{\gamma_n^i}) &= \frac{m_1 m_2}{4\pi^2 n_1 n_2} \sum_{x \in G_{mh}} \sum_{k \in J_{R_{nh}^{mh}}} r_{k,mh} v_{nh}(x + knh) \sum_{\beta \in \beta_n^j} e^{-\imath n\theta_{\beta}^{\gamma_n^i} \cdot x/(nh)} \\ &= \frac{m_1 m_2}{4\pi^2 n_1 n_2} \sum_{\beta \in \beta_n^j} \sum_{k \in J_{R_{nh}^{mh}}} r_{k,mh} e^{\imath n\theta_{\beta}^{\gamma_n^i} \cdot k} \sum_{x \in G_{mh}} v_{nh}(x + knh) e^{-\imath n\theta_{\beta}^{\gamma_n^i} \cdot (x+knh)/(nh)} \\ &= \frac{m_1 m_2}{4\pi^2 n_1 n_2} \sum_{\beta \in \beta_n^j} \sum_{k \in J_{R_{nh}^{mh}}} r_{k,mh} e^{\imath n\theta_{\beta}^{\gamma_n^i} \cdot k} \sum_{x \in G_{mh}} \Big(v_{nh}(x + knh) e^{-\imath n\theta_{\beta}^{\gamma_n^i} \cdot (x+knh)/(nh)} \\ &+ \sum_{l \in l_n} v_{nh}(x + knh + lnh) e^{-\imath n\theta_{\beta}^{\gamma_n^i} \cdot (x+knh+lnh)/(nh)} \\ &- \sum_{l \in l_n} v_{nh}(x + knh + lnh) e^{-\imath n\theta_{\beta}^{\gamma_n^i} \cdot (x+knh+lnh)/(nh)} \Big) \\ &= \frac{m_1 m_2}{4\pi^2 n_1 n_2} \sum_{\beta \in \beta_n^j} \sum_{k \in J_{R_{nh}^{mh}}} r_{k,mh} e^{\imath n\theta_{\beta}^{\gamma_n^i} \cdot k} \sum_{x \in G_{nh}} v_{nh}(x) e^{-\imath n\theta_{\beta}^{\gamma_n^i} \cdot x/(nh)} \\ &- \frac{m_1 m_2}{4\pi^2 n_1 n_2} \sum_{\beta \in \beta_n^j} \sum_{k \in J_{R_{nh}^{mh}}} r_{k,mh} \sum_{x \in G_{mh}} \sum_{l \in l_n} v_{nh}(x + knh + lnh) e^{-\imath n\theta_{\beta}^{\gamma_n^i} \cdot (x+lnh)/(nh)} \\ &= \sum_{\beta \in \beta_n^j} \sum_{k \in J_{R_{nh}^{mh}}} r_{k,mh} e^{\imath n\theta_{\beta}^{\gamma_n^i} \cdot k} \frac{m_1 m_2}{4\pi^2 n_1 n_2} \sum_{x \in G_{nh}} v_{nh}(x) e^{-\imath n\theta_{\beta}^{\gamma_n^i} \cdot x/(nh)} \\ &- \frac{m_1 m_2}{4\pi^2 n_1 n_2} \sum_{k \in J_{R_{nh}^{mh}}} r_{k,mh} \sum_{x \in G_{mh}} \sum_{l \in l_n} v_{nh}(x + knh + lnh) \sum_{\beta \in \beta_n^j} e^{-\imath n\theta_{\beta}^{\gamma_n^i} \cdot (x+lnh)/(nh)}\end{aligned} \tag{4.26}$$

with $l_n := \alpha_n^1 \backslash \gamma_n^1$. Using (4.10) we obtain for $x \in G_{mh}$, hence $x = jmh, j \in \mathbb{Z}^2$, that

$$\begin{aligned}
\sum_{\beta\in\beta_n^j} e^{-\imath n\theta_\beta^{\gamma_n^i}\cdot(x+lnh)/(nh)} &= \sum_{\beta\in\beta_n^j} e^{-\imath n\theta_\beta^{\gamma_n^i}\cdot(jmh+lnh)/(nh)} \\
&= \sum_{\beta\in\beta_n^j} e^{-\imath n(\theta_\beta^{00}-(\bar\gamma_1\,\mathrm{sign}((\theta_\beta^{00})_1),\bar\gamma_2\,\mathrm{sign}((\theta_\beta^{00})_2))\pi)\cdot(j\frac{m}{n}+l)} \\
&= \sum_{\beta\in\beta_n^j} e^{-\imath n\theta_\beta^{00}\cdot(j\frac{m}{n}+l)} e^{\imath n\pi(\bar\gamma_1\,\mathrm{sign}((\theta_\beta^{00})_1),\bar\gamma_2\,\mathrm{sign}((\theta_\beta^{00})_2))\cdot(j\frac{m}{n}+l)} \\
&= \sum_{\beta\in\beta_n^j} e^{-\imath n(\theta_{00}^{00}-(\bar\beta_1\mathrm{sign}((\theta_{00}^{00})_1),\bar\beta_2\mathrm{sign}((\theta_{00}^{00})_2))\pi)\cdot(j\frac{m}{n}+l)} \\
&\qquad e^{\imath n\pi(\bar\gamma_1\,\mathrm{sign}((\theta_\beta^{00})_1),\bar\gamma_2\,\mathrm{sign}((\theta_\beta^{00})_2))\cdot(j\frac{m}{n}+l)}, \qquad (4.27)
\end{aligned}$$

with $\beta = (\bar\beta_1, \bar\beta_2)$ and $\gamma_n^i = (\bar\gamma_1, \bar\gamma_2)$. The last summation in (4.27) can be further evaluated as

- For uniform coarsening we have $\beta_{(2,2)}^1 = \{(0,0),(\frac12,\frac12),(\frac12,0),(0,\frac12)\}$, $\gamma_{(2,2)}^1 = (0,0)$ and $l_{(2,2)} = \{(1,1),(1,0),(0,1)\}$ and we obtain

$$\begin{aligned}
&\sum_{\beta\in\beta_{(2,2)}^1} e^{-\imath n(\theta_{00}^{00}-(\bar\beta_1\mathrm{sign}((\theta_{00}^{00})_1),\bar\beta_2\mathrm{sign}((\theta_{00}^{00})_2))\pi)\cdot(j\frac{m}{n}+l)} e^{\imath n\pi(\bar\gamma_1\,\mathrm{sign}((\theta_\beta^{00})_1),\bar\gamma_2\,\mathrm{sign}((\theta_\beta^{00})_2))\cdot(j\frac{m}{n}+l)} \\
&\qquad = \sum_{\beta\in\beta_{(2,2)}^1} e^{-\imath n(\theta_{00}^{00}-(\bar\beta_1\mathrm{sign}((\theta_{00}^{00})_1),\bar\beta_2\mathrm{sign}((\theta_{00}^{00})_2))\pi)\cdot(2j_1+l_1,2j_2+l_2)} \\
&\qquad = e^{-\imath\theta_{00}^{00}\cdot(4j_1+2l_1,4j_2+2l_2)} \sum_{\beta\in\beta_{(2,2)}^1} e^{\imath\pi(\bar\beta_1\mathrm{sign}((\theta_{00}^{00})_1),\bar\beta_2\mathrm{sign}((\theta_{00}^{00})_2))\cdot(4j_1+2l_1,4j_2+2l_2)}
\end{aligned}$$

  Use now the relations

$$\begin{aligned}
&\sum_{\beta\in\beta_{(2,2)}^1} e^{\imath\pi(\bar\beta_1\mathrm{sign}((\theta_{00}^{00})_1),\bar\beta_2\mathrm{sign}((\theta_{00}^{00})_2))\cdot(4j_1+2l_1,4j_2+2l_2)} \\
&\qquad = 1 + e^{\imath\pi\mathrm{sign}((\theta_{00}^{00})_1)} + e^{\imath\pi\mathrm{sign}((\theta_{00}^{00})_1)} + 1 = 0, && \text{if } l = (1,0), \\
&\qquad = 1 + e^{\imath\pi\mathrm{sign}((\theta_{00}^{00})_2)} + 1 + e^{\imath\pi\mathrm{sign}((\theta_{00}^{00})_2)} = 0, && \text{if } l = (0,1), \\
&\qquad = 1 + e^{\imath\pi\mathrm{sign}((\theta_{00}^{00})_1)} e^{\imath\pi\mathrm{sign}((\theta_{00}^{00})_2)} + e^{\imath\pi\mathrm{sign}((\theta_{00}^{00})_1)} \\
&\qquad + e^{\imath\pi\mathrm{sign}((\theta_{00}^{00})_2)} = 0, && \text{if } l = (1,1).
\end{aligned}$$

- For semi-coarsening in the x_1-direction we have two cases: $\beta_{(2,1)}^1 = \{(0,0),(\frac12,0)\}$, $\gamma_{(2,1)}^1 = (0,0)$ and $\beta_{(2,1)}^2 = \{(\frac12,\frac12),(0,\frac12)\}$, $\gamma_{(2,1)}^2 = (0,1)$ with $l_{(2,1)} = \{(1,0)\}$, which can be further evaluated as

$$\begin{aligned}
&\sum_{\beta\in\beta_{(2,1)}^1} e^{-\imath n(\theta_{00}^{00}-(\bar\beta_1\mathrm{sign}((\theta_{00}^{00})_1),\bar\beta_2\mathrm{sign}((\theta_{00}^{00})_2))\pi)\cdot(j\frac{m}{n}+l)} e^{\imath n\pi(\bar\gamma_1\,\mathrm{sign}((\theta_\beta^{00})_1),\bar\gamma_2\,\mathrm{sign}((\theta_\beta^{00})_2))\cdot(j\frac{m}{n}+l)} \\
&\qquad = \sum_{\beta\in\beta_{(2,1)}^1} e^{-\imath(\theta_{00}^{00}-(\bar\beta_1\mathrm{sign}((\theta_{00}^{00})_1),\bar\beta_2\mathrm{sign}((\theta_{00}^{00})_2))\pi)\cdot(4j_1+2,j_2)} \\
&\qquad = e^{-\imath\theta_{00}^{00}\cdot(4j_1+2,j_2)}(1 + e^{\imath\pi\mathrm{sign}((\theta_{00}^{00})_1)}) \\
&\qquad = 0.
\end{aligned}$$

$$\sum_{\beta\in\beta^2_{(2,1)}} e^{-\imath n(\theta^{00}_{00}-(\bar\beta_1\mathrm{sign}((\theta^{00}_{00})_1),\bar\beta_2\mathrm{sign}((\theta^{00}_{00})_2))\pi)\cdot(j\frac{m}{n}+l)} e^{\imath n\pi(\bar\gamma_1\,\mathrm{sign}((\theta^{00}_{\beta})_1),\bar\gamma_2\,\mathrm{sign}((\theta^{00}_{\beta})_2))\cdot(j\frac{m}{n}+l)}$$

$$= e^{-\imath\theta^{00}_{00}\cdot(4j_1+2,j_2)} \sum_{\beta\in\beta^2_{(2,1)}} e^{\imath\pi(\bar\beta_1\mathrm{sign}((\theta^{00}_{00})_1),\bar\beta_2\mathrm{sign}((\theta^{00}_{00})_2))\cdot(4j_1+2,j_2))}$$

$$e^{\imath\pi(0,\bar\gamma_2\,\mathrm{sign}((\theta^{00}_{\beta})_2))\cdot(4j_1+2,j_2)}$$

$$= e^{-\imath\theta^{00}_{00}\cdot(4j_1+2,j_2)}\big(e^{\imath\pi(\frac{1}{2}\mathrm{sign}((\theta^{00}_{00})_1),\frac{1}{2}\mathrm{sign}((\theta^{00}_{00})_2))\cdot(4j_1+2,j_2)} e^{\imath\pi(0,\mathrm{sign}((\theta^{00}_{\beta})_2))\cdot(4j_1+2,j_2)}$$

$$+ e^{\imath\pi(0,\frac{1}{2}\mathrm{sign}((\theta^{00}_{00})_2)\cdot(4j_1+2,j_2)} e^{\imath\pi(0,\mathrm{sign}((\theta^{00}_{\beta})_2))\cdot(4j_1+2,j_2)}\big)$$

$$= e^{-\imath\theta^{00}_{00}\cdot(4j_1+2,j_2)} e^{\frac{1}{2}\imath\pi\mathrm{sign}((\theta^{00}_{00})_2)j_2} e^{\imath\pi\mathrm{sign}((\theta^{00}_{\beta})_2)j_2}\big(e^{\imath\pi\mathrm{sign}((\theta^{00}_{00})_1)}+1\big)$$

$$= 0.$$

- For semi-coarsening in the x_2-direction we have two cases: $\beta^1_{(1,2)} = \{(0,0),(0,\frac{1}{2})\}$, $\gamma^1_{(1,2)} = (0,0)$ and $\beta^2_{(1,2)} = \{(\frac{1}{2},\frac{1}{2}),(\frac{1}{2},0)\}$, $\gamma^2_{(1,2)} = (1,0)$ with $l_{(1,2)} = \{(0,1)\}$, which can be further evaluated as

$$\sum_{\beta\in\beta^1_{(1,2)}} e^{-\imath n(\theta^{00}_{00}-(\bar\beta_1\mathrm{sign}((\theta^{00}_{00})_1),\bar\beta_2\mathrm{sign}((\theta^{00}_{00})_2))\pi)\cdot(j\frac{m}{n}+l)} e^{\imath n\pi(\bar\gamma_1\,\mathrm{sign}((\theta^{00}_{\beta})_1),\bar\gamma_2\,\mathrm{sign}((\theta^{00}_{\beta})_2))\cdot(j\frac{m}{n}+l)}$$

$$= \sum_{\beta\in\beta^1_{(1,2)}} e^{-\imath(\theta^{00}_{00}-(\bar\beta_1\mathrm{sign}((\theta^{00}_{00})_1),\bar\beta_2\mathrm{sign}((\theta^{00}_{00})_2))\pi)\cdot(j_1,4j_2+2)}$$

$$= e^{-\imath\theta^{00}_{00}\cdot(j_1,4j_2+2)}\big(1+e^{\imath\pi\mathrm{sign}((\theta^{00}_{00})_2)}\big)$$

$$= 0.$$

$$\sum_{\beta\in\beta^2_{(1,2)}} e^{-\imath n(\theta^{00}_{00}-(\bar\beta_1\mathrm{sign}((\theta^{00}_{00})_1),\bar\beta_2\mathrm{sign}((\theta^{00}_{00})_2))\pi)\cdot(j\frac{m}{n}+l)} e^{\imath n\pi(\bar\gamma_1\,\mathrm{sign}((\theta^{00}_{\beta})_1),\bar\gamma_2\,\mathrm{sign}((\theta^{00}_{\beta})_2))\cdot(j\frac{m}{n}+l)}$$

$$= e^{-\imath\theta^{00}_{00}\cdot(j_1,4j_2+2)} \sum_{\beta\in\beta^2_{(1,2)}} e^{\imath\pi(\bar\beta_1\mathrm{sign}((\theta^{00}_{00})_1),\bar\beta_2\mathrm{sign}((\theta^{00}_{00})_2))\cdot(j_1,4j_2+2))}$$

$$e^{\imath\pi(\bar\gamma_1\mathrm{sign}((\theta^{00}_{\beta})_1),0)\cdot(j_1,4j_2+2)}$$

$$= e^{-\imath\theta^{00}_{00}\cdot(j_1,4j_2+2)}\big(e^{\imath\pi(\frac{1}{2}\mathrm{sign}((\theta^{00}_{00})_1),\frac{1}{2}\mathrm{sign}((\theta^{00}_{00})_2))\cdot(j_1,4j_2+2)} e^{\imath\pi(\mathrm{sign}((\theta^{00}_{\beta})_1),0)\cdot(j_1,4j_2+2)}$$

$$+ e^{\imath\pi(\frac{1}{2}\mathrm{sign}((\theta^{00}_{00})_1),0)\cdot(j_1,4j_2+2)} e^{\imath\pi(\mathrm{sign}((\theta^{00}_{\beta})_1),0)\cdot(j_1,4j_2+2)}\big)$$

$$= e^{-\imath\theta^{00}_{00}\cdot(j_1,4j_2+2)} e^{\frac{1}{2}\imath\pi\mathrm{sign}((\theta^{00}_{00})_1)j_1} e^{\imath\pi\mathrm{sign}((\theta^{00}_{\beta})_1)j_1}\big(e^{\imath\pi\mathrm{sign}((\theta^{00}_{00})_2)}+1\big)$$

$$= 0.$$

Combining all contributions in (4.27) we obtain that

$$\sum_{\beta\in\beta^j_n} e^{-\imath\theta^{\gamma^i_n}_{\beta}\cdot(x+lnh)/(nh)} = 0, \qquad i,j\in s_n.$$

The relation for the discrete Fourier transform of the restriction operator, given by (4.26), therefore can be further evaluated into

$$\widehat{R^{mh}_{nh}v_{nh}}(m\theta^{\gamma^i_n}_{\delta^j_n}) = \sum_{\beta\in\beta^j_n} \widehat{R^{mh}_{nh}}(n\theta^{\gamma^i_n}_{\beta})\widehat{v_{nh}}(n\theta^{\gamma^i_n}_{\beta}), \quad i,j\in s_n, \tag{4.28}$$

with the Fourier symbol $\widehat{R_{nh}^{mh}}(n\theta_\beta^{\gamma_n^i})$ defined as

$$\widehat{R_{nh}^{mh}}(n\theta_\beta^{\gamma_n^i}) = \sum_{k \in J_{R_{nh}^{mh}}} r_{k,mh} e^{\imath n \theta_\beta^{\gamma_n^i} \cdot k}, \qquad \forall \beta \in \beta_n^j,\ i,j \in s_n.$$

Relation (4.28) shows that the restriction operator couples the modes with frequency $\theta_\beta^{\gamma_n^i}$ on the grid G_{nh} to the coarse grid modes with frequency $\theta_{\delta_n^j}^{\gamma_n^i}$ on the grid G_{mh}. Using relation (4.28) we can transform (4.23) into

$$(R_{nh}^{mh} v_{nh})(x) = \sum_{i \in s_n} \sum_{j \in s_n} \int_{\theta \in \Pi_{(4,4)}} \left(\sum_{\beta \in \beta_n^j} \widehat{R_{nh}^{mh}}(n\theta_\beta^{\gamma_n^i}) \widehat{v_{nh}}(n\theta_\beta^{\gamma_n^i}) \right) e^{\imath m \theta_{\delta_j^{\gamma_n^i}} \cdot x/(mh)} d\theta.$$

In the special case of two-level analysis the restriction operator is related to its discrete Fourier transform through the relation

$$(R_h^{nh} v_h)(x) = \sum_{i \in s_n} \int_{\theta \in \Pi_{(2,2)}} \widehat{R_h^{nh} v_h}(n\theta^{\gamma_n^i}) e^{\imath n \theta^{\gamma_n^i} \cdot x/(nh)} d\theta. \tag{4.29}$$

The discrete transform of the restriction operator then reduces to

$$\widehat{R_h^{nh} v_h}(n\theta^{\gamma_n^i}) = \sum_{\alpha \in \alpha_n^i} \widehat{R_h^{nh}}(\theta^\alpha) \widehat{v_h}(\theta^\alpha), \quad i \in s_n, \tag{4.30}$$

with the Fourier symbol $\widehat{R_h^{nh}}(\theta^\alpha)$ defined as

$$\widehat{R_h^{nh}}(\theta^\alpha) = \sum_{k \in J_{R_h^{nh}}} r_{k,nh} e^{\imath \theta^\alpha \cdot k}.$$

Relation (4.30) shows that the restriction operator couples the modes θ^α, with $\alpha \in \alpha_n^i$, $i \in s_n$, on the grid G_h to the coarse grid modes $\theta^{\gamma_n^i}$ on the grid G_{nh}. Using relation (4.30) we obtain from (4.29) the following relation for the restriction operator in two-level analysis

$$(R_h^{nh} v_h)(x) = \sum_{i \in s_n} \int_{\theta \in \Pi_{(2,2)}} \left(\sum_{\alpha \in \alpha_n^i} \widehat{R_h^{nh}}(\theta^\alpha) \widehat{v_h}(\theta^\alpha) \right) e^{\imath n \theta^{\gamma_n^i} \cdot x/(nh)} d\theta, \tag{4.31}$$

with $\theta^\alpha = \theta^\alpha(\theta)$ given by (4.12).

4.2.5 h-Multigrid prolongation operators

The definition of the prolongation operator $P_{nh}^h : \mathcal{F}(G_{nh}) \to \mathcal{F}(G_h)$, with $n \in \{(2,2),(2,1),(1,2)\}$, requires the introduction of subsets of the mesh G_h. Define the meshes G_h^κ as

$$G_h^\kappa := \{(x_1,x_2) \in \mathbb{R}^2 \mid (x_1,x_2) = ((n_1 j_1 + \bar{\kappa}_1)h_1, (n_2 j_2 + \bar{\kappa}_2)h_2),\ j \in \mathbb{Z}^2\},$$

with $\kappa \in \kappa_n := \{\kappa = (\bar{\kappa}_1, \bar{\kappa}_2) \,|\, \bar{\kappa}_i \in \{0, n_i - 1\}, i = 1,2\}$, see Figure 4.4. The prolongation operator related to the mesh G_h^κ then is equal to

$$(P_{nh}^h v_h)(x) = \sum_{k \in J_{P_{nh}^h}^\kappa} p_{k,h} v_{nh}(x + kh), \qquad x \in G_h^\kappa,\ x + kh \in G_{nh},$$

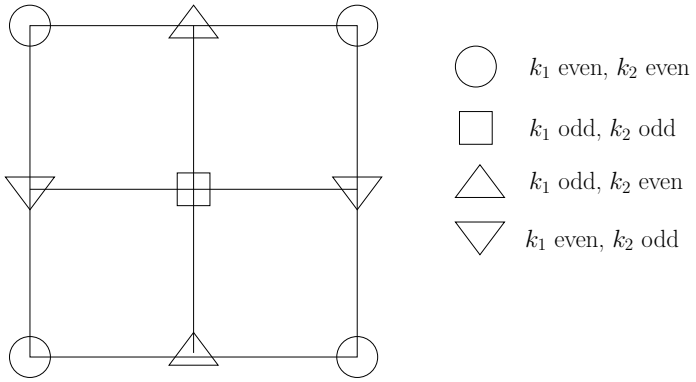

Figure 4.4: Different meshes G_h^α used in the definition of the prolongation operator for uniformly coarsened meshes

where the index set $J^\kappa_{P^h_{nh}} \subset \mathbb{Z}^2$ is used to define the prolongation operator on each mesh.

We consider now the prolongation operator $P^n_{nh} : \mathcal{F}(G_{nh}) \to \mathcal{F}(G_h)$, with $n \in \{(2,2),(2,1),(1,2)\}$. The prolongation operator P^h_{nh} is related to its discrete Fourier transform through the relation

$$\begin{aligned}\left(P^h_{nh} v_{nh}\right)(x) &= \int_{\theta\in\Pi_{(1,1)}} \widehat{P^h_{nh} v_{nh}}(\theta) e^{\imath\theta\cdot x/h} d\theta \\ &= \sum_{i\in s_n}\sum_{j\in s_n}\sum_{\alpha\in\alpha_n^i}\sum_{\beta\in\beta_n^j} \int_{\theta\in\Pi_{(4,4)}} \widehat{P^h_{nh} v_{nh}}(\theta^\alpha_\beta) e^{\imath\theta^\alpha_\beta\cdot x/h} d\theta, \end{aligned} \tag{4.32}$$

with $\theta^\alpha_\beta = \theta^\alpha_\beta(\theta)$ given by (4.10). The discrete Fourier transform $\widehat{P^h_{nh} v_{nh}}(\theta^\alpha_\beta)$, with $\alpha \in \alpha^i_n$, $\beta \in \beta^j_n$, $i,j \in s_n$, can be further evaluated as

$$\begin{aligned}\widehat{P^h_{nh} v_{nh}}(\theta^\alpha_\beta) &= \frac{1}{4\pi^2}\sum_{x\in G_h} (P^h_{nh} v_{nh})(x) e^{-\imath\theta^\alpha_\beta\cdot x/h} \\ &= \frac{1}{4\pi^2}\sum_{\kappa\in\kappa_n}\sum_{x\in G^\kappa_h} (P^h_{nh} v_{nh})(x) e^{-\imath\theta^\alpha_\beta\cdot x/h} \\ &= \frac{1}{4\pi^2}\sum_{\kappa\in\kappa_n}\sum_{k\in J^\kappa_{P^h_{nh}}} p_{k,h} e^{\imath\theta^\alpha_\beta\cdot k}\sum_{x\in G^\kappa_h} v_{nh}(x+kh) e^{-\imath\theta^\alpha_\beta\cdot(x+kh)/h} \\ &= \frac{1}{n_1 n_2}\sum_{\kappa\in\kappa_n}\sum_{k\in J^\kappa_{P^h_{nh}}} p_{k,h} e^{\imath\theta^\alpha_\beta\cdot k}\,\frac{n_1 n_2}{4\pi^2}\sum_{x\in G_{nh}} v_{nh}(x) e^{-\imath n\theta^{\gamma^i_n}_\beta\cdot x/(nh)} \\ &= \widehat{P^h_{nh}}(\theta^\alpha_\beta)\widehat{v_{nh}}(n\theta^{\gamma^i_n}_\beta). \end{aligned} \tag{4.33}$$

Note, in the fourth step in (4.33) we used that $x+kh \in G_{nh}$ for all $k \in J^\kappa_{P^h_{nh}}$ and that modes with frequency θ^α_β, $\alpha \in \alpha^i_n$, on the mesh G_h alias to modes with frequency $\theta^{\gamma^i_n}_\beta$ on the mesh

G_{nh}. The Fourier symbol $\widehat{P^h_{nh}}(\theta^\alpha_\beta)$ is defined as

$$\widehat{P^h_{nh}}(\theta^\alpha_\beta) = \frac{1}{n_1 n_2} \sum_{\kappa\in\kappa_n} \sum_{k\in J^\kappa_{P^h_{nh}}} p_{k,h} e^{\imath\theta^\alpha_\beta\cdot k}.$$

Hence, we obtain for $x \in G_h$ the expression

$$\left(P^h_{nh} v_{nh}\right)(x) = \sum_{i\in s_n}\sum_{j\in s_n}\sum_{\alpha\in\alpha^i_n}\sum_{\beta\in\beta^j_n} \int_{\theta\in\Pi_{(4,4)}} \widehat{P^h_{nh}}(\theta^\alpha_\beta)\widehat{v_{nh}}(n\theta^{\gamma_n^i}_\beta) e^{\imath\theta^\alpha_\beta\cdot x/h} d\theta.$$

Next, the prolongation operator $P^{nh}_{mh} : \mathcal{F}(G_{mh}) \to \mathcal{F}(G_{nh})$, with $n \in \{(2,2),(2,1),(1,2)\}$, $m \in \{(4,4),(4,1),(1,4)\}$, is considered. The definition of the prolongation operator requires the introduction of subsets of the mesh G_{nh}. Define the meshes G^κ_{nh} as

$$G^\kappa_{nh} := \{(x_1,x_2) \in \mathbb{R}^2 \mid (x_1,x_2) = ((m_1 j_1 + \bar\kappa_1)h_1, (m_2 j_2 + \bar\kappa_2)h_2),\ j \in \mathbb{Z}^2\},$$

with $\kappa \in \kappa_m := \{\kappa = (\bar\kappa_1,\bar\kappa_2) \,|\, \bar\kappa_i \in \{0,(2m_i-2)/3\}, i = 1,2\}$. The prolongation operator related to the mesh G^κ_{nh} then is equal to

$$(P^{nh}_{mh} v_{mh})(x) = \sum_{k\in J^\kappa_{P^{nh}_{mh}}} p_{k,nh} v_{mh}(x+knh), \qquad x \in G^\kappa_{nh},\ x+knh \in G_{mh}.$$

The prolongation operator P^{nh}_{mh} is related to its discrete Fourier transform through the relation

$$\begin{aligned}\left(P^{nh}_{mh} v_{mh}\right)(x) &= \int_{\theta\in\Pi_n} \widehat{P^{nh}_{mh} v_{mh}}(n\theta) e^{\imath n\theta\cdot x/(nh)} d\theta \\ &= \sum_{i\in s_n}\sum_{j\in s_n}\sum_{\beta\in\beta^j_n} \int_{\theta\in\Pi_{(4,4)}} \widehat{P^{nh}_{mh} v_{mh}}(n\theta^{\gamma_n^i}_\beta) e^{\imath n\theta^{\gamma_n^i}_\beta\cdot x/(nh)} d\theta. \end{aligned} \tag{4.34}$$

The discrete Fourier transform $\widehat{P^{nh}_{mh} v_{mh}}(n\theta^{\gamma_n^i}_\beta)$, with $\beta \in \beta^j_n$, $j \in s_n$, can be further evaluated as

$$\begin{aligned}\widehat{P^{nh}_{mh} v_{mh}}(n\theta^{\gamma_n^i}_\beta) &= \frac{n_1 n_2}{4\pi^2} \sum_{x\in G_{nh}} (P^{nh}_{mh} v_{mh})(x) e^{-\imath n\theta^{\gamma_n^i}_\beta\cdot x/(nh)} \\ &= \frac{n_1 n_2}{4\pi^2} \sum_{\kappa\in\kappa_m}\sum_{x\in G^\kappa_{nh}} (P^{nh}_{mh} v_{mh})(x) e^{-\imath n\theta^{\gamma_n^i}_\beta\cdot x/(nh)} \\ &= \frac{n_1 n_2}{4\pi^2} \sum_{\kappa\in\kappa_m}\sum_{k\in J^\kappa_{P^{nh}_{mh}}} p_{k,nh} e^{\imath n\theta^{\gamma_n^i}_\beta\cdot k} \sum_{x\in G^\kappa_{nh}} v_{mh}(x+knh) e^{-\imath n\theta^{\gamma_n^i}_\beta\cdot(x+knh)/(nh)} \\ &= \frac{n_1 n_2}{m_1 m_2} \sum_{\kappa\in\kappa_m}\sum_{k\in J^\kappa_{P^{nh}_{mh}}} p_{k,nh} e^{\imath n\theta^{\gamma_n^i}_\beta\cdot k} \frac{m_1 m_2}{4\pi^2} \sum_{x\in G_{mh}} v_{mh}(x) e^{-\imath m\theta^{\gamma_n^i}_{\delta^j_n}\cdot x/(mh)} \\ &= \widehat{P^{nh}_{mh}}(n\theta^{\gamma_n^i}_\beta)\widehat{v_{mh}}(m\theta^{\gamma_n^i}_{\delta^j_n}). \end{aligned} \tag{4.35}$$

Note, in the fourth step in (4.35) we used that $x + knh \in G_{mh}$ for all $k \in J^\kappa_{P^{nh}_{mh}}$ and that modes with frequency $\theta^{\gamma_n^i}_\beta$, with $\beta \in \beta^j_n$, on the mesh G_{nh} alias to modes with frequency

$\theta_{\delta_n^j}^{\gamma_n^i}$, $i,j \in s_n$ on the mesh G_{mh}. The Fourier symbol $\widehat{P_{mh}^{nh}}(n\theta_\beta^{\gamma_n^i})$ is defined as

$$\widehat{P_{mh}^{nh}}(n\theta_\beta^{\gamma_n^i}) = \frac{n_1 n_2}{m_1 m_2} \sum_{\kappa \in \kappa_m} \sum_{k \in J^\kappa_{P_{mh}^{nh}}} p_{k,nh} e^{\imath n \theta_\beta^{\gamma_n^i} \cdot k}.$$

Hence, we obtain for $x \in G_{nh}$ expression

$$\left(P_{mh}^{nh} v_{mh}\right)(x) = \sum_{i \in s_n} \sum_{j \in s_n} \sum_{\beta \in \beta_n^j} \int_{\theta \in \Pi_{(4,4)}} \widehat{P_{mh}^{nh}}(n\theta_\beta^{\gamma_n^i}) \widehat{v_{mh}}(m\theta_{\delta_n^j}^{\gamma_n^i}) e^{\imath n \theta_\beta^{\gamma_n^i} \cdot x/(nh)} d\theta.$$

In the special case of two-level analysis the Fourier symbol of the prolongation operator is related to its discrete Fourier transform through the relation

$$\left(P_{nh}^{h} v_{nh}\right)(x) = \sum_{i \in s_n} \sum_{\alpha \in \alpha_n^i} \int_{\theta \in \Pi_{(2,2)}} \widehat{P_{nh}^{h} v_{nh}}(\theta^\alpha) e^{\imath \theta^\alpha \cdot x/h} d\theta, \tag{4.36}$$

The discrete Fourier transform of $\widehat{P_{nh}^{h} v_{nh}}(\theta^\alpha)$ reduces to

$$\widehat{P_{nh}^{h} v_{nh}}(\theta^\alpha) = \widehat{P_{nh}^{h}}(\theta^\alpha) \widehat{v_{nh}}(n\theta^{\gamma_n^i}) \tag{4.37}$$

with $\widehat{P_{nh}^{h}}(\theta^\alpha)$ defined as

$$\widehat{P_{nh}^{h}}(\theta^\alpha) = \frac{1}{n_1 n_2} \sum_{\kappa \in \kappa_n} \sum_{k \in J^\kappa_{P_{nh}^{h}}} p_{k,h} e^{\imath \theta^\alpha \cdot k}.$$

Hence, we obtain for $x \in G_h$ the expression

$$\left(P_{nh}^{h} v_{nh}\right)(x) = \sum_{i \in s_n} \sum_{\alpha \in \alpha_n^i} \int_{\theta \in \Pi_{(2,2)}} \widehat{P_{nh}^{h}}(\theta^\alpha) \widehat{v_{nh}}(n\theta^{\gamma_n^i}) e^{\imath \theta^\alpha \cdot x/h} d\theta.$$

4.2.6 p-Multigrid restriction and prolongation operators

Define the p-multigrid prolongation operators $T_{h,p-1}^{p} : \mathcal{F}(G_h) \to \mathcal{F}(G_h)$ in stencil notation as

$$(T_{h,p-1}^{p} v_{h,p-1})(\bar{x}) = t_{h,p} v_{h,p}(\bar{x}), \qquad \bar{x} \in G_h,$$

where $t_{h,p} \in \mathbb{R}^{m_p \times m_p}$ is the matrix defining the p-multigrid prolongation operator in an element. Since this is a purely element based operator it immediately follows that its Fourier symbol is equal to

$$\widehat{T_{h,p-1}^{p}} = t_{h,p}. \tag{4.38}$$

The p-multigrid restriction operator $Q_{h,p}^{p-1} : \mathcal{F}(G_h) \to \mathcal{F}(G_h)$ is equal to the transposed of the p-multigrid prolongation operator. The Fourier symbol of the p-restriction operator then is equal to

$$\widehat{Q_{h,p}^{p-1}} = (\widehat{T_{h,p-1}^{p}})^T. \tag{4.39}$$

4.3 Two-level Fourier analysis

In two-level analysis the Fourier symbols $\widehat{L_h}(\theta)$ and $\widehat{L_{nh}}(n\theta)$ can be zero for certain values of θ. The frequencies of these Fourier harmonics are removed from $\mathcal{F}_h^2(\theta)$ through the definition of

$$\mathcal{F}_h^{2g} := \{\mathcal{F}_h^2(\theta^\alpha) \mid \theta \in \Pi_{(2,2)} \setminus \Psi_n, \forall \alpha \in \alpha_n^i, i \in s_n\}$$

with

$$\Psi_n := \left\{\theta \in \Pi_{(2,2)} \mid \det(\widehat{L_h}(\theta^\alpha)) = 0 \text{ or } \det(\widehat{L_{nh}}(n\theta^{\gamma_n^i})) = 0, \forall \alpha \in \alpha_n^i, i \in s_n\right\}, \tag{4.40}$$

and $\theta^\alpha = \theta^\alpha(\theta)$. The set s_n is defined as $s_n = \{1, 2\}$ if $n = (2, 1)$ or $(1, 2)$ and $s_n = \{1\}$ if $n = (2, 2)$. The error e_h^D on the mesh G_h after one iteration of a two-grid multigrid cycle is derived in Section 3.1, equation (3.7), and equal to

$$e_h^D = M_h^{2g} e_h^A, \tag{4.41}$$

with e_h^A the initial error and M_h^{2g} the two level multigrid error transformation operator defined as

$$M_h^{2g} = S_h^{\nu_2}\left(I_h - P_{nh}^h L_{nh}^{-1} R_h^{nh} L_h\right) S_h^{\nu_1}, \tag{4.42}$$

where L_h denotes the discrete approximation of the spatial operator L, S_h the multigrid smoother, R_h^{nh} the restriction operator, P_{nh}^h the prolongation operator, ν_1, ν_2 the number of pre- and post-smoothing iterations, and I_h the identity operator. The error e_h^D has for $x \in G_h$ the Fourier decomposition

$$\begin{aligned} e_h^D(x) &= \int_{\theta \in \Pi_{(1,1)}} \widehat{e_h^D}(\theta) e^{\imath\theta \cdot x/h} d\theta \\ &= \sum_{i \in s_n} \sum_{\alpha \in \alpha_n^i} \int_{\theta \in \Pi_{(2,2)}} \left(\widehat{M_h^{2g} e_h^A}\right)(\theta^\alpha) e^{\imath\theta^\alpha \cdot x/h} d\theta. \end{aligned} \tag{4.43}$$

In order to compute the Fourier symbol of the error transformation operator M_h^{2g} we first compute the discrete Fourier transform of $S_h^{\nu_2} P_{nh}^h L_{nh}^{-1} R_h^{nh} L_h S_h^{\nu_1} e_h^A$ for each group of modes with $\alpha \in \alpha_n^i$, $i \in s_n$ using the following steps:

1. Using (4.13), (4.14), (4.15) and (4.16) we obtain

$$\begin{aligned} \left(L_h S_h^{\nu_1} e_h^A\right)(x) &= \sum_{i \in s_n} \sum_{\alpha \in \alpha_n^i} \int_{\theta \in \Pi_{(2,2)}} \left(\widehat{L_h S_h^{\nu_1} e_h^A}\right)(\theta^\alpha) e^{\imath\theta^\alpha \cdot x/h} d\theta \\ &= \sum_{i \in s_n} \sum_{\alpha \in \alpha_n^i} \int_{\theta \in \Pi_{(2,2)}} \widehat{L_h}(\theta^\alpha) \left(\widehat{S_h}(\theta^\alpha)\right)^{\nu_1} \widehat{e_h^A}(\theta^\alpha) e^{\imath\theta^\alpha \cdot x/h} d\theta \end{aligned}$$

   hence

$$\left(\widehat{L_h S_h^{\nu_1} e_h^A}\right)(\theta^\alpha) = \widehat{L_h}(\theta^\alpha)\left(\widehat{S_h}(\theta^\alpha)\right)^{\nu_1} \widehat{e_h^A}(\theta^\alpha), \quad \forall \alpha \in \alpha_n^i, i \in s_n. \tag{4.44}$$

2. Using (4.29) and (4.30) we obtain for $x \in G_{nh}$

$$\begin{aligned}\big(R_h^{nh} L_h S_h^{\nu_1} e_h^A\big)(x) &= \sum_{i\in s_n} \int_{\theta\in\Pi_{(2,2)}} \widehat{\big(R_h^{nh} L_h S_h^{\nu_1} e_h^A\big)}(n\theta^{\gamma_n^i}) e^{\imath n\theta^{\gamma_n^i}\cdot x/(nh)} d\theta \\ &= \sum_{i\in s_n} \int_{\theta\in\Pi_{(2,2)}} \sum_{\alpha_2\in\alpha_n^i} \widehat{R_h^{nh}}(\theta^{\alpha_2}) \widehat{\big(L_h S_h^{\nu_1} e_h^A\big)}(\theta^{\alpha_2}) e^{\imath n\theta^{\gamma_n^i}\cdot x/(nh)} d\theta \\ &= \sum_{i\in s_n} \int_{\theta\in\Pi_{(2,2)}} \sum_{\alpha_2\in\alpha_n^i} \widehat{R_h^{nh}}(\theta^{\alpha_2}) \widehat{L_h}(\theta^{\alpha_2}) \big(\widehat{S_h}(\theta^{\alpha_2})\big)^{\nu_1} \widehat{e_h^A}(\theta^{\alpha_2}) e^{\imath n\theta^{\gamma_n^i}\cdot x/(nh)} d\theta,\end{aligned}$$

where $\widehat{(\,\cdot\,)}$ is used to indicate the Fourier symbol of the product of a number of variables, hence

$$\widehat{\big(R_h^{nh} L_h S_h^{\nu_1} e_h^A\big)}(n\theta^{\gamma_n^i}) = \sum_{\alpha_2\in\alpha_n^i} \widehat{R_h^{nh}}(\theta^{\alpha_2}) \widehat{L_h}(\theta^{\alpha_2}) \big(\widehat{S_h}(\theta^{\alpha_2})\big)^{\nu_1} \widehat{e_h^A}(\theta^{\alpha_2}), \quad i \in s_n. \tag{4.45}$$

3. Using (4.13), (4.14), and (4.45) we obtain

$$\begin{aligned}\big(L_{nh}^{-1} R_h^{nh} L_h S_h^{\nu_1} e_h^A\big)(x) &= \sum_{i\in s_n} \int_{\theta\in\Pi_{(2,2)}} \widehat{\big(L_{nh}^{-1} R_h^{nh} L_h S_h^{\nu_1} e_h^A\big)}(n\theta^{\gamma_n^i}) e^{\imath n\theta^{\gamma_n^i}\cdot x/(nh)} d\theta \\ &= \sum_{i\in s_n} \int_{\theta\in\Pi_{(2,2)}} \Big(\widehat{L_{nh}}(n\theta^{\gamma_n^i})\Big)^{-1} \widehat{\big(R_h^{nh} L_h S_h^{\nu_1} e_h^A\big)}(n\theta^{\gamma_n^i}) e^{\imath n\theta^{\gamma_n^i}\cdot x/(nh)} d\theta \\ &= \sum_{i\in s_n} \int_{\theta\in\Pi_{(2,2)}} \Big(\widehat{L_{nh}}(n\theta^{\gamma_n^i})\Big)^{-1} \sum_{\alpha_2\in\alpha_n^i} \widehat{R_h^{nh}}(\theta^{\alpha_2}) \widehat{L_h}(\theta^{\alpha_2}) \big(\widehat{S_h}(\theta^{\alpha_2})\big)^{\nu_1} \\ &\qquad \widehat{e_h^A}(\theta^{\alpha_2}) e^{\imath n\theta^{\gamma_n^i}\cdot x/(nh)} d\theta\end{aligned}$$

hence

$$\begin{aligned}\widehat{\big(L_{nh}^{-1} R_h^{nh} L_h S_h^{\nu_1} e_h^A\big)}(n\theta^{\gamma_n^i}) = \Big(\widehat{L_{nh}}(n\theta^{\gamma_n^i})\Big)^{-1} &\sum_{\alpha_2\in\alpha_n^i} \widehat{R_h^{nh}}(\theta^{\alpha_2}) \widehat{L_h}(\theta^{\alpha_2}) \\ &\big(\widehat{S_h}(\theta^{\alpha_2})\big)^{\nu_1} \widehat{e_h^A}(\theta^{\alpha_2}), \qquad i \in s_n.\end{aligned} \tag{4.46}$$

4. Using (4.36), (4.37) and (4.46) we obtain

$$\begin{aligned}\big(P_{nh}^h L_{nh}^{-1} R_h^{nh} L_h S_h^{\nu_1} e_h^A\big)(x) &= \sum_{i\in s_n} \sum_{\alpha\in\alpha_n^i} \int_{\theta\in\Pi_{(2,2)}} \widehat{\big(P_{nh}^h L_{nh}^{-1} R_h^{nh} L_h S_h^{\nu_1} e_h^A\big)}(\theta^{\alpha}) e^{\imath\theta^{\alpha}\cdot x/h} d\theta \\ &= \sum_{i\in s_n} \sum_{\alpha\in\alpha_n^i} \int_{\theta\in\Pi_{(2,2)}} \widehat{P_{nh}^h}(\theta^{\alpha}) \widehat{\big(L_{nh}^{-1} R_h^{nh} L_h S_h^{\nu_1} e_h^A\big)}(n\theta^{\gamma_n^i}) e^{\imath\theta^{\alpha}\cdot x/h} d\theta \\ &= \sum_{i\in s_n} \sum_{\alpha\in\alpha_n^i} \int_{\theta\in\Pi_{(2,2)}} \widehat{P_{nh}^h}(\theta^{\alpha}) \Big(\widehat{L_{nh}}(n\theta^{\gamma_n^i})\Big)^{-1} \sum_{\alpha_2\in\alpha_n^i} \widehat{R_h^{nh}}(\theta^{\alpha_2}) \\ &\qquad \widehat{L_h}(\theta^{\alpha_2}) \big(\widehat{S_h}(\theta^{\alpha_2})\big)^{\nu_1} \widehat{e_h^A}(\theta^{\alpha_2}) e^{\imath\theta^{\alpha}\cdot x/h} d\theta\end{aligned}$$

hence

$$\begin{aligned}\widehat{\big(P_{nh}^h L_{nh}^{-1} R_h^{nh} L_h S_h^{\nu_1} e_h^A\big)}(\theta^{\alpha}) = &\widehat{P_{nh}^h}(\theta^{\alpha}) \Big(\widehat{L_{nh}}(n\theta^{\gamma_n^i})\Big)^{-1} \sum_{\alpha_2\in\alpha_n^i} \widehat{R_h^{nh}}(\theta^{\alpha_2}) \widehat{L_h}(\theta^{\alpha_2}) \\ &\big(\widehat{S_h}(\theta^{\alpha_2})\big)^{\nu_1} \widehat{e_h^A}(\theta^{\alpha_2}), \quad \forall \alpha \in \alpha_n^i,\ i \in s_n.\end{aligned} \tag{4.47}$$

5. Using (4.15), (4.16) and (4.47) we obtain

$$\begin{aligned}
&\big(S_h^{\nu_2} P_{nh}^h L_{nh}^{-1} R_h^{nh} L_h S_h^{\nu_1} e_h^A\big)(x) \\
&\quad = \sum_{i\in s_n} \sum_{\alpha\in\alpha_n^i} \int_{\theta\in\Pi_{(2,2)}} \widehat{\big(S_h^{\nu_2} P_{nh}^h L_{nh}^{-1} R_h^{nh} L_h S_h^{\nu_1} e_h^A\big)}(\theta^\alpha) e^{\imath\theta^\alpha\cdot x/h} d\theta \\
&\quad = \sum_{i\in s_n} \sum_{\alpha\in\alpha_n^i} \int_{\theta\in\Pi_{(2,2)}} (\widehat{S_h}(\theta^\alpha))^{\nu_2} \widehat{\big(P_{nh}^h L_{nh}^{-1} R_h^{nh} L_h S_h^{\nu_1} e_h^A\big)}(\theta^\alpha) e^{\imath\theta^\alpha\cdot x/h} d\theta \\
&\quad = \sum_{i\in s_n} \sum_{\alpha\in\alpha_n^i} \int_{\theta\in\Pi_{(2,2)}} \big(\widehat{S_h}(\theta^\alpha)\big)^{\nu_2} \widehat{P_{nh}^h}(\theta^\alpha) \Big(\widehat{L_{nh}}(n\theta^{\gamma_n^i})\Big)^{-1} \sum_{\alpha_2\in\alpha_n^i} \widehat{R_h^{nh}}(\theta^{\alpha_2}) \widehat{L_h}(\theta^{\alpha_2}) \\
&\qquad\qquad \big(\widehat{S_h}(\theta^{\alpha_2})\big)^{\nu_1} \widehat{e_h^A}(\theta^{\alpha_2}) e^{\imath\theta^\alpha\cdot x/h} d\theta
\end{aligned}$$

hence

$$\begin{aligned}
\widehat{\big(S_h^{\nu_2} P_{nh}^h L_{nh}^{-1} R_h^{nh} L_h S_h^{\nu_1} e_h^A\big)}(\theta^\alpha) =& (\widehat{S_h}(\theta^\alpha))^{\nu_2} \widehat{P_{nh}^h}(\theta^\alpha) \Big(\widehat{L_{nh}}(n\theta^{\gamma_n^i})\Big)^{-1} \sum_{\alpha_2\in\alpha_n^i} \widehat{R_h^{nh}}(\theta^{\alpha_2}) \\
& \widehat{L_h}(\theta^{\alpha_2}) \big(\widehat{S_h}(\theta^{\alpha_2})\big)^{\nu_1} \widehat{e_h^A}(\theta^{\alpha_2}), \quad \forall\alpha\in\alpha_n^i,\ i\in s_n.
\end{aligned} \tag{4.48}$$

Using (4.16), (4.42) and (4.48) the error in the two-level multigrid algorithm in (4.43) can now be expressed as

$$\begin{aligned}
e_h^D(x) &= \sum_{i\in s_n} \sum_{\alpha\in\alpha_n^i} \int_{\theta\in\Pi_{(2,2)}} \big(S_h^{\nu_2}(I_h - P_{nh}^h L_{nh}^{-1} R_h^{nh} L_h) S_h^{\nu_1} e_h^A\big)\widehat{\ }(\theta^\alpha) e^{\imath\theta^\alpha\cdot x/h} d\theta \\
&= \sum_{i\in s_n} \sum_{\alpha\in\alpha_n^i} \int_{\theta\in\Pi_{(2,2)}} \Big(\big(\widehat{S_h}(\theta^\alpha)\big)^{\nu_1+\nu_2} \widehat{e}_h^A(\theta^\alpha) - \big(\widehat{S_h}(\theta^\alpha)\big)^{\nu_2} \widehat{P_{nh}^h}(\theta^\alpha) \Big(\widehat{L_{nh}}(n\theta^{\gamma_n^i})\Big)^{-1} \\
&\qquad \sum_{\alpha_2\in\alpha_n^i} \widehat{R_h^{nh}}(\theta^{\alpha_2}) \widehat{L_h}(\theta^{\alpha_2}) \big(\widehat{S_h}(\theta^{\alpha_2})\big)^{\nu_1} \widehat{e_h^A}(\theta^{\alpha_2}) \Big) e^{\imath\theta^\alpha\cdot x/h} d\theta.
\end{aligned}$$

The discrete Fourier transform of $M_h^{2g} e_h^A(\theta^\alpha)$ for the group of modes with $\alpha\in\alpha_n^i$, $i\in s_n$, denoted by $\widehat{M_h^n e_h^A}(\theta^\alpha)$, can now be defined as

$$\begin{aligned}
\widehat{M_h^n e_h^A}(\theta^\alpha) =& \big(\widehat{S_h}(\theta^\alpha)\big)^{\nu_1+\nu_2} \widehat{e}_h^A(\theta^\alpha) - \big(\widehat{S_h}(\theta^\alpha)\big)^{\nu_2} \widehat{P_{nh}^h}(\theta^\alpha) \Big(\widehat{L_{nh}}(n\theta^{\gamma_n^i})\Big)^{-1} \\
& \sum_{\alpha_2\in\alpha_n^i} \widehat{R_h^{nh}}(\theta^{\alpha_2}) \widehat{L_h}(\theta^{\alpha_2}) \big(\widehat{S_h}(\theta^{\alpha_2})\big)^{\nu_1} \widehat{e_h^A}(\theta^{\alpha_2}), \qquad \forall\alpha\in\alpha_n^i,\ i\in s_n.
\end{aligned} \tag{4.49}$$

The expression for the discrete Fourier transform of the error transformation operator M_h^{2g} can be simplified using matrix notation. Define for each group of modes $\alpha_n^i, i\in s_n$ the matrices

$$\widehat{L}_h^n(\theta^{\alpha_n^i}) := \text{bdiag}\,(\widehat{L_h}(\theta^{\alpha_1}),\cdots,\widehat{L_h}(\theta^{\alpha_r}))\in\mathbb{C}^{qr\times qr}, \tag{4.50}$$

$$\widehat{S}_h^n(\theta^{\alpha_n^i}) := \text{bdiag}\,(\widehat{S_h}(\theta^{\alpha_1}),\cdots,\widehat{S_h}(\theta^{\alpha_r}))\in\mathbb{C}^{qr\times qr}, \tag{4.51}$$

$$\widehat{R}_h^{nh}(\theta^{\alpha_n^i}) := (\widehat{R_h^{nh}}(\theta^{\alpha_1}),\cdots,\widehat{R_h^{nh}}(\theta^{\alpha_r}))\in\mathbb{C}^{q\times qr}, \tag{4.52}$$

$$\widehat{P}_{nh}^h(\theta^{\alpha_n^i}) := (\widehat{P_{nh}^h}(\theta^{\alpha_1}),\cdots,\widehat{P_{nh}^h}(\theta^{\alpha_r}))^T\in\mathbb{C}^{qr\times q}, \tag{4.53}$$

$$\widehat{e}_h^n(\theta^{\alpha_n^i}) := (\widehat{e_h^A}(\theta^{\alpha_1}),\cdots,\widehat{e_h^A}(\theta^{\alpha_r}))^T\in\mathbb{C}^{qr\times 1}, \tag{4.54}$$

with $\theta^{\alpha_n^i} := (\theta^{\alpha_1}, \cdots, \theta^{\alpha_r})^T$, $\alpha_1, \cdots, \alpha_r \in \alpha_n^i$, $r := \mathrm{Car}(\alpha_n^i)$, with $\mathrm{Car}(\alpha_n^i)$ the cardinality of the set α_n^i, and bdiag refers to a block diagonal matrix consisting of $q \times q$ blocks with $q \geq 1$. Note, the same ordering of mode indices α_i for each group of modes defined in Section 4.2.1 must be used in all vectors and matrices defined in (4.50)-(4.54). For example, for uniform coarsening we have

$$\widehat{L}_h^{(2,2)}(\theta^{00}, \theta^{11}, \theta^{10}, \theta^{01}) = \mathrm{bdiag}\,(\widehat{L_h}(\theta^{00}), \widehat{L_h}(\theta^{11}), \widehat{L_h}(\theta^{10}), \widehat{L_h}(\theta^{01})),$$

for semi-coarsening in the x_1-direction

$$\begin{aligned}\widehat{L}_h^{(2,1)}(\theta^{00}, \theta^{10}) =& \mathrm{bdiag}\,(\widehat{L_h}(\theta^{00}), \widehat{L_h}(\theta^{10})),\\ \widehat{L}_h^{(2,1)}(\theta^{11}, \theta^{01}) =& \mathrm{bdiag}\,(\widehat{L_h}(\theta^{11}), \widehat{L_h}(\theta^{01})),\end{aligned}$$

and for semi-coarsening in the x_2-direction

$$\begin{aligned}\widehat{L}_h^{(1,2)}(\theta^{00}, \theta^{01}) =& \mathrm{bdiag}\,(\widehat{L_h}(\theta^{00}), \widehat{L_h}(\theta^{01})),\\ \widehat{L}_h^{(1,2)}(\theta^{11}, \theta^{10}) =& \mathrm{bdiag}\,(\widehat{L_h}(\theta^{11}), \widehat{L_h}(\theta^{10})).\end{aligned}$$

If we use (4.50)-(4.54) and consider uniform coarsening then we obtain for all $\alpha \in \alpha^1_{(2,2)}$ the relation

$$\begin{aligned}&\big(\widehat{S_h}(\theta^{\alpha})\big)^{\nu_2}\widehat{P_{2h}^h}(\theta^{\alpha})\left(\widehat{L_{2h}}(2\theta^{00})\right)^{-1}\sum_{\alpha_2\in\alpha^1_{(2,2)}}\widehat{R_h^{2h}}(\theta^{\alpha_2})\widehat{L_h}(\theta^{\alpha_2})\big(\widehat{S_h}(\theta^{\alpha_2})\big)^{\nu_1}\widehat{e_h^A}(\theta^{\alpha_2})\\
&\quad= \begin{pmatrix} \widehat{S_h}(\theta^{00}) & 0 & 0 & 0\\ 0 & \widehat{S_h}(\theta^{11}) & 0 & 0\\ 0 & 0 & \widehat{S_h}(\theta^{10}) & 0\\ 0 & 0 & 0 & \widehat{S_h}(\theta^{01}) \end{pmatrix}^{\nu_2}\begin{pmatrix} \widehat{P_{2h}^h}(\theta^{00})\\ \widehat{P_{2h}^h}(\theta^{11})\\ \widehat{P_{2h}^h}(\theta^{10})\\ \widehat{P_{2h}^h}(\theta^{01}) \end{pmatrix}\\
&\qquad\left(\widehat{L_{2h}}(2\theta^{00})\right)^{-1}\begin{pmatrix} \widehat{R_h^{2h}}(\theta^{00}) & \widehat{R_h^{2h}}(\theta^{11}) & \widehat{R_h^{2h}}(\theta^{10}) & \widehat{R_h^{2h}}(\theta^{01}) \end{pmatrix}\\
&\qquad\begin{pmatrix} \widehat{L_h}(\theta^{00}) & 0 & 0 & 0\\ 0 & \widehat{L_h}(\theta^{11}) & 0 & 0\\ 0 & 0 & \widehat{L_h}(\theta^{10}) & 0\\ 0 & 0 & 0 & \widehat{L_h}(\theta^{01}) \end{pmatrix}\\
&\qquad\begin{pmatrix} \widehat{S_h}(\theta^{00}) & 0 & 0 & 0\\ 0 & \widehat{S_h}(\theta^{11}) & 0 & 0\\ 0 & 0 & \widehat{S_h}(\theta^{10}) & 0\\ 0 & 0 & 0 & \widehat{S_h}(\theta^{01}) \end{pmatrix}^{\nu_1}\begin{pmatrix} \widehat{e_h^A}(\theta^{00})\\ \widehat{e_h^A}(\theta^{11})\\ \widehat{e_h^A}(\theta^{10})\\ \widehat{e_h^A}(\theta^{01}) \end{pmatrix}\\
&= \big(\widehat{S}_h^{(2,2)}(\theta^{\alpha^1_{(2,2)}})\big)^{\nu_2}\widehat{P}_{2h}^h(\theta^{\alpha^1_{(2,2)}})\left(\widehat{L_{2h}}(2\theta^{00})\right)^{-1}\widehat{R}_h^{2h}(\theta^{\alpha^1_{(2,2)}})\widehat{L}_h^{(2,2)}(\theta^{\alpha^1_{(2,2)}})\\
&\quad\big(\widehat{S}_h^{(2,2)}(\theta^{\alpha^1_{(2,2)}})\big)^{\nu_1}\widehat{e}_h^{(2,2)}(\theta^{\alpha^1_{(2,2)}}).\end{aligned}$$

Analogously, for semi-coarsening in the x_1-direction we obtain for all $\alpha \in \alpha^1_{(2,1)}$ the relation

$$\begin{aligned}
&\big(\widehat{S_h}(\theta^{\alpha})\big)^{\nu_2} \widehat{P^{(h_1,h_2)}_{(2h_1,h_2)}}(\theta^{\alpha}) \Big(\widehat{L_{(2h_1,h_2)}}\big(2(\theta^{00})_1,(\theta^{00})_2\big)\Big)^{-1} \sum_{\alpha_2 \in \alpha^1_{(2,2)}} \widehat{R^{(2h_1,h_2)}_{(h_1,h_2)}}(\theta^{\alpha_2}) \widehat{L_h}(\theta^{\alpha_2}) \\
&\qquad \big(\widehat{S_h}(\theta^{\alpha_2})\big)^{\nu_1} \widehat{e^A_h}(\theta^{\alpha_2}) = \\
&\qquad = \begin{pmatrix} \widehat{S_h}(\theta^{00}) & 0 \\ 0 & \widehat{S_h}(\theta^{10}) \end{pmatrix}^{\nu_2} \begin{pmatrix} \widehat{P^{(h_1,h_2)}_{(2h_1,h_2)}}(\theta^{00}) \\ \widehat{P^{(h_1,h_2)}_{(2h_1,h_2)}}(\theta^{10}) \end{pmatrix} \\
&\qquad \Big(\widehat{L_{(2h_1,h_2)}}\big(2(\theta^{00})_1,(\theta^{00})_2\big)\Big)^{-1} \begin{pmatrix} \widehat{R^{(2h_1,h_2)}_{(h_1,h_2)}}(\theta^{00}) & \widehat{R^{(2h_1,h_2)}_{(h_1,h_2)}}(\theta^{10}) \end{pmatrix} \\
&\qquad \begin{pmatrix} \widehat{L_h}(\theta^{00}) & 0 \\ 0 & \widehat{L_h}(\theta^{10}) \end{pmatrix} \begin{pmatrix} \widehat{S_h}(\theta^{00}) & 0 \\ 0 & \widehat{S_h}(\theta^{10}) \end{pmatrix}^{\nu_1} \begin{pmatrix} \widehat{e^A_h}(\theta^{00}) \\ \widehat{e^A_h}(\theta^{10}) \end{pmatrix} \\
&\qquad = \big(\widehat{S}^{(2,1)}_h(\theta^{\alpha^1_{(2,1)}})\big)^{\nu_2} \widehat{P}^h_{(2h_1,h_2)}(\theta^{\alpha^1_{(2,1)}}) \Big(\widehat{L_{(2h_1,h_2)}}\big((2,1)\theta^{\gamma^1_{(2,1)}}\big)\Big)^{-1} \\
&\qquad \widehat{R}^{(2h_1,h_2)}_h(\theta^{\alpha^1_{(2,1)}}) \widehat{L}^{(2,1)}_h(\theta^{\alpha^1_{(2,1)}}) \big(\widehat{S}^{(2,1)}_h(\theta^{\alpha^1_{(2,1)}})\big)^{\nu_1} \widehat{e}^{(2,1)}_h(\theta^{\alpha^1_{(2,1)}}).
\end{aligned}$$

with a similar result for $\alpha \in \alpha^2_{(2,1)}$ and also for semi-coarsening in the x_2-direction.

For each group of modes α^i_n, $i \in s_n$ the discrete Fourier transform of the two-level multigrid error transformation operator then is equal to

$$\begin{aligned}
\widehat{M}^n_h(\theta^{\alpha^i_n}) = \big(\widehat{S}^{n,i}_h(\theta^{\alpha^i_n})\big)^{\nu_2} \Big(I^{qr} - \widehat{P}^h_{nh}(\theta^{\alpha^i_n}) \Big(\widehat{L_{nh}}(n\theta^{\gamma^i_n})\Big)^{-1} \widehat{R}^{nh}_h(\theta^{\alpha^i_n}) \widehat{L}^n_h(\theta^{\alpha^i_n})\Big) \\
\big(\widehat{S}^n_h(\theta^{\alpha^i_n})\big)^{\nu_1} \in \mathbb{C}^{qr \times qr},
\end{aligned} \tag{4.55}$$

with I^{qr} the $qr \times qr$ identity matrix and $r = \mathrm{Car}(\alpha^i_n)$.

The two-level error transformation operator is now obtained by combining the contributions from the different groups of modes α^i_n, $i \in s_n$. The multigrid error transformation operator for uniform coarsening then is equal to

$$\widehat{M}^{(2,2)}_h(\theta^{\alpha_{(2,2)}}) = \widehat{M}^{(2,2)}_h(\theta^{\alpha^1_{(2,2)}}) \in \mathbb{C}^{4q \times 4q},$$

with $\theta^{\alpha_{(2,2)}} = \theta^{\alpha^1_{(2,2)}} = (\theta^{00}, \theta^{11}, \theta^{10}, \theta^{01})^T$. The error after one two-level multigrid cycle with uniform coarsening can now be expressed as

$$\widehat{e^D_h}(\theta^{\alpha_{(2,2)}}) = \widehat{M}^{(2,2)}_h(\theta^{\alpha_{(2,2)}}) \widehat{e^A_h}(\theta^{\alpha_{(2,2)}}),$$

with $\widehat{e^{A,D}_h}(\theta^{\alpha_{(2,2)}}) = (\widehat{e^{A,D}_h}(\theta^{00}), \widehat{e^{A,D}_h}(\theta^{11}), \widehat{e^{A,D}_h}(\theta^{10}), \widehat{e^{A,D}_h}(\theta^{01}))^T$.
The multigrid error transformation operator for semi-coarsening in the x_1-direction is

$$\widehat{M}^{(2,1)}_h(\theta^{\alpha_{(2,1)}}) = \begin{pmatrix} \widehat{M}^{(2,1)}_h(\theta^{\alpha^1_{(2,1)}}) & 0 \\ 0 & \widehat{M}^{(2,1)}_h(\theta^{\alpha^2_{(2,1)}}) \end{pmatrix} \in \mathbb{C}^{4q \times 4q},$$

with $\theta^{\alpha_{(2,1)}} = (\theta^{\alpha^1_{(2,1)}}, \theta^{\alpha^2_{(2,1)}})^T$, $\theta^{\alpha^1_{(2,1)}} = (\theta^{00}, \theta^{10})^T$ and $\theta^{\alpha^2_{(2,1)}} = (\theta^{11}, \theta^{01})^T$. Note, however, that the error vectors $\widehat{e^{A,D}_h}(\theta^{\alpha_{(2,1)}}) = \big(\widehat{e^{A,D}_h}(\theta^{00}), \widehat{e^{A,D}_h}(\theta^{10}), \widehat{e^{A,D}_h}(\theta^{11}), \widehat{e^{A,D}_h}(\theta^{01})\big)^T$ have a

different ordering than the error components for uniform coarsening, $\widehat{e_h^{A,D}}(\theta^{\alpha_{(2,2)}})$. The ordering of the components of the error vectors is not important for the computation of the operator norms and the spectral radius of the error transformation operator, which are discussed in Chapter 5. For the coupling of different multigrid algorithms, such as uniform and semi-coarsening, it is, however, essential that the same ordering of the components of the error vectors is used. This can be easily accomplished using the permutation matrix $P_h^{(2,1)} \in \mathbb{R}^{4q\times 4q}$, which reorders the vector $\widehat{e_h^{A,D}}(\theta^{\alpha_{(2,1)}})$ to $\widehat{e_h^{A,D}}(\theta^{\alpha_{(2,2)}})$ and is defined as

$$P_h^{(2,1)} = \begin{pmatrix} I^q & 0 & 0 & 0 \\ 0 & 0 & I^q & 0 \\ 0 & I^q & 0 & 0 \\ 0 & 0 & 0 & I^q \end{pmatrix}.$$

The error after one two-level multigrid cycle with semi-coarsening in the x_2-direction can now be expressed as

$$\widehat{e_h^D}(\theta^{\alpha_{(2,2)}}) = \left(P_h^{(2,1)}\right)^{-1} \widehat{M}_h^{(2,1)}(\theta^{\alpha_{(2,1)}}) P_h^{(2,1)} \widehat{e_h^A}(\theta^{\alpha_{(2,2)}}).$$

Finally, the multigrid error transformation operator for semi-coarsening in the x_2-direction is

$$\widehat{M}_h^{(1,2)}(\theta^{\alpha_{(1,2)}}) = \begin{pmatrix} \widehat{M}_h^{(1,2)}(\theta^{\alpha^1_{(1,2)}}) & 0 \\ 0 & \widehat{M}_h^{(1,2)}(\theta^{\alpha^2_{(1,2)}}) \end{pmatrix} \in \mathbb{C}^{4q\times 4q},$$

with $\theta^{\alpha_{(1,2)}} = (\theta^{\alpha^1_{(1,2)}}, \theta^{\alpha^2_{(1,2)}})^T$, $\theta^{\alpha^1_{(1,2)}} = (\theta^{00}, \theta^{01})^T$ and $\theta^{\alpha^2_{(1,2)}} = (\theta^{11}, \theta^{10})^T$. The permutation matrix $P_h^{(1,2)} \in \mathbb{R}^{4q\times 4q}$ for semi-coarsening in the x_2-direction is defined as

$$P_h^{(1,2)} = \begin{pmatrix} I^q & 0 & 0 & 0 \\ 0 & 0 & 0 & I^q \\ 0 & I^q & 0 & 0 \\ 0 & 0 & I^q & 0 \end{pmatrix}.$$

The error after one two-level multigrid cycle with semi-coarsening in the x_2-direction can now be expressed as

$$\widehat{e_h^D}(\theta^{\alpha_{(2,2)}}) = \left(P_h^{(1,2)}\right)^{-1} \widehat{M}_h^{(1,2)}(\theta^{\alpha_{(1,2)}}) P_h^{(1,2)} \widehat{e_h^A}(\theta^{\alpha_{(2,2)}}).$$

4.4 Three-grid Fourier analysis

In three-level analysis the Fourier symbols $\widehat{L_h}(\theta)$, $\widehat{L_{nh}}(n\theta)$ and $\widehat{L_{mh}}(m\theta)$, $n \in \{(2,2),(2,1),(1,2)\}$, $m \in \{(4,4),(4,1),(1,4)\}$, can be zero for certain values of θ. The frequencies of these Fourier harmonics are removed from $\mathcal{F}_h^3(\theta)$ through the definition of

$$\mathcal{F}_h^{3g} := \{\mathcal{F}_h^3(\theta_\beta^\alpha) \mid \theta \in \Pi_{(4,4)} \backslash \Psi_{n,m}, \forall \alpha \in \alpha_n^i, \forall \beta \in \beta_n^j,\ i,j \in s_n\},$$

with

$$\Psi_{n,m} := \big\{\theta \in \Pi_{(4,4)} \mid \det(\widehat{L_h}(\theta_\beta^\alpha)) = 0 \text{ or } \det(\widehat{L_{nh}}(n\theta_\beta^{\gamma_n^i})) = 0 \text{ or } \det(\widehat{L_{mh}}(m\theta_{\delta_n^j}^{\gamma_n^i})) = 0, \\ \forall \alpha \in \alpha_n^i, \forall \beta \in \beta_n^j,\ i,j \in s_n\big\}, \tag{4.56}$$

and $\theta_\beta^\alpha = \theta_\beta^\alpha(\theta)$. The set s_n is defined as $s_n = \{1,2\}$ if $n = (2,1)$ or $(1,2)$ and $s_n = \{1\}$ if $n = (2,2)$. The error transformation resulting from a three-level multigrid cycle is derived in Section 3.1 and equal to

$$e_h^D = M_h^{3g} e_h^A. \tag{4.57}$$

The three-level multigrid error transformation operator is defined as

$$M_h^{3g} = S_h^{\nu_2} \left(I_h - P_{nh}^h \left(I_{nh} - (M_{nh}^m)^\gamma\right) L_{nh}^{-1} R_h^{nh} L_h\right) S_h^{\nu_1} \tag{4.58}$$

with coarse grid correction

$$M_{nh}^m = S_{nh}^{\nu_4} \left(I_{nh} - P_{mh}^{nh} L_{mh}^{-1} R_{nh}^{mh} L_{nh}\right) S_{nh}^{\nu_3}. \tag{4.59}$$

Here L_h, L_{nh} and L_{mh} denote the discrete approximation of the spatial operator L on the meshes G_h, G_{nh} and G_{mh}, respectively, S_h, S_{nh} the multigrid smoothers on the meshes G_h and G_{nh}, respectively, R_h^{nh}, R_{nh}^{mh} the restriction operators, P_{nh}^h, P_{mh}^{nh} the prolongation operators, I_h and I_{nh} the identity operators on G_h and G_{nh}, respectively, ν_1, ν_2 denote the number of pre- and post-smoothing iterations on G_h, ν_3, ν_4 the number of pre- and post-smoothing iterations on G_{nh} and γ the number of applications of the coarse grid correction operator M_{nh}^m.

In order to compute the discrete Fourier transform of M_h^{3g} we first compute the discrete Fourier transform of the two-level operator M_{nh}^m on the mesh G_{nh} using the following steps

1. Using (4.13), (4.14), (4.15) and (4.16) we obtain

$$\begin{aligned}\left(L_{nh} S_{nh}^{\nu_3} e_{nh}^A\right)(x) &= \sum_{i\in s_n} \sum_{j\in s_n} \sum_{\beta\in\beta_n^j} \int_{\theta\in\Pi_{(4,4)}} \widehat{\left(L_{nh} S_{nh}^{\nu_3} e_{nh}^A\right)}(n\theta_\beta^{\gamma_n^i}) e^{\imath n\theta_\beta^{\gamma_n^i}\cdot x/(nh)} d\theta \\ &= \sum_{i\in s_n} \sum_{j\in s_n} \sum_{\beta\in\beta_n^j} \int_{\theta\in\Pi_{(4,4)}} \widehat{L_{nh}}(n\theta_\beta^{\gamma_n^i}) \left(\widehat{S_{nh}}(n\theta_\beta^{\gamma_n^i})\right)^{\nu_3} \widehat{e_{nh}^A}(n\theta_\beta^{\gamma_n^i}) e^{\imath n\theta_\beta^{\gamma_n^i}\cdot x/(nh)} d\theta\end{aligned}$$

   hence

$$\widehat{\left(L_{nh} S_{nh}^{\nu_3} e_{nh}^A\right)}(n\theta_\beta^{\gamma_n^i}) = \widehat{L_{nh}}(n\theta_\beta^{\gamma_n^i}) \left(\widehat{S_{nh}}(n\theta_\beta^{\gamma_n^i})\right)^{\nu_3} \widehat{e_{nh}^A}(n\theta_\beta^{\gamma_n^i}), \quad \forall \beta \in \beta_n^j,\ i,j \in s_n. \tag{4.60}$$

2. Using (4.23), (4.28) and (4.60) we obtain

$$\begin{aligned}\left(R_{nh}^{mh} L_{nh} S_{nh}^{\nu_3} e_{nh}^A\right)(x) &= \sum_{i\in s_n} \sum_{j\in s_n} \int_{\theta\in\Pi_{(4,4)}} \widehat{\left(R_{nh}^{mh} L_{nh} S_{nh}^{\nu_3} e_{nh}^A\right)}(m\theta_{\delta_n^j}^{\gamma_n^i}) e^{\imath m\theta_{\delta_n^j}^{\gamma_n^i}\cdot x/(mh)} d\theta \\ &= \sum_{i\in s_n} \sum_{j\in s_n} \int_{\theta\in\Pi_{(4,4)}} \sum_{\beta_2\in\beta_n^j} \widehat{R_{nh}^{mh}}(n\theta_{\beta_2}^{\gamma_n^i}) \widehat{\left(L_{nh} S_{nh}^{\nu_3} e_{nh}^A\right)}(n\theta_{\beta_2}^{\gamma_n^i}) e^{\imath m\theta_{\delta_n^j}^{\gamma_n^i}\cdot x/(mh)} d\theta \\ &= \sum_{i\in s_n} \sum_{j\in s_n} \int_{\theta\in\Pi_{(4,4)}} \sum_{\beta_2\in\beta_n^j} \widehat{R_{nh}^{mh}}(n\theta_{\beta_2}^{\gamma_n^i}) \widehat{L_{nh}}(n\theta_{\beta_2}^{\gamma_n^i}) \left(\widehat{S_{nh}}(n\theta_{\beta_2}^{\gamma_n^i})\right)^{\nu_3} \\ &\qquad \widehat{e_{nh}^A}(n\theta_{\beta_2}^{\gamma_n^i}) e^{\imath m\theta_{\delta_n^j}^{\gamma_n^i}\cdot x/(mh)} d\theta\end{aligned}$$

   hence

$$\widehat{\left(R_{nh}^{mh} L_{nh} S_{nh}^{\nu_3} e_{nh}^A\right)}(m\theta_{\delta_n^j}^{\gamma_n^i}) = \sum_{\beta_2\in\beta_n^j} \widehat{R_{nh}^{mh}}(n\theta_{\beta_2}^{\gamma_n^i}) \widehat{L_{nh}}(n\theta_{\beta_2}^{\gamma_n^i}) \left(\widehat{S_{nh}}(n\theta_{\beta_2}^{\gamma_n^i})\right)^{\nu_3} \widehat{e_{nh}^A}(n\theta_{\beta_2}^{\gamma_n^i}),$$

$$i,j \in s_n. \tag{4.61}$$

3. Using (4.13), (4.14), and (4.61) we obtain

$$\begin{aligned}
&\big(L_{mh}^{-1}R_{nh}^{mh}L_{nh}S_{nh}^{\nu_3}e_{nh}^{A}\big)(x)\\
&\quad=\sum_{i\in s_n}\sum_{j\in s_n}\int_{\theta\in\Pi_{(4,4)}}\widehat{\big(L_{mh}^{-1}R_{nh}^{mh}L_{nh}S_{nh}^{\nu_3}e_{nh}^{A}\big)}(m\theta_{\delta_n^j}^{\gamma_n^i})e^{\imath m\theta_{\delta_n^j}^{\gamma_n^i}\cdot x/(mh)}d\theta\\
&\quad=\sum_{i\in s_n}\sum_{j\in s_n}\int_{\theta\in\Pi_{(4,4)}}\Big(\widehat{L_{mh}}(m\theta_{\delta_n^j}^{\gamma_n^i})\Big)^{-1}\widehat{\big(R_{nh}^{mh}L_{nh}S_{nh}^{\nu_3}e_{nh}^{A}\big)}(m\theta_{\delta_n^j}^{\gamma_n^i})e^{\imath m\theta_{\delta_n^j}^{\gamma_n^i}\cdot x/(mh)}d\theta\\
&\quad=\sum_{i\in s_n}\sum_{j\in s_n}\int_{\theta\in\Pi_{(4,4)}}\Big(\widehat{L_{mh}}(m\theta_{\delta_n^j}^{\gamma_n^i})\Big)^{-1}\sum_{\beta_2\in\beta_n^j}\widehat{R_{nh}^{mh}}(n\theta_{\beta_2}^{\gamma_n^i})\widehat{L_{nh}}(n\theta_{\beta_2}^{\gamma_n^i})\big(\widehat{S_{nh}}(n\theta_{\beta_2}^{\gamma_n^i})\big)^{\nu_3}\\
&\qquad\qquad\widehat{e_{nh}^{A}}(n\theta_{\beta_2}^{\gamma_n^i})e^{\imath m\theta_{\delta_n^j}^{\gamma_n^i}\cdot x/(mh)}d\theta
\end{aligned}$$

hence

$$\begin{aligned}
\big(L_{mh}^{-1}R_{nh}^{mh}L_{nh}S_{nh}^{\nu_3}e_{nh}^{A}\big)\widehat{\ }(m\theta_{\delta_n^j}^{\gamma_n^i})&=\Big(\widehat{L_{mh}}(m\theta_{\delta_n^j}^{\gamma_n^i})\Big)^{-1}\sum_{\beta_2\in\beta_n^j}\widehat{R_{nh}^{mh}}(n\theta_{\beta_2}^{\gamma_n^i})\widehat{L_{nh}}(n\theta_{\beta_2}^{\gamma_n^i})\\
&\qquad\big(\widehat{S_{nh}}(n\theta_{\beta_2}^{\gamma_n^i})\big)^{\nu_3}\widehat{e_{nh}^{A}}(n\theta_{\beta_2}^{\gamma_n^i}),\qquad i,j\in s_n.
\end{aligned}\tag{4.62}$$

4. Using (4.34), (4.35) and (4.62) we obtain

$$\begin{aligned}
&\Big(P_{mh}^{nh}L_{mh}^{-1}R_{nh}^{mh}L_{nh}S_{nh}^{\nu_3}e_{nh}^{A}\Big)(x)=\\
&\quad=\sum_{i\in s_n}\sum_{j\in s_n}\sum_{\beta\in\beta_n^j}\int_{\theta\in\Pi_{(4,4)}}\widehat{\big(P_{mh}^{nh}L_{mh}^{-1}R_{nh}^{mh}L_{nh}S_{nh}^{\nu_3}e_{nh}^{A}\big)}(n\theta_{\beta}^{\gamma_n^i})e^{\imath n\theta_{\beta}^{\gamma_n^i}\cdot x/(nh)}d\theta\\
&\quad=\sum_{i\in s_n}\sum_{j\in s_n}\sum_{\beta\in\beta_n^j}\int_{\theta\in\Pi_{(4,4)}}\widehat{P_{mh}^{nh}}(n\theta_{\beta}^{\gamma_n^i})\,\widehat{\big(L_{mh}^{-1}R_{nh}^{mh}L_{nh}S_{nh}^{\nu_3}e_{nh}^{A}\big)}(m\theta_{\delta_n^j}^{\gamma_n^i})e^{\imath n\theta_{\beta}^{\gamma_n^i}\cdot x/(nh)}d\theta\\
&\quad=\sum_{i\in s_n}\sum_{j\in s_n}\sum_{\beta\in\beta_n^j}\int_{\theta\in\Pi_{(4,4)}}\widehat{P_{mh}^{nh}}(n\theta_{\beta}^{\gamma_n^i})\Big(\widehat{L_{mh}}(m\theta_{\delta_n^j}^{\gamma_n^i})\Big)^{-1}\sum_{\beta_2\in\beta_n^j}\widehat{R_{nh}^{mh}}(n\theta_{\beta_2}^{\gamma_n^i})\\
&\qquad\qquad\widehat{L_{nh}}(n\theta_{\beta_2}^{\gamma_n^i})\big(\widehat{S_{nh}}(n\theta_{\beta_2}^{\gamma_n^i})\big)^{\nu_3}\widehat{e_{nh}^{A}}(n\theta_{\beta_2}^{\gamma_n^i})e^{\imath n\theta_{\beta}^{\gamma_n^i}\cdot x/(nh)}d\theta
\end{aligned}$$

hence

$$\begin{aligned}
\big(P_{mh}^{nh}L_{mh}^{-1}R_{nh}^{mh}L_{nh}S_{nh}^{\nu_3}e_{nh}^{A}\big)\widehat{\ }(n\theta_{\beta}^{\gamma_n^i})&=\widehat{P_{mh}^{nh}}(n\theta_{\beta}^{\gamma_n^i})\Big(\widehat{L_{mh}}(m\theta_{\delta_n^j}^{\gamma_n^i})\Big)^{-1}\sum_{\beta_2\in\beta_n^j}\widehat{R_{nh}^{mh}}(n\theta_{\beta_2}^{\gamma_n^i})\\
&\qquad\widehat{L_{nh}}(n\theta_{\beta_2}^{\gamma_n^i})\big(\widehat{S_{nh}}(n\theta_{\beta_2}^{\gamma_n^i})\big)^{\nu_3}\widehat{e_{nh}^{A}}(n\theta_{\beta_2}^{\gamma_n^i}),\qquad\forall\beta\in\beta_n^j,\ i,j\in s_n.
\end{aligned}\tag{4.63}$$

5. Using (4.15), (4.16) and (4.63) we obtain

$$\begin{aligned}
&\left(S_{nh}^{\nu_4}P_{mh}^{nh}L_{mh}^{-1}R_{nh}^{mh}L_{nh}S_{nh}^{\nu_3}e_{nh}^{A}\right)(x) = \\
&= \sum_{i\in s_n}\sum_{j\in s_n}\sum_{\beta\in\beta_n^j}\int_{\theta\in\Pi_{(4,4)}} \widehat{\left(S_{nh}^{\nu_4}P_{mh}^{nh}L_{mh}^{-1}R_{nh}^{mh}L_{nh}S_{nh}^{\nu_3}e_{nh}^{A}\right)}(n\theta_\beta^{\gamma_i^n})e^{\imath n\theta_\beta^{\gamma_i^n}\cdot x/(nh)}d\theta \\
&= \sum_{i\in s_n}\sum_{j\in s_n}\sum_{\beta\in\beta_n^j}\int_{\theta\in\Pi_{(4,4)}} (\widehat{S_{nh}}(n\theta_\beta^{\gamma_i^n}))^{\nu_4}\widehat{\left(P_{mh}^{nh}L_{mh}^{-1}R_{nh}^{mh}L_{nh}S_{nh}^{\nu_3}e_{nh}^{A}\right)}(n\theta_\beta^{\gamma_i^n})e^{\imath n\theta_\beta^{\gamma_i^n}\cdot x/(nh)}d\theta \\
&= \sum_{i\in s_n}\sum_{j\in s_n}\sum_{\beta\in\beta_n^j}\int_{\theta\in\Pi_{(4,4)}} \left(\widehat{S_{nh}}(n\theta_\beta^{\gamma_i^n})\right)^{\nu_4}\widehat{P_{mh}^{nh}}(n\theta_\beta^{\gamma_i^n})\left(\widehat{L_{mh}}(m\theta_{\delta_n^j}^{\gamma_i^n})\right)^{-1}\sum_{\beta_2\in\beta_n^j}\widehat{R_{nh}^{mh}}(n\theta_{\beta_2}^{\gamma_i^n}) \\
&\qquad\qquad \widehat{L_{nh}}(n\theta_{\beta_2}^{\gamma_i^n})\left(\widehat{S_{nh}}(n\theta_{\beta_2}^{\gamma_i^n})\right)^{\nu_3}\widehat{e_{nh}^{A}}(n\theta_{\beta_2}^{\gamma_i^n})e^{\imath n\theta_\beta^{\gamma_i^n}\cdot x/(nh)}d\theta
\end{aligned}$$

hence

$$\begin{aligned}
\widehat{\left(S_{nh}^{\nu_4}P_{mh}^{nh}L_{mh}^{-1}R_{nh}^{mh}L_{nh}S_{nh}^{\nu_3}e_{nh}^{A}\right)}(n\theta_\beta^{\gamma_i^n}) &= (\widehat{S_{nh}}(n\theta_\beta^{\gamma_i^n}))^{\nu_4}\widehat{P_{mh}^{nh}}(n\theta_\beta^{\gamma_i^n})\left(\widehat{L_{mh}}(m\theta_{\delta_n^j}^{\gamma_i^n})\right)^{-1} \\
&\sum_{\beta_2\in\beta_n^j}\widehat{R_{nh}^{mh}}(n\theta_{\beta_2}^{\gamma_i^n})\widehat{L_{nh}}(n\theta_{\beta_2}^{\gamma_i^n})\left(\widehat{S_{nh}}(n\theta_{\beta_2}^{\gamma_i^n})\right)^{\nu_3}\widehat{e_{nh}^{A}}(n\theta_{\beta_2}^{\gamma_i^n}), \quad \forall\beta\in\beta_n^j,\ i,j\in s_n.
\end{aligned}$$

Using (4.59) the error in the multigrid algorithm at the mesh G_{nh} can now be expressed as

$$\begin{aligned}
e_{nh}^{D}(x) &= \sum_{i\in s_n}\sum_{j\in s_n}\sum_{\beta\in\beta_n^j}\int_{\theta\in\Pi_{(4,4)}}\widehat{\left(S_{nh}^{\nu_4}(I_{nh}-P_{mh}^{nh}L_{mh}^{-1}R_{nh}^{mh}L_{nh})S_{nh}^{\nu_3}e_{nh}^{A}\right)}(n\theta_\beta^{\gamma_i^n}) \\
&\qquad\qquad e^{\imath n\theta_\beta^{\gamma_i^n}\cdot x/(nh)}d\theta \\
&= \sum_{i\in s_n}\sum_{j\in s_n}\sum_{\beta\in\beta_n^j}\int_{\theta\in\Pi_{(4,4)}}\Big(\left(\widehat{S_{nh}}(n\theta_\beta^{\gamma_i^n})\right)^{\nu_3+\nu_4}\widehat{e_{nh}^{A}}(n\theta_\beta^{\gamma_i^n}) - \left(\widehat{S_{nh}}(n\theta_\beta^{\gamma_i^n})\right)^{\nu_4}\widehat{P_{mh}^{nh}}(n\theta_\beta^{\gamma_i^n}) \\
&\qquad\qquad \left(\widehat{L_{mh}}(m\theta_{\delta_n^j}^{\gamma_i^n})\right)^{-1}\sum_{\beta_2\in\beta_n^j}\widehat{R_{nh}^{mh}}(n\theta_{\beta_2}^{\gamma_i^n})\widehat{L_{nh}}(n\theta_{\beta_2}^{\gamma_i^n})\left(\widehat{S_{nh}}(n\theta_{\beta_2}^{\gamma_i^n})\right)^{\nu_3} \\
&\qquad\qquad \widehat{e_{nh}^{A}}(n\theta_{\beta_2}^{\gamma_i^n})\Big)e^{\imath n\theta_\beta^{\gamma_i^n}\cdot x/(nh)}d\theta.
\end{aligned}$$

The discrete Fourier transform of $M_{nh}^{m}e_{nh}^{A}$ is thus equal to

$$\begin{aligned}
\widehat{M_{nh}^{m}e_{nh}^{A}}(n\theta_\beta^{\gamma_i^n}) &= \left(\widehat{S_{nh}}(n\theta_\beta^{\gamma_i^n})\right)^{\nu_3+\nu_4}\widehat{e_{nh}^{A}}(n\theta_\beta^{\gamma_i^n}) - \left(\widehat{S_{nh}}(n\theta_\beta^{\gamma_i^n})\right)^{\nu_4}\widehat{P_{mh}^{nh}}(n\theta_\beta^{\gamma_i^n})\left(\widehat{L_{mh}}(m\theta_{\delta_n^j}^{\gamma_i^n})\right)^{-1} \\
&\sum_{\beta_2\in\beta_n^j}\widehat{R_{nh}^{mh}}(n\theta_{\beta_2}^{\gamma_i^n})\widehat{L_{nh}}(n\theta_{\beta_2}^{\gamma_i^n})\left(\widehat{S_{nh}}(n\theta_{\beta_2}^{\gamma_i^n})\right)^{\nu_3}\widehat{e_{nh}^{A}}(n\theta_{\beta_2}^{\gamma_i^n}), \qquad \forall\beta\in\beta_n^j,\ i,j\in s_n.
\end{aligned} \tag{4.64}$$

We can also obtain this result directly using the fact that the modes $\phi_h(\theta_\beta^\alpha, x)$ on the mesh G_h alias to $\phi_{nh}(n\theta_\beta^{\gamma_i^n}, x)$, with $\alpha\in\alpha_n^i, \beta\in\beta_n^j,\ i,j\in s_n$, on the mesh G_{nh}. If we use the discrete Fourier transform of the two-level error transformation operator (4.49) and replace θ^α with $n\theta_\beta^{\gamma_i^n}$, θ^{α_2} with $n\theta_{\beta_2}^{\gamma_i^n}$, $n\theta^{\gamma_i^n}$ with $m\theta_{\delta_n^j}^{\gamma_i^n}$, nh with mh, and h with nh then we also obtain (4.64).

Define now the coarse grid correction operator

$$\widetilde{M}_{nh}^m = I_{nh} - (M_{nh}^m)^\gamma. \tag{4.65}$$

If we introduce the matrices $\left(\widehat{\widetilde{M}_{nh}^m}\right)_\beta \in \mathbb{C}^{qr\times qr}$, with $\beta \in \beta_n^j$, $i,j \in s_n$ and $r = \mathrm{Car}(\alpha_n^i) = \mathrm{Car}(\beta_n^j)$, we can write the discrete Fourier transform of $\widetilde{M}_{nh}^m e_{nh}^A$ as

$$\widehat{\widetilde{M}_{nh}^m e_{nh}^A}(n\theta_\beta^{\gamma_n^i}) = \sum_{\beta_2\in\beta_n^j} \left(\widehat{\widetilde{M}_{nh}^m}\right)_{\beta_2}(n\theta_\beta^{\gamma_n^i};\gamma)\widehat{e_{nh}^A}(n\theta_{\beta_2}^{\gamma_n^i}), \qquad \forall \beta\in\beta_n^j,\ i,j\in s_n, \tag{4.66}$$

where an explicit expression of $\left(\widehat{\widetilde{M}_{nh}^n}\right)_{\beta_2}(n\theta_\beta^{\gamma_n^i};\gamma)$ can be obtained using (4.64). For instance, if $\gamma = 1$ then

$$\begin{aligned}
\left(\widehat{\widetilde{M}_{nh}^m}\right)_{\beta_2}(n\theta_\beta^{\gamma_n^i};1) =& I^{qr} - \left(\widehat{S_{nh}}(n\theta_\beta^{\gamma_n^i})\right)^{\nu_3+\nu_4} + \left(\widehat{S_{nh}}(n\theta_\beta^{\gamma_n^i})\right)^{\nu_4}\widehat{P_{mh}^{nh}}(n\theta_\beta^{\gamma_n^i})\left(\widehat{L_{mh}}(m\theta_{\delta_n^j}^{\gamma_n^i})\right)^{-1}\\
&\widehat{R_{nh}^{mh}}(n\theta_\beta^{\gamma_n^i})\widehat{L_{nh}}(n\theta_\beta^{\gamma_n^i})\left(\widehat{S_{nh}}(n\theta_\beta^{\gamma_n^i})\right)^{\nu_3}, && \text{if } \beta_2=\beta\\
=&\left(\widehat{S_{nh}}(n\theta_\beta^{\gamma_n^i})\right)^{\nu_4}\widehat{P_{mh}^{nh}}(n\theta_\beta^{\gamma_n^i})\left(\widehat{L_{mh}}(m\theta_{\delta_n^j}^{\gamma_n^i})\right)^{-1}\widehat{R_{nh}^{mh}}(n\theta_{\beta_2}^{\gamma_n^i})\\
&\widehat{L_{nh}}(n\theta_{\beta_2}^{\gamma_n^i})\left(\widehat{S_{nh}}(n\theta_{\beta_2}^{\gamma_n^i})\right)^{\nu_3}, && \text{if } \beta_2\neq\beta.
\end{aligned}$$

Next, we compute the Fourier symbol of the operator M_h^{3g}. We first derive for each group of modes α_n^i, β_n^j, $i,j\in s_n$ the discrete Fourier transform of $S_h^{\nu_2}P_{nh}^h\widetilde{M}_{nh}^m L_{nh}^{-1}R_h^{nh}L_hS_h^{\nu_1}e_h^A$ using the following steps:

1. Using (4.13), (4.14), (4.15) and (4.16) we obtain

$$\begin{aligned}
\left(L_hS_h^{\nu_1}e_h^A\right)(x) &= \sum_{i\in s_n}\sum_{j\in s_n}\sum_{\alpha\in\alpha_n^i}\sum_{\beta\in\beta_n^j}\int_{\theta\in\Pi_{(4,4)}}\widehat{\left(L_hS_h^{\nu_1}e_h^A\right)}(\theta_\beta^\alpha)e^{\imath\theta_\beta^\alpha\cdot x/h}d\theta\\
&= \sum_{i\in s_n}\sum_{j\in s_n}\sum_{\alpha\in\alpha_n^i}\sum_{\beta\in\beta_n^j}\int_{\theta\in\Pi_{(4,4)}}\widehat{L_h}(\theta_\beta^\alpha)\left(\widehat{S_h}(\theta_\beta^\alpha)\right)^{\nu_1}\widehat{e_h^A}(\theta_\beta^\alpha)e^{\imath\theta_\beta^\alpha\cdot x/h}d\theta
\end{aligned}$$

   hence

$$\widehat{\left(L_hS_h^{\nu_1}e_h^A\right)}(\theta_\beta^\alpha) = \widehat{L_h}(\theta_\beta^\alpha)\left(\widehat{S_h}(\theta_\beta^\alpha)\right)^{\nu_1}\widehat{e_h^A}(\theta_\beta^\alpha), \quad \forall\alpha\in\alpha_n^i,\forall\beta\in\beta_n^j,\ i,j\in s_n. \tag{4.67}$$

2. Using (4.17), (4.22) and (4.67) we obtain

$$\begin{aligned}
\left(R_h^{nh}L_hS_h^{\nu_1}e_h^A\right)(x) &= \sum_{i\in s_n}\sum_{j\in s_n}\sum_{\beta\in\beta_n^j}\int_{\theta\in\Pi_{(4,4)}}\widehat{\left(R_h^{nh}L_hS_h^{\nu_1}e_h^A\right)}(n\theta_\beta^{\gamma_n^i})e^{\imath n\theta_\beta^{\gamma_n^i}\cdot x/(nh)}d\theta\\
&= \sum_{i\in s_n}\sum_{j\in s_n}\sum_{\beta\in\beta_n^j}\int_{\theta\in\Pi_{(4,4)}}\sum_{\alpha_2\in\alpha_n^i}\widehat{R_h^{nh}}(\theta_\beta^{\alpha_2})\widehat{\left(L_hS_h^{\nu_1}e_h^A\right)}(\theta_\beta^{\alpha_2})e^{\imath n\theta_\beta^{\gamma_n^i}\cdot x/(nh)}d\theta\\
&= \sum_{i\in s_n}\sum_{j\in s_n}\sum_{\beta\in\beta_n^j}\int_{\theta\in\Pi_{(4,4)}}\sum_{\alpha_2\in\alpha_n^i}\widehat{R_h^{nh}}(\theta_\beta^{\alpha_2})\widehat{L_h}(\theta_\beta^{\alpha_2})\left(\widehat{S_h}(\theta_\beta^{\alpha_2})\right)^{\nu_1}\widehat{e_h^A}(\theta_\beta^{\alpha_2})\\
&\qquad\qquad e^{\imath n\theta_\beta^{\gamma_n^i}\cdot x/(nh)}d\theta
\end{aligned}$$

hence

$$\big(R_h^{nh}L_hS_h^{\nu_1}e_h^A\big)\widehat{\ }(n\theta_\beta^{\gamma_n^i}) = \sum_{\alpha_2\in\alpha_n^i}\widehat{R_h^{nh}}(\theta_\beta^{\alpha_2})\widehat{L_h}(\theta_\beta^{\alpha_2})\big(\widehat{S_h}(\theta_\beta^{\alpha_2})\big)^{\nu_1}\widehat{e_h^A}(\theta_\beta^{\alpha_2}),$$
$$\forall\beta\in\beta_n^j,\ i,j\in s_n. \quad (4.68)$$

3. Using (4.13), (4.14), and (4.68) we obtain

$$\begin{aligned}
&\big(L_{nh}^{-1}R_h^{nh}L_hS_h^{\nu_1}e_h^A\big)(x)\\
&\quad=\sum_{i\in s_n}\sum_{j\in s_n}\sum_{\beta\in\beta_n^j}\int_{\theta\in\Pi_{(4,4)}}\big(L_{nh}^{-1}R_h^{nh}L_hS_h^{\nu_1}e_h^A\big)\widehat{\ }(n\theta_\beta^{\gamma_n^i})e^{\imath n\theta_\beta^{\gamma_n^i}\cdot x/(nh)}d\theta\\
&\quad=\sum_{i\in s_n}\sum_{j\in s_n}\sum_{\beta\in\beta_n^j}\int_{\theta\in\Pi_{(4,4)}}\Big(\widehat{L_{nh}}(n\theta_\beta^{\gamma_n^i})\Big)^{-1}\big(R_h^{nh}L_hS_h^{\nu_1}e_h^A\big)\widehat{\ }(n\theta_\beta^{\gamma_n^i})e^{\imath n\theta_\beta^{\gamma_n^i}\cdot x/(nh)}d\theta\\
&\quad=\sum_{i\in s_n}\sum_{j\in s_n}\sum_{\beta\in\beta_n^j}\int_{\theta\in\Pi_{(4,4)}}\Big(\widehat{L_{nh}}(n\theta_\beta^{\gamma_n^i})\Big)^{-1}\sum_{\alpha_2\in\alpha_n^i}\widehat{R_h^{nh}}(\theta_\beta^{\alpha_2})\widehat{L_h}(\theta_\beta^{\alpha_2})\big(\widehat{S_h}(\theta_\beta^{\alpha_2})\big)^{\nu_1}\\
&\qquad\qquad\widehat{e_h^A}(\theta_\beta^{\alpha_2})e^{\imath n\theta_\beta^{\gamma_n^i}\cdot x/(nh)}d\theta
\end{aligned}$$

hence

$$\big(L_{nh}^{-1}R_h^{nh}L_hS_h^{\nu_1}e_h^A\big)\widehat{\ }(n\theta_\beta^{\gamma_n^i}) = \Big(\widehat{L_{nh}}(n\theta_\beta^{\gamma_n^i})\Big)^{-1}\sum_{\alpha_2\in\alpha_n^i}\widehat{R_h^{nh}}(\theta_\beta^{\alpha_2})\widehat{L_h}(\theta_\beta^{\alpha_2})\big(\widehat{S_h}(\theta_\beta^{\alpha_2})\big)^{\nu_1}$$
$$\widehat{e_h^A}(\theta_\beta^{\alpha_2}),\qquad \forall\beta\in\beta_n^j,\ i,j\in s_n. \quad (4.69)$$

4. Using (4.66) and (4.69) we obtain

$$\begin{aligned}
&\Big(\widetilde{M}_{nh}^mL_{nh}^{-1}R_h^{nh}L_hS_h^{\nu_1}e_h^A\Big)(x) =\\
&=\sum_{i\in s_n}\sum_{j\in s_n}\sum_{\beta\in\beta_n^j}\int_{\theta\in\Pi_{(4,4)}}\big(\widetilde{M}_{nh}^mL_{nh}^{-1}R_h^{nh}L_hS_h^{\nu_1}e_h^A\big)\widehat{\ }(n\theta_\beta^{\gamma_n^i})e^{\imath n\theta_\beta^{\gamma_n^i}\cdot x/(nh)}d\theta =\\
&=\sum_{i\in s_n}\sum_{j\in s_n}\sum_{\beta\in\beta_n^j}\int_{\theta\in\Pi_{(4,4)}}\sum_{\beta_2\in\beta_n^j}\big(\widehat{\widetilde{M}_{nh}^m}\big)_{\beta_2}(n\theta_\beta^{\gamma_n^i};\gamma)\big(L_{nh}^{-1}R_h^{nh}L_hS_h^{\nu_1}e_h^A\big)\widehat{\ }(n\theta_{\beta_2}^{\gamma_n^i})\\
&\qquad\qquad e^{\imath n\theta_\beta^{\gamma_n^i}\cdot x/(nh)}d\theta\\
&=\sum_{i\in s_n}\sum_{j\in s_n}\sum_{\beta\in\beta_n^j}\int_{\theta\in\Pi_{(4,4)}}\sum_{\beta_2\in\beta_n^j}\big(\widehat{\widetilde{M}_{nh}^m}\big)_{\beta_2}(n\theta_\beta^{\gamma_n^i};\gamma)\Big(\widehat{L_{nh}}(n\theta_{\beta_2}^{\gamma_n^i})\Big)^{-1}\sum_{\alpha_2\in\alpha_n^i}\widehat{R_h^{nh}}(\theta_{\beta_2}^{\alpha_2})\\
&\qquad\qquad\widehat{L_h}(\theta_{\beta_2}^{\alpha_2})\big(\widehat{S_h}(\theta_{\beta_2}^{\alpha_2})\big)^{\nu_1}\widehat{e_h^A}(\theta_{\beta_2}^{\alpha_2})e^{\imath n\theta_\beta^{\gamma_n^i}\cdot x/(nh)}d\theta
\end{aligned}$$

hence

$$\big(\widetilde{M}_{nh}^mL_{nh}^{-1}R_h^{nh}L_hS_h^{\nu_1}e_h^A\big)\widehat{\ }(n\theta_\beta^{\gamma_n^i}) = \sum_{\beta_2\in\beta_n^j}\big(\widehat{\widetilde{M}_{nh}^m}\big)_{\beta_2}(n\theta_\beta^{\gamma_n^i};\gamma)\Big(\widehat{L_{nh}}(n\theta_{\beta_2}^{\gamma_n^i})\Big)^{-1}$$
$$\sum_{\alpha_2\in\alpha_n^i}\widehat{R_h^{nh}}(\theta_{\beta_2}^{\alpha_2})\widehat{L_h}(\theta_{\beta_2}^{\alpha_2})\big(\widehat{S_h}(\theta_{\beta_2}^{\alpha_2})\big)^{\nu_1}\widehat{e_h^A}(\theta_{\beta_2}^{\alpha_2}),\quad \forall\beta\in\beta_n^j,\ i,j\in s_n. \quad (4.70)$$

5. Using (4.32), (4.33) and (4.70) we obtain

$$\begin{aligned}
&\left(P_{nh}^{h}\widetilde{M}_{nh}^{m}L_{nh}^{-1}R_{h}^{nh}L_{h}S_{h}^{\nu_1}e_{h}^{A}\right)(x)=\\
&=\sum_{i\in s_n}\sum_{j\in s_n}\sum_{\alpha\in\alpha_n^i}\sum_{\beta\in\beta_n^j}\int_{\theta\in\Pi_{(4,4)}}\widehat{\left(P_{nh}^{h}\widetilde{M}_{nh}^{m}L_{nh}^{-1}R_{h}^{nh}L_{h}S_{h}^{\nu_1}e_{h}^{A}\right)}(\theta_\beta^\alpha)e^{\imath\theta_\beta^\alpha\cdot x/h}d\theta\\
&=\sum_{i\in s_n}\sum_{j\in s_n}\sum_{\alpha\in\alpha_n^i}\sum_{\beta\in\beta_n^j}\int_{\theta\in\Pi_{(4,4)}}\widehat{P_{nh}^{h}}(\theta_\beta^\alpha)\widehat{\left(\widetilde{M}_{nh}^{m}L_{nh}^{-1}R_{h}^{nh}L_{h}S_{h}^{\nu_1}e_{h}^{A}\right)}(n\theta_\beta^{\gamma_n^i})e^{\imath\theta_\beta^\alpha\cdot x/h}d\theta\\
&=\sum_{i\in s_n}\sum_{j\in s_n}\sum_{\alpha\in\alpha_n^i}\sum_{\beta\in\beta_n^j}\int_{\theta\in\Pi_{(4,4)}}\widehat{P_{nh}^{h}}(\theta_\beta^\alpha)\sum_{\beta_2\in\beta_n^j}\widehat{\left(\widetilde{M}_{nh}^{m}\right)}_{\beta_2}(n\theta_\beta^{\gamma_n^i};\gamma)\left(\widehat{L_{nh}}(n\theta_{\beta_2}^{\gamma_n^i})\right)^{-1}\\
&\qquad\sum_{\alpha_2\in\alpha_n^i}\widehat{R_{h}^{nh}}(\theta_{\beta_2}^{\alpha_2})\widehat{L_h}(\theta_{\beta_2}^{\alpha_2})\left(\widehat{S_h}(\theta_{\beta_2}^{\alpha_2})\right)^{\nu_1}\widehat{e_h^A}(\theta_{\beta_2}^{\alpha_2})e^{\imath\theta_\beta^\alpha\cdot x/h}d\theta
\end{aligned}$$

hence

$$\begin{aligned}
\widehat{\left(P_{nh}^{h}\widetilde{M}_{nh}^{m}L_{nh}^{-1}R_{h}^{nh}L_{h}S_{h}^{\nu_1}e_{h}^{A}\right)}(\theta_\beta^\alpha)=&\widehat{P_{nh}^{h}}(\theta_\beta^\alpha)\sum_{\beta_2\in\beta_n^j}\widehat{\left(\widetilde{M}_{nh}^{m}\right)}_{\beta_2}(n\theta_\beta^{\gamma_n^i};\gamma)\left(\widehat{L_{nh}}(n\theta_{\beta_2}^{\gamma_n^i})\right)^{-1}\\
&\sum_{\alpha_2\in\alpha_n^i}\widehat{R_{h}^{nh}}(\theta_{\beta_2}^{\alpha_2})\widehat{L_h}(\theta_{\beta_2}^{\alpha_2})\left(\widehat{S_h}(\theta_{\beta_2}^{\alpha_2})\right)^{\nu_1}\widehat{e_h^A}(\theta_{\beta_2}^{\alpha_2}),\\
&\forall\alpha\in\alpha_n^i,\forall\beta\in\beta_n^j,\ i,j\in s_n.
\end{aligned}\tag{4.71}$$

6. Using (4.15), (4.16) and (4.71) we obtain

$$\begin{aligned}
&\left(S_h^{\nu_2}P_{nh}^{h}\widetilde{M}_{nh}^{m}L_{nh}^{-1}R_{h}^{nh}L_{h}S_{h}^{\nu_1}e_{h}^{A}\right)(x)=\\
&=\sum_{i\in s_n}\sum_{j\in s_n}\sum_{\alpha\in\alpha_n^i}\sum_{\beta\in\beta_n^j}\int_{\theta\in\Pi_{(4,4)}}\widehat{\left(S_h^{\nu_2}P_{nh}^{h}\widetilde{M}_{nh}^{m}L_{nh}^{-1}R_{h}^{nh}L_{h}S_{h}e_{h}^{A}\right)}(\theta_\beta^\alpha)e^{\imath\theta_\beta^\alpha\cdot x/h}d\theta\\
&=\sum_{i\in s_n}\sum_{j\in s_n}\sum_{\alpha\in\alpha_n^i}\sum_{\beta\in\beta_n^j}\int_{\theta\in\Pi_{(4,4)}}(\widehat{S_h}(\theta_\beta^\alpha))^{\nu_2}\widehat{\left(P_{nh}^{h}\widetilde{M}_{nh}^{m}L_{nh}^{-1}R_{h}^{nh}L_{h}S_{h}e_{h}^{A}\right)}(\theta_\beta^\alpha)e^{\imath\theta_\beta^\alpha\cdot x/h}d\theta\\
&=\sum_{i\in s_n}\sum_{j\in s_n}\sum_{\alpha\in\alpha_n^i}\sum_{\beta\in\beta_n^j}\int_{\theta\in\Pi_{(4,4)}}\left(\widehat{S_h}(\theta_\beta^\alpha)\right)^{\nu_2}\widehat{P_{nh}^{h}}(\theta_\beta^\alpha)\sum_{\beta_2\in\beta_n^j}\widehat{\left(\widetilde{M}_{nh}^{m}\right)}_{\beta_2}(n\theta_\beta^{\gamma_n^i};\gamma)\\
&\qquad\left(\widehat{L_{nh}}(n\theta_{\beta_2}^{\gamma_n^i})\right)^{-1}\sum_{\alpha_2\in\alpha_n^i}\widehat{R_{h}^{nh}}(\theta_{\beta_2}^{\alpha_2})\widehat{L_h}(\theta_{\beta_2}^{\alpha_2})\left(\widehat{S_h}(\theta_{\beta_2}^{\alpha_2})\right)^{\nu_1}\\
&\qquad\widehat{e_h^A}(\theta_{\beta_2}^{\alpha_2})e^{\imath\theta_\beta^\alpha\cdot x/h}d\theta
\end{aligned}$$

hence

$$\begin{aligned}
\widehat{\left(S_h^{\nu_2}P_{nh}^{h}\widetilde{M}_{nh}^{m}L_{nh}^{-1}R_{h}^{nh}L_{h}S_{h}^{\nu_1}e_{h}^{A}\right)}(\theta_\beta^\alpha)=&(\widehat{S_h}(\theta_\beta^\alpha))^{\nu_2}\widehat{P_{nh}^{h}}(\theta_\beta^\alpha)\sum_{\beta_2\in\beta_n^j}\widehat{\left(\widetilde{M}_{nh}^{m}\right)}_{\beta_2}(n\theta_\beta^{\gamma_n^i};\gamma)\\
&\left(\widehat{L_{nh}}(n\theta_{\beta_2}^{\gamma_n^i})\right)^{-1}\sum_{\alpha_2\in\alpha_n^i}\widehat{R_{h}^{nh}}(\theta_{\beta_2}^{\alpha_2})\widehat{L_h}(\theta_{\beta_2}^{\alpha_2})\left(\widehat{S_h}(\theta_{\beta_2}^{\alpha_2})\right)^{\nu_1}\widehat{e_h^A}(\theta_{\beta_2}^{\alpha_2}),\\
&\forall\alpha\in\alpha_n^i,\forall\beta\in\beta_n^j,\ i,j\in s_n.
\end{aligned}\tag{4.72}$$

Using (4.58) and (4.72) the error in the three-level multigrid algorithm can now be expressed as

$$\begin{aligned} e_h^D(x) &= \sum_{i\in s_n}\sum_{j\in s_n}\sum_{\alpha\in\alpha_n^i}\sum_{\beta\in\beta_n^j}\int_{\theta\in\Pi_{(4,4)}} \Big(S_h^{\nu_2}(I_h - P_{nh}^h\widetilde{M}_{nh}^m L_{nh}^{-1}R_h^{nh}L_h)S_h^{\nu_1}e_h^A\Big)\widehat{\ }(\theta_\beta^\alpha)\, e^{\imath\theta_\beta^\alpha\cdot x/h}d\theta \\ &= \sum_{i\in s_n}\sum_{j\in s_n}\sum_{\alpha\in\alpha_n^i}\sum_{\beta\in\beta_n^j}\int_{\theta\in\Pi_{(4,4)}} \Big(\big(\widehat{S_h}(\theta_\beta^\alpha)\big)^{\nu_1+\nu_2}\widehat{e_h^A}(\theta_\beta^\alpha) - \big(\widehat{S_h}(\theta_\beta^\alpha)\big)^{\nu_2}\widehat{P_{nh}^h}(\theta_\beta^\alpha) \\ &\qquad \sum_{\beta_2\in\beta_n^j}\big(\widehat{\widetilde{M}_{nh}^m}\big)_{\beta_2}(n\theta_\beta^{\gamma_i^n};\gamma)\Big(\widehat{L_{nh}}(n\theta_{\beta_2}^{\gamma_i^n})\Big)^{-1} \\ &\qquad \sum_{\alpha_2\in\alpha_n^i}\widehat{R_h^{nh}}(\theta_{\beta_2}^{\alpha_2})\widehat{L_h}(\theta_{\beta_2}^{\alpha_2})\big(\widehat{S_h}(\theta_{\beta_2}^{\alpha_2})\big)^{\nu_1}\widehat{e_h^A}(\theta_{\beta_2}^{\alpha_2})\Big)e^{\imath\theta_\beta^\alpha\cdot x/h}d\theta. \end{aligned}$$

The discrete Fourier transform of $M_h^{3g}e_h^A$ is thus equal to

$$\begin{aligned} \widehat{M_h^{3g}e_h^A}(\theta_\beta^\alpha) =& \big(\widehat{S_h}(\theta_\beta^\alpha)\big)^{\nu_1+\nu_2}\widehat{e^A}(\theta_\beta^\alpha) - \big(\widehat{S_h}(\theta_\beta^\alpha)\big)^{\nu_2}\widehat{P_{nh}^h}(\theta_\beta^\alpha)\sum_{\beta_2\in\beta_n^j}\big(\widehat{\widetilde{M}_{nh}^m}\big)_{\beta_2}(n\theta_\beta^{\gamma_i^n};\gamma) \\ &\Big(\widehat{L_{nh}}(n\theta_{\beta_2}^{\gamma_i^n})\Big)^{-1}\sum_{\alpha_2\in\alpha_n^i}\widehat{R_h^{nh}}(\theta_{\beta_2}^{\alpha_2})\widehat{L_h}(\theta_{\beta_2}^{\alpha_2})\big(\widehat{S_h}(\theta_{\beta_2}^{\alpha_2})\big)^{\nu_1}\widehat{e_h^A}(\theta_{\beta_2}^{\alpha_2}), \\ &\forall\alpha\in\alpha_n^i,\forall\beta\in\beta_n^j,\ i,j\in s_n. \end{aligned}$$

The expressions for the discrete Fourier transform of the error transformation operator can be simplified using a matrix representation. On the mesh G_{nh} we introduce the matrices

$$\widehat{L}_{nh}^n(n\theta_{\beta_n^j}^{\gamma_i^n}) = \text{bdiag}\ (\widehat{L_{nh}}(n\theta_{\beta_1}^{\gamma_i^n}),\cdots,\widehat{L_{nh}}(n\theta_{\beta_r}^{\gamma_i^n}))\in\mathbb{C}^{qr\times qr}, \tag{4.73}$$

$$\widehat{S}_{nh}^n(n\theta_{\beta_n^j}^{\gamma_i^n}) = \text{bdiag}\ (\widehat{S_{nh}}(n\theta_{\beta_1}^{\gamma_i^n}),\cdots,\widehat{S_{nh}}(n\theta_{\beta_r}^{\gamma_i^n}))\in\mathbb{C}^{qr\times qr}, \tag{4.74}$$

$$\widehat{R}_{nh}^{mh}(n\theta_{\beta_n^j}^{\gamma_i^n}) = (\widehat{R_{nh}^{mh}}(n\theta_{\beta_1}^{\gamma_i^n}),\cdots,\widehat{R_{nh}^{mh}}(n\theta_{\beta_r}^{\gamma_i^n}))\in\mathbb{C}^{q\times qr}, \tag{4.75}$$

$$\widehat{P}_{mh}^{nh}(n\theta_{\beta_n^j}^{\gamma_i^n}) = (\widehat{P_{mh}^{nh}}(n\theta_{\beta_1}^{\gamma_i^n}),\cdots,\widehat{P_{mh}^{nh}}(n\theta_{\beta_r}^{\gamma_i^n}))^T\in\mathbb{C}^{qr\times q}, \tag{4.76}$$

with $\theta_{\beta_n^j}^{\gamma_i^n} = (\theta_{\beta_1}^{\gamma_i^n},\cdots,\theta_{\beta_r}^{\gamma_i^n})^T$, $\beta_1,\cdots,\beta_r\in\beta_n^j$, $r=\text{Car}(\alpha_n^i)=\text{Car}(\beta_n^j)$, $i,j\in s_n$ and bdiag refers to a block diagonal matrix consisting of $q\times q$ blocks with $q\geq 1$. For each group of modes β_n^j, $j\in s_n$, the discrete Fourier transform of the coarse grid multigrid error transformation operator $\widehat{M}_{nh}^m$ can be directly obtained from (4.64) resulting in

$$\begin{aligned} \widehat{M}_{nh}^m(n\theta_{\beta_n^j}^{\gamma_i^n}) =& \big(\widehat{S}_{nh}^n(n\theta_{\beta_n^j}^{\gamma_i^n})\big)^{\nu_4}\Big(I^{qr} - \widehat{P}_{mh}^{nh}(n\theta_{\beta_n^j}^{\gamma_i^n})\big(\widehat{L_{mh}}(m\theta_{\delta_n^j}^{\gamma_i^n})\big)^{-1}\widehat{R}_{nh}^{mh}(n\theta_{\beta_n^j}^{\gamma_i^n})\widehat{L}_{nh}^n(n\theta_{\beta_n^j}^{\gamma_i^n})\Big) \\ &\big(\widehat{S}_{nh}^n(n\theta_{\beta_n^j}^{\gamma_i^n})\big)^{\nu_3}\in\mathbb{C}^{qr\times qr},\qquad i,j\in s_n, \end{aligned}$$

with $I^{qr}\in\mathbb{R}^{qr\times qr}$ the identity matrix. The matrices representing the discrete Fourier transform of the coarse grid operator (4.65) then are equal to

$$\widehat{\widetilde{M}_{nh}^m}(n\theta_{\beta_n^j}^{\gamma_i^n};\gamma) = I^{qr} - (\widehat{M}_{nh}^m(n\theta_{\beta_n^j}^{\gamma_i^n}))^\gamma\in\mathbb{C}^{qr\times qr}.$$

Next, we introduce the matrices

$$\widetilde{L}_h^n(\theta_{\beta_k}^{\alpha_n^i}) = \mathrm{bdiag}\left(\widehat{L_h}(\theta_{\beta_k}^{\alpha_1}),\cdots,\widehat{L_h}(\theta_{\beta_k}^{\alpha_r})\right) \in \mathbb{C}^{qr\times qr} \tag{4.77}$$

$$\bar{L}_h^n(\theta_{\beta_n^j}^{\alpha_n^i}) = \mathrm{bdiag}\left(\widetilde{L}_h^n(\theta_{\beta_1}^{\alpha_n^i}),\cdots,\widetilde{L}_h^n(\theta_{\beta_r}^{\alpha_n^i})\right) \in \mathbb{C}^{qr^2\times qr^2} \tag{4.78}$$

$$\widetilde{S}_h^n(\theta_{\beta_k}^{\alpha_n^i}) = \mathrm{bdiag}\left(\widehat{S_h}(\theta_{\beta_k}^{\alpha_1}),\cdots,\widehat{S_h}(\theta_{\beta_k}^{\alpha_r})\right) \in \mathbb{C}^{qr\times qr} \tag{4.79}$$

$$\bar{S}_h^n(\theta_{\beta_n^j}^{\alpha_n^i}) = \mathrm{bdiag}\left(\widetilde{S}_h^n(\theta_{\beta_1}^{\alpha_n^i}),\cdots,\widetilde{S}_h^n(\theta_{\beta_r}^{\alpha_n^i})\right) \in \mathbb{C}^{qr^2\times qr^2} \tag{4.80}$$

$$\widetilde{R}_h^{nh}(\theta_{\beta_k}^{\alpha_n^i}) = \left(\widehat{R_h^{nh}}(\theta_{\beta_k}^{\alpha_1}),\cdots,\widehat{R_h^{nh}}(\theta_{\beta_k}^{\alpha_r})\right) \in \mathbb{C}^{q\times qr} \tag{4.81}$$

$$\bar{R}_h^{nh}(\theta_{\beta_n^j}^{\alpha_n^i}) = \mathrm{bdiag}\big(\widetilde{R}_h^{nh}(\theta_{\beta_1}^{\alpha_n^i}),\cdots,\widetilde{R}_h^{nh}(\theta_{\beta_r}^{\alpha_n^i})\big) \in \mathbb{C}^{qr\times qr^2} \tag{4.82}$$

$$\widetilde{P}_{nh}^h(\theta_{\beta_k}^{\alpha_n^i}) = \left(\widehat{P_{nh}^h}(\theta_{\beta_k}^{\alpha_1}),\cdots,\widehat{P_{nh}^h}(\theta_{\beta_k}^{\alpha_r})\right)^T \in \mathbb{C}^{qr\times q} \tag{4.83}$$

$$\bar{P}_{nh}^h(\theta_{\beta_n^j}^{\alpha_n^i}) = \mathrm{bdiag}\left(\widetilde{P}_{nh}^h(\theta_{\beta_1}^{\alpha_n^i}),\cdots,\widetilde{P}_{nh}^h(\theta_{\beta_r}^{\alpha_n^i})\right) \in \mathbb{C}^{qr^2\times qr} \tag{4.84}$$

$$\bar{Q}_{nh}^n(n\theta_{\beta_n^j}^{\gamma_n^i}) = \left(\mathrm{bdiag}\left(\widehat{L_{nh}}(n\theta_{\beta_1}^{\gamma_n^i}),\cdots,\widehat{L_{nh}}(n\theta_{\beta_r}^{\gamma_n^i})\right)\right)^{-1} \in \mathbb{C}^{qr\times qr} \tag{4.85}$$

$$\widetilde{e}_h^n(\theta_{\beta_k}^{\alpha_n^i}) = (\widehat{e_h^A}(\theta_{\beta_k}^{\alpha_1}),\cdots,\widehat{e_h^A}(\theta_{\beta_k}^{\alpha_r}))^T \in \mathbb{C}^{qr\times 1},$$

$$\bar{e}_h^n(\theta_{\beta_n^j}^{\alpha_n^i}) = (\widetilde{e}_h^n(\theta_{\beta_1}^{\alpha_n^i}),\cdots,\widetilde{e}_h^n(\theta_{\beta_r}^{\alpha_n^i}))^T \in \mathbb{C}^{qr^2\times 1},$$

with $\theta_{\beta_k}^{\alpha_n^i} = (\theta_{\beta_k}^{\alpha_1},\cdots,\theta_{\beta_k}^{\alpha_r})^T$, $\theta_{\beta_n^j}^{\alpha_n^i} = (\theta_{\beta_1}^{\alpha_n^i},\cdots,\theta_{\beta_r}^{\alpha_n^i})^T$, $\alpha_1,\cdots,\alpha_r \in \alpha_n^i$, $\beta_1,\cdots,\beta_r \in \beta_n^j$, Using these matrix representations, we obtain now for $\forall\alpha_k \in \alpha_n^i$, $\forall\beta_l \in \beta_n^j$, $i,j \in s_n$

$$\begin{aligned}
&\left(\widehat{S_h}(\theta_{\beta_l}^{\alpha_k})\right)^{\nu_2}\widehat{P_{nh}^h}(\theta_{\beta_l}^{\alpha_k})\sum_{\beta_2\in\beta_n^j}\left(\widehat{\widetilde{M}_{nh}^m}\right)_{\beta_2}(n\theta_{\beta_l}^{\gamma_n^i};\gamma)\left(\widehat{L_{nh}}(n\theta_{\beta_2}^{\gamma_n^i})\right)^{-1}\sum_{\alpha_2\in\alpha_n^i}\widehat{R_h^{nh}}(\theta_{\beta_2}^{\alpha_2})\widehat{L_h}(\theta_{\beta_2}^{\alpha_2})\\
&\quad\left(\widehat{S_h}(\theta_{\beta_2}^{\alpha_2})\right)^{\nu_1}\widehat{e_h^A}(\theta_{\beta_2}^{\alpha_2})\\
&=\left(\widehat{S_h}(\theta_{\beta_l}^{\alpha_k})\right)^{\nu_2}\widehat{P_{nh}^h}(\theta_{\beta_l}^{\alpha_k})\sum_{\beta_2\in\beta_n^j}\left(\widehat{\widetilde{M}_{nh}^m}\right)_{\beta_2}(n\theta_{\beta_l}^{\gamma_n^i};\gamma)\left(\widehat{L_{nh}}(n\theta_{\beta_2}^{\gamma_n^i})\right)^{-1}\widetilde{R}_h^n(\theta_{\beta_2}^{\alpha_n^i})\widetilde{L}_h^n(\theta_{\beta_2}^{\alpha_n^i})\\
&\quad\left(\widetilde{S}_h^n(\theta_{\beta_2}^{\alpha_n^i})\right)^{\nu_1}\widetilde{e}_h^n(\theta_{\beta_2}^{\alpha_n^i})\\
&=\begin{pmatrix}\widetilde{S}_h^n(\theta_{\beta_1}^{\alpha_n^i}) & \dots & 0\\ \vdots & \ddots & \vdots\\ 0 & \dots & \widetilde{S}_h^n(\theta_{\beta_r}^{\alpha_n^i})\end{pmatrix}^{\nu_2}\begin{pmatrix}\widetilde{P}_{nh}^h(\theta_{\beta_1}^{\alpha_n^i}) & \dots & 0\\ \vdots & \ddots & \vdots\\ 0 & \dots & \widetilde{P}_{nh}^h(\theta_{\beta_r}^{\alpha_n^i})\end{pmatrix}\\
&\quad\begin{pmatrix}\left(\widehat{\widetilde{M}_{nh}^m}\right)_{\beta_1}(n\theta_{\beta_1}^{\gamma_n^i};\gamma) & \dots & \left(\widehat{\widetilde{M}_{nh}^m}\right)_{\beta_r}(n\theta_{\beta_1}^{\gamma_n^i};\gamma)\\ \vdots & \ddots & \vdots\\ \left(\widehat{\widetilde{M}_{nh}^m}\right)_{\beta_1}(n\theta_{\beta_r}^{\gamma_n^i};\gamma) & \dots & \left(\widehat{\widetilde{M}_{nh}^m}\right)_{\beta_r}(n\theta_{\beta_r}^{\gamma_n^i};\gamma)\end{pmatrix}\begin{pmatrix}\widehat{L_{nh}}(n\theta_{\beta_1}^{\gamma_n^i}) & \dots & 0\\ \vdots & \ddots & \vdots\\ 0 & \dots & \widehat{L_{nh}}(n\theta_{\beta_r}^{\gamma_n^i})\end{pmatrix}^{-1}\\
&\quad\begin{pmatrix}\widetilde{R}_h^{nh}(\theta_{\beta_1}^{\alpha_n^i}) & \dots & 0\\ \vdots & \ddots & \vdots\\ 0 & \dots & \widetilde{R}_h^{nh}(\theta_{\beta_r}^{\alpha_n^i})\end{pmatrix}\begin{pmatrix}\widetilde{L}_h^n(\theta_{\beta_1}^{\alpha_n^i}) & \dots & 0\\ \vdots & \ddots & \vdots\\ 0 & \dots & \widetilde{L}_h^n(\theta_{\beta_r}^{\alpha_n^i})\end{pmatrix}
\end{aligned}$$

$$\begin{pmatrix} \widetilde{S}_h^n(\theta_{\beta_1}^{\alpha_n^i}) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \widetilde{S}_h^n(\theta_{\beta_r}^{\alpha_n^i}) \end{pmatrix}^{\nu_1} \begin{pmatrix} \widetilde{e}_h^n(\theta_{\beta_1}^{\alpha_n^i}) \\ \vdots \\ \widetilde{e}_h^n(\theta_{\beta_r}^{\alpha_n^i}) \end{pmatrix}$$

$$= \big(\bar{S}_h^n(\theta_{\beta_n^j}^{\alpha_n^i})\big)^{\nu_2} \bar{P}_{nh}^h(\theta_{\beta_n^j}^{\alpha_n^i}) \widehat{\widetilde{M}_{nh}^m}(n\theta_{\beta_n^j}^{\gamma_n^i};\gamma) \bar{Q}_{nh}^n(n\theta_{\beta_n^j}^{\gamma_n^i}) \bar{R}_h^{nh}(\theta_{\beta_n^j}^{\alpha_n^i}) \bar{L}_h^n(\theta_{\beta_n^j}^{\alpha_n^i}) \big(\bar{S}_h^n(\theta_{\beta_n^j}^{\alpha_n^i})\big)^{\nu_1} \bar{e}_h^n(\theta_{\beta_n^j}^{\alpha_n^i}).$$

The discrete Fourier transform for each group of modes of the error transformation operator for a three-level multigrid cycle $\widehat{M}_h^n(\theta_{\beta_n^j}^{\alpha_n^i}) \in \mathbb{C}^{r^2q \times r^2q}$, with $i,j \in s_n$, can now be expressed as

$$\widehat{M}_h^n(\theta_{\beta_n^j}^{\alpha_n^i}) = \big(\bar{S}_h^n(\theta_{\beta_n^j}^{\alpha_n^i})\big)^{\nu_2} \Big(I^{r^2q} - \bar{P}_{nh}^h(\theta_{\beta_n^j}^{\alpha_n^i}) \widehat{\widetilde{M}_{nh}^m}(n\theta_{\beta_n^j}^{\gamma_n^i};\gamma) \bar{Q}_{nh}^n(n\theta_{\beta_n^j}^{\gamma_n^i}) \bar{R}_h^{nh}(\theta_{\beta_n^j}^{\alpha_n^i}) \bar{L}_h^n(\theta_{\beta_n^j}^{\alpha_n^i})\Big) \big(\bar{S}_h^n(\theta_{\beta_n^j}^{\alpha_n^i})\big)^{\nu_1}. \tag{4.86}$$

The three-level error transformation operator is now obtained by combining the contributions from the different groups of modes. For uniform coarsening the multigrid error transformation operator is equal to

$$\widehat{M}_h^{(2,2)}(\theta_\beta^\alpha) = \widehat{M}_h^{(2,2)}(\theta_{\beta_{(2,2)}^1}^{\alpha_{(2,2)}^1}) \in \mathbb{C}^{16q \times 16q}, \tag{4.87}$$

with $\theta_\beta^\alpha = \theta_{\beta_{(2,2)}^1}^{\alpha_{(2,2)}^1}$. The error after one three-level multigrid cycle with uniform coarsening can now be expressed as

$$\widehat{e_h^D}(\theta_\beta^\alpha) = \widehat{M}_h^{(2,2)}(\theta_\beta^\alpha) \widehat{e_h^A}(\theta_\beta^\alpha),$$

with

$$\widehat{e_h^{A,D}}(\theta_\beta^\alpha) = \big(\widehat{e_h^D}(\theta_{\beta_1}^{\alpha_1}), \cdots, \widehat{e_h^{A,D}}(\theta_{\beta_1}^{\alpha_4}), \widehat{e_h^{A,D}}(\theta_{\beta_2}^{\alpha_1}), \cdots, \widehat{e_h^{A,D}}(\theta_{\beta_2}^{\alpha_4}), \cdots, \widehat{e_h^{A,D}}(\theta_{\beta_4}^{\alpha_1}),$$
$$\cdots, \widehat{e_h^{A,D}}(\theta_{\beta_4}^{\alpha_4}))^T, \qquad \alpha_1, \cdots, \alpha_4 \in \alpha_{(2,2)}^1, \beta_1, \cdots, \beta_4 \in \beta_{(2,2)}^1.$$

The multigrid error transformation operator for semi-coarsening in the x_1-direction is

$$\widehat{M}_h^{(2,1)}(\theta_{\beta_{(2,1)}}^{\alpha_{(2,1)}}) = \mathrm{bdiag}\big(\widehat{M}_h^{(2,1)}(\theta_{\beta_{(2,1)}^1}^{\alpha_{(2,1)}^1}), \widehat{M}_h^{(2,1)}(\theta_{\beta_{(2,1)}^1}^{\alpha_{(2,1)}^2}), \widehat{M}_h^{(2,1)}(\theta_{\beta_{(2,1)}^2}^{\alpha_{(2,1)}^1}),$$
$$\widehat{M}_h^{(2,1)}(\theta_{\beta_{(2,1)}^2}^{\alpha_{(2,1)}^2})\big) \in \mathbb{C}^{16q \times 16q},$$

with $\theta_{\beta_{(2,1)}}^{\alpha_{(2,1)}} = (\theta_{\beta_{(2,1)}^1}^{\alpha_{(2,1)}^1}, \theta_{\beta_{(2,1)}^1}^{\alpha_{(2,1)}^2}, \theta_{\beta_{(2,1)}^2}^{\alpha_{(2,1)}^1}, \theta_{\beta_{(2,1)}^2}^{\alpha_{(2,1)}^2})^T$. The frequencies $\theta_{\beta_{(2,1)}^j}^{\alpha_{(2,1)}^i}$, $i,j \in s_n$, are defined as

$$\theta_{\beta_{(2,1)}^1}^{\alpha_{(2,1)}^1} = (\theta_{00}^{00}, \theta_{00}^{10}, \theta_{\frac12 0}^{00}, \theta_{\frac12 0}^{10})^T, \qquad \theta_{\beta_{(2,1)}^1}^{\alpha_{(2,1)}^2} = (\theta_{00}^{11}, \theta_{00}^{01}, \theta_{\frac12 0}^{11}, \theta_{\frac12 0}^{01})^T,$$
$$\theta_{\beta_{(2,1)}^2}^{\alpha_{(2,1)}^1} = (\theta_{\frac12\frac12}^{00}, \theta_{\frac12\frac12}^{10}, \theta_{0\frac12}^{00}, \theta_{0\frac12}^{10})^T, \qquad \theta_{\beta_{(2,1)}^2}^{\alpha_{(2,1)}^2} = (\theta_{\frac12\frac12}^{11}, \theta_{\frac12\frac12}^{01}, \theta_{0\frac12}^{11}, \theta_{0\frac12}^{01})^T.$$

Note, however, that the error vectors for semi-coarsening in the x_1 and x_2-direction have a different ordering than the error components for uniform coarsening $\widehat{e_h^{A,D}}(\theta_\beta^\alpha)$. The ordering of the components of the error vectors is not important for the computation of the operator norms and the spectral radius of the error transformation operator, which are discussed in

Chapter 5. For the coupling of different multigrid algorithms, such as uniform and semi-coarsening, it is essential that the same ordering of the components of the error vectors is used. This can be easily accomplished using the permutation matrix $P_h^{(2,1)} \in \mathbb{R}^{16q\times 16q}$, which reorders the vector $\widehat{e_h^{A,D}}(\theta^{\alpha_{(2,1)}}_{\beta_{(2,1)}})$ to $\widehat{e_h^{A,D}}(\theta^{\alpha}_{\beta})$ and is defined as

$$P_h^{(2,1)} = \begin{pmatrix}
I^q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I^q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^q & 0 & 0 & 0 & 0 & 0 \\
0 & I^q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I^q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^q & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^q & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I^q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I^q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^q & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^q & 0 \\
0 & 0 & 0 & 0 & 0 & I^q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I^q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^q & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^q
\end{pmatrix}. \tag{4.88}$$

The error after one three-level multigrid cycle with semi-coarsening in the x_1-direction can now be expressed as

$$\widehat{e_h^D}(\theta^{\alpha}_{\beta}) = \big(P_h^{(2,1)}\big)^{-1}\widehat{M}_h^{(2,1)}(\theta^{\alpha_{(2,1)}}_{\beta_{(2,1)}})P_h^{(2,1)}\widehat{e_h^A}(\theta^{\alpha}_{\beta}).$$

Finally, the multigrid error transformation operator for semi-coarsening in the x_2-direction is

$$\widehat{M}_h^{(1,2)}(\theta^{\alpha_{(1,2)}}_{\beta_{(1,2)}}) = \mathrm{bdiag}\big(\widehat{M}_h^{(1,2)}(\theta^{\alpha^1_{(1,2)}}_{\beta^1_{(1,2)}}), \widehat{M}_h^{(1,2)}(\theta^{\alpha^2_{(1,2)}}_{\beta^1_{(1,2)}}), \widehat{M}_h^{(1,2)}(\theta^{\alpha^1_{(1,2)}}_{\beta^2_{(1,2)}}),$$
$$\widehat{M}_h^{(1,2)}(\theta^{\alpha^2_{(1,2)}}_{\beta^2_{(1,2)}})\big) \in \mathbb{C}^{16q\times 16q},$$

with $\theta^{\alpha_{(1,2)}}_{\beta_{(1,2)}} = (\theta^{\alpha^1_{(1,2)}}_{\beta^1_{(1,2)}}, \theta^{\alpha^2_{(1,2)}}_{\beta^1_{(1,2)}}, \theta^{\alpha^1_{(1,2)}}_{\beta^2_{(1,2)}}, \theta^{\alpha^2_{(1,2)}}_{\beta^2_{(1,2)}})^T$. The frequencies $\theta^{\alpha^i_{(1,2)}}_{\beta^j_{(1,2)}}$ are defined as

$$\theta^{\alpha^1_{(1,2)}}_{\beta^1_{(1,2)}} = (\theta^{00}_{00}, \theta^{01}_{00}, \theta^{00}_{0\frac{1}{2}}, \theta^{01}_{0\frac{1}{2}})^T, \qquad \theta^{\alpha^2_{(1,2)}}_{\beta^1_{(1,2)}} = (\theta^{11}_{00}, \theta^{10}_{00}, \theta^{11}_{0\frac{1}{2}}, \theta^{10}_{0\frac{1}{2}})^T,$$
$$\theta^{\alpha^1_{(1,2)}}_{\beta^2_{(1,2)}} = (\theta^{00}_{\frac{1}{2}\frac{1}{2}}, \theta^{01}_{\frac{1}{2}\frac{1}{2}}, \theta^{00}_{\frac{1}{2}0}, \theta^{01}_{\frac{1}{2}0})^T, \qquad \theta^{\alpha^2_{(1,2)}}_{\beta^2_{(1,2)}} = (\theta^{11}_{\frac{1}{2}\frac{1}{2}}, \theta^{10}_{\frac{1}{2}\frac{1}{2}}, \theta^{11}_{\frac{1}{2}0}, \theta^{10}_{\frac{1}{2}0})^T.$$

the permutation matrix $P_h^{(1,2)} \in \mathbb{R}^{16q\times 16q}$, which reorders the vector $\widehat{e_h^{A,D}}(\theta^{\alpha_{(1,2)}}_{\beta_{(1,2)}})$ to

$\widehat{e_h^{A,D}}(\theta_\beta^\alpha)$ and is defined as

$$P_h^{(1,2)} = \begin{pmatrix}
I^q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I^q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^q & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^q \\
0 & I^q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I^q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^q & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^q & 0 \\
0 & 0 & 0 & 0 & I^q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I^q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^q & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I^q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I^q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^q & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^q & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. \tag{4.89}$$

The error after one three-level multigrid cycle with semi-coarsening in the x_2-direction can now be expressed as

$$\widehat{e_h^D}(\theta_\beta^\alpha) = \left(P_h^{(1,2)}\right)^{-1} \widehat{M}_h^{(1,2)}(\theta^{\alpha_{(1,2)}}_{\beta_{(1,2)}}) P_h^{(1,2)} \widehat{e_h^A}(\theta_\beta^\alpha).$$

4.5 Discrete Fourier Analysis *hp-MGS* algorithm

In this section we derive the discrete Fourier transform of the error transformation operator $\widehat{M}_{h,3}(\theta_\beta^\alpha)$, with $\theta_\beta^\alpha = \theta^{\alpha^1_{(2,2)}}_{\beta^1_{(2,2)}}$, for the *hp-MGS* multigrid algorithm for a polynomial order $p = 3$ and three (semi)-coarsened mesh levels, given by (3.11). We will use the shorthand notation $\alpha = \alpha^1_{(2,2)}$ and $\beta = \beta^1_{(2,2)}$ for the Fourier mode indices in uniform mesh coarsening. The first part of the *hp-MGS* algorithm consists of p-multigrid. Since there is no coupling between modes on different meshes in the p-multigrid the discrete Fourier transform of the p-multigrid part of the *hp-MGS* algorithm can be computed straightforwardly, resulting in

$$\begin{aligned}
\widehat{M}_{h,3}(\theta_\beta^\alpha) =& \left(\widehat{HU}_{h,3}(\theta_\beta^\alpha)\right)^{\gamma_2} \Big(I^{16q_3} - \bar{T}^3_{h,2}(\theta_\beta^\alpha)\left(I^{16q_2} - \widehat{M}_{h,2}(\theta_\beta^\alpha)\right) \left(\bar{L}^{(2,2)}_{h,2}(\theta_\beta^\alpha)\right)^{-1} \\
& \bar{Q}^2_{h,3}(\theta_\beta^\alpha) \bar{L}^{(2,2)}_{h,3}(\theta_\beta^\alpha) \Big) \left(\widehat{HU}_{h,3}(\theta_\beta^\alpha)\right)^{\gamma_1} \in \mathbb{C}^{16q_3 \times 16q_3}.
\end{aligned} \tag{4.90}$$

with the contribution from the $p = 2$ level given by

$$\begin{aligned}
\widehat{M}_{h,2}(\theta_\beta^\alpha) =& \left(\widehat{HU}_{h,2}(\theta_\beta^\alpha)\right)^{\gamma_2} \Big(I^{16q_2} - \bar{T}^2_{h,1}(\theta_\beta^\alpha)(I^{16q_1} - \widehat{HU}_{h,1}(\theta_\beta^\alpha)) \left(\bar{L}^{(2,2)}_{h,1}(\theta_\beta^\alpha)\right)^{-1} \\
& \bar{Q}^1_{h,2}(\theta_\beta^\alpha) \bar{L}^{(2,2)}_{h,2}(\theta_\beta^\alpha) \Big) \left(\widehat{HU}_{h,2}(\theta_\beta^\alpha)\right)^{\gamma_1} \in \mathbb{C}^{16q_2 \times 16q_2},
\end{aligned}$$

where $\theta_{\beta}^{\alpha} = \theta_{\beta^1_{(2,2)}}^{\alpha^1_{(2,2)}}$. Here q_p refers to the size of the blocks in the matrices for polynomial order p. The p-multigrid prolongation operator $\bar{T}_{h,p}^{p+1}$ is defined using (4.38) as

$$\bar{T}_{h,p}^{p+1}(\theta_{\beta}^{\alpha}) = \text{bdiag}\big(\widehat{T}_{h,p}^{p+1}(\theta_{\beta_1}^{\alpha_1}), \cdots, \widehat{T}_{h,p}^{p+1}(\theta_{\beta_1}^{\alpha_4}), \widehat{T}_{h,p}^{p+1}(\theta_{\beta_2}^{\alpha_1}), \cdots, \widehat{T}_{h,p}^{p+1}(\theta_{\beta_2}^{\alpha_4}),$$
$$\widehat{T}_{h,p}^{p+1}(\theta_{\beta_4}^{\alpha_1}), \cdots, \widehat{T}_{h,p}^{p+1}(\theta_{\beta_4}^{\alpha_4})\big) \in \mathbb{C}^{16q_p \times 16q_p},$$

and the restriction operator is equal to $\bar{Q}_{h,p+1}^{p} = \big(\bar{T}_{h,p}^{p+1}\big)^T$. Note, frequently the p-multigrid restriction and prolongation operators are purely element based in which case their discrete Fourier transform is not a function of θ_{β}^{α}.

The discrete Fourier transform of the *hp-MGS* error transformation operator depends on the discrete Fourier transform of the three-level *h-MGS* smoothers $\widehat{HU}_{h,p}(\theta_{\beta}^{\alpha})$, $p \in \{1,2,3\}$. The discrete Fourier transform of these operators can be obtained using the thee-level analysis discussed in Section 4.4. In order to describe the discrete Fourier transform we extend the matrices defined in (4.73)-(4.76) and (4.77)-(4.85) to include also the polynomial order p. Using the result for the three-level error transformation operator given by (4.87) we obtain the discrete Fourier transform of the *h-MGS* error transformation operator for each polynomial order p

$$\widehat{HU}_{h,p}(\theta_{\beta}^{\alpha}) = \big(\widehat{HS}_{h,p}^{(2,1)}(\theta_{\beta}^{\alpha})\widehat{HS}_{h,p}^{(1,2)}(\theta_{\beta}^{\alpha})\big)^{\nu_2}\Big(I^{16q_p} - \bar{P}_{2h,p}^{h}(\theta_{\beta}^{\alpha})\widehat{\widetilde{M}_{2h,p}^{(4,4)}}(2\theta_{\beta}^{00};1)\bar{Q}_{2h,p}^{(2,2)}(2\theta_{\beta}^{00})$$
$$\bar{R}_{h,p}^{2h}(\theta_{\beta}^{\alpha})\bar{L}_{h,p}^{(2,2)}(\theta_{\beta}^{\alpha})\Big)\big(\widehat{HS}_{h,p}^{(1,2)}(\theta_{\beta}^{\alpha})\widehat{HS}_{h,p}^{(2,1)}(\theta_{\beta}^{\alpha})\big)^{\nu_1} \in \mathbb{C}^{16q_p \times 16q_p}, \tag{4.91}$$

with $\widehat{HS}_{h,p}^{(2,1)}(\theta_{\beta}^{\alpha})$ and $\widehat{HS}_{h,p}^{(1,2)}(\theta_{\beta}^{\alpha})$ the discrete Fourier transform of the error transformation operator of the semi-coarsening multigrid smoothers in, respectively, the x_1 and x_2-direction. The coarse grid contribution $\widehat{\widetilde{M}_{2h,p}^{(4,4)}}(2\theta_{\beta}^{00};1)$ from the mesh G_{2h} in (4.91) is given by

$$\widehat{\widetilde{M}_{2h,p}^{(4,4)}}(2\theta_{\beta}^{00};1) = I^{4q_p} - \widehat{M}_{2h,p}^{(4,4)}(2\theta_{\beta}^{00}) \in \mathbb{C}^{4q_p \times 4q_p},$$

with

$$\widehat{M}_{2h,p}^{(4,4)}(2\theta_{\beta}^{00}) = \big(\widehat{HS}_{2h,p}^{(2,1)}(2\theta_{\beta}^{00})\widehat{HS}_{2h,p}^{(1,2)}(2\theta_{\beta}^{00})\big)^{\nu_2}\Big(I^{4q_p} - \widehat{P}_{4h,p}^{2h}(2\theta_{\beta}^{00})\big(\widehat{L_{4h,p}}(4\theta_{00}^{00})\big)^{-1}$$
$$\widehat{R}_{2h,p}^{4h}(2\theta_{\beta}^{00})\widehat{L}_{2h,p}^{(2,2)}(2\theta_{\beta}^{00})\Big)\big(\widehat{HS}_{2h,p}^{(1,2)}(2\theta_{\beta}^{00})\widehat{HS}_{2h,p}^{(2,1)}(2\theta_{\beta}^{00})\big)^{\nu_1} \in \mathbb{C}^{4q_p \times 4q_p}.$$

The discrete Fourier transform of the semi-coarsening smoother in the local x_1-direction is given by

$$\widehat{HS}_{h,p}^{(2,1)}(\theta_{\beta}^{\alpha}) = (P_h^{(2,1)})^{-1}\text{bdiag}\big(\widehat{M}_{h,p}^{(2,1)}(\theta_{\beta^1_{(2,1)}}^{\alpha^1_{(2,1)}}), \widehat{M}_{h,p}^{(2,1)}(\theta_{\beta^1_{(2,1)}}^{\alpha^2_{(2,1)}}), \widehat{M}_{h,p}^{(2,1)}(\theta_{\beta^2_{(2,1)}}^{\alpha^1_{(2,1)}}),$$
$$\widehat{M}_{h,p}^{(2,1)}(\theta_{\beta^2_{(2,1)}}^{\alpha^2_{(2,1)}})\big)P_h^{(2,1)} \in \mathbb{C}^{16q_p \times 16q_p},$$

with the permutation matrix $P_h^{(2,1)} \in \mathbb{C}^{16q_p \times 16q_p}$ given by (4.88) and

$$\theta_{\beta^1_{(2,1)}}^{\alpha^1_{(2,1)}} = (\theta_{00}^{00}, \theta_{00}^{10}, \theta_{\frac{1}{2}0}^{00}, \theta_{\frac{1}{2}0}^{10})^T, \qquad \theta_{\beta^1_{(2,1)}}^{\alpha^2_{(2,1)}} = (\theta_{00}^{11}, \theta_{00}^{01}, \theta_{\frac{1}{2}0}^{11}, \theta_{\frac{1}{2}0}^{01})^T,$$
$$\theta_{\beta^2_{(2,1)}}^{\alpha^1_{(2,1)}} = (\theta_{\frac{1}{2}\frac{1}{2}}^{00}, \theta_{\frac{1}{2}\frac{1}{2}}^{10}, \theta_{0\frac{1}{2}}^{00}, \theta_{0\frac{1}{2}}^{10})^T, \qquad \theta_{\beta^2_{(2,1)}}^{\alpha^2_{(2,1)}} = (\theta_{\frac{1}{2}\frac{1}{2}}^{11}, \theta_{\frac{1}{2}\frac{1}{2}}^{01}, \theta_{0\frac{1}{2}}^{11}, \theta_{0\frac{1}{2}}^{01})^T.$$

The discrete Fourier transform of the semi-coarsening smoother in the local x_2-direction is

$$\widehat{HS}_{h,p}^{(1,2)}(\theta_\beta^\alpha) = (P_h^{(1,2)})^{-1}\mathrm{bdiag}\big(\widehat{M}_{h,p}^{(1,2)}(\theta_{\beta^1_{(1,2)}}^{\alpha^1_{(1,2)}}), \widehat{M}_{h,p}^{(1,2)}(\theta_{\beta^1_{(1,2)}}^{\alpha^2_{(1,2)}}), \widehat{M}_{h,p}^{(1,2)}(\theta_{\beta^2_{(1,2)}}^{\alpha^1_{(1,2)}}),$$
$$\widehat{M}_{h,p}^{(1,2)}(\theta_{\beta^2_{(1,2)}}^{\alpha^2_{(1,2)}})\big) P_h^{(1,2)} \in \mathbb{C}^{16q_p\times 16q_p},$$

the permutation matrix $P_h^{(1,2)} \in \mathbb{C}^{16q_p\times 16q_p}$ given by (4.89) and

$$\theta_{\beta^1_{(1,2)}}^{\alpha^1_{(1,2)}} = (\theta_{00}^{00}, \theta_{00}^{01}, \theta_{0\frac12}^{00}, \theta_{0\frac12}^{01})^T, \qquad \theta_{\beta^1_{(1,2)}}^{\alpha^2_{(1,2)}} = (\theta_{00}^{11}, \theta_{00}^{10}, \theta_{0\frac12}^{11}, \theta_{0\frac12}^{10})^T,$$
$$\theta_{\beta^2_{(1,2)}}^{\alpha^1_{(1,2)}} = (\theta_{\frac12\frac12}^{00}, \theta_{\frac12\frac12}^{01}, \theta_{\frac12 0}^{00}, \theta_{\frac12 0}^{01})^T, \qquad \theta_{\beta^2_{(1,2)}}^{\alpha^2_{(1,2)}} = (\theta_{\frac12\frac12}^{11}, \theta_{\frac12\frac12}^{10}, \theta_{\frac12 0}^{11}, \theta_{\frac12 0}^{10})^T.$$

Note, the permutation matrices are necessary in order to combine the error transformation operators for the different types of mesh coarsening which use a different ordering of the Fourier modes. The contribution to the error transformation operators from the different groups of modes in the semi-coarsening smoothers $\widehat{HS}_{h,p}^{(2,1)}(\theta_\beta^\alpha)$ and $\widehat{HS}_{h,p}^{(1,2)}(\theta_\beta^\alpha)$ is now given for $i, j \in s_n$ by

$$\widehat{M}_{h,p}^n(\theta_{\beta_n^j}^{\alpha_n^i}) = \big(\bar{S}_{h,p}^n(\theta_{\beta_n^j}^{\alpha_n^i})\big)^{\mu_2}\Big(I^{4q_p} - \bar{P}_{nh,p}^h(\theta_{\beta_n^j}^{\alpha_n^i})\widehat{\widetilde{M}_{nh,p}^m}(n\theta_{\beta_n^j}^{\gamma_n^i};1)\bar{Q}_{nh,p}^n(n\theta_{\beta_n^j}^{\gamma_n^i})\bar{R}_{h,p}^{nh}(\theta_{\beta_n^j}^{\alpha_n^i})$$
$$\bar{L}_{h,p}^n(\theta_{\beta_n^j}^{\alpha_n^i})\Big)\big(\bar{S}_{h,p}^n(\theta_{\beta_n^j}^{\alpha_n^i})\big)^{\mu_1} \in \mathbb{C}^{4q_p\times 4q_p},$$

with the coarse grid contributions

$$\widehat{\widetilde{M}_{nh,p}^m}(n\theta_{\beta_n^j}^{\gamma_n^i};1) = I^{2q_p} - \widehat{M}_{nh,p}^m(n\theta_{\beta_n^j}^{\gamma_n^i}) \in \mathbb{C}^{2q_p\times 2q_p},$$

and

$$\widehat{M}_{nh,p}^m(n\theta_{\beta_n^j}^{\gamma_n^i}) = \big(\widehat{S}_{nh,p}^n(n\theta_{\beta_n^j}^{\gamma_n^i})\big)^{\mu_2}\Big(I^{2q_p} - \widehat{P}_{mh,p}^{nh}(n\theta_{\beta_n^j}^{\gamma_n^i})\big(I^{q_p} - \big(\widehat{S_{mh,p}}(m\theta_{\delta_n^j}^{\gamma_n^i})\big)^{\mu_3}\big)$$
$$\Big(\widehat{L_{mh,p}}(m\theta_{\delta_n^j}^{\gamma_n^i})\Big)^{-1}\widehat{R}_{nh,p}^{mh}(n\theta_{\beta_n^j}^{\gamma_n^i})\widehat{L}_{nh,p}^n(n\theta_{\beta_n^j}^{\gamma_n^i})\Big)\big(\widehat{S}_{nh,p}^n(n\theta_{\beta_n^j}^{\gamma_n^i})\big)^{\mu_1} \in \mathbb{C}^{2q_p\times 2q_p},$$

with $n = (2,1)$, $m = (4,1)$ for $HS_{h,p}^{(2,1)}$ and $n = (1,2)$, $m = (1,4)$ for $HS_{h,p}^{(1,2)}$. The smoothers $\bar{S}_{h,p}^n$, $\widehat{S}_{nh,p}^n$ and $\widehat{S_{mh,p}}(m\theta_{\delta_n^j}^{\gamma_n^i})$ are either the point implicit or the semi-implicit pseudo-time Runge-Kutta smoother in the local direction x_1 for $n = (2,1)$ or local x_2 for $n = (1,2)$, which are defined in Section 4.2.3.

The contribution of the semi-coarsening smoothers at the mesh level $2h$ is equal to

$$\widehat{HS}_{2h,p}^n(2\theta_\beta^{00}) = (P_{2h}^n)^{-1}\mathrm{bdiag}\big(\widehat{M}_{2h,p}^n(2\theta_{\beta_n^1}^{00}), \widehat{M}_{2h,p}^n(2\theta_{\beta_n^2}^{00})\big)P_{2h}^n \in \mathbb{C}^{4q_p\times 4q_p},$$

with

$$\widehat{M}_{2h,p}^n(2\theta_{\beta_n^j}^{00}) = \big(\widehat{S}_{2h,p}^n(2\theta_{\beta_n^j}^{00})\big)^{\mu_2}\Big(I^{2q_p} - \widehat{P}_{2nh,p}^{2h}(2\theta_{\beta_n^j}^{00})(I^{q_p} - \big(\widehat{S_{2nh,p}}(2n\theta_{\delta_n^j}^{00})\big)^{\mu_3})$$
$$\Big(\widehat{L_{2nh,p}}(2n\theta_{\delta_n^j}^{00})\Big)^{-1}\widehat{R}_{2h,p}^{2nh}(2\theta_{\beta_n^j}^{00})\widehat{L}_{2h,p}^n(2\theta_{\beta_n^j}^{00})\Big)\big(\widehat{S}_{2h,p}^n(2\theta_{\beta_n^j}^{00})\big)^{\mu_1} \in \mathbb{C}^{2q_p\times 2q_p},$$

where $n = (2,1)$ for $HS^{(2,1)}_{2h,p}$ and $n = (1,2)$ for $HS^{(1,2)}_{2h,p}$. The permutation matrices are defined as

$$P^{(2,1)}_{2h} = \begin{pmatrix} I^q & 0 & 0 & 0 \\ 0 & 0 & I^q & 0 \\ 0 & I^q & 0 & 0 \\ 0 & 0 & 0 & I^q \end{pmatrix} \qquad P^{(1,2)}_{2h} = \begin{pmatrix} I^q & 0 & 0 & 0 \\ 0 & 0 & 0 & I^q \\ 0 & I^q & 0 & 0 \\ 0 & 0 & I^q & 0 \end{pmatrix}.$$

Chapter 5

Definition of Convergence Rates

The performance of the multigrid scheme is measured with two parameters.

1. The *cycle convergence factor*, which is defined as

$$\lambda = \sup_{e_h^A \neq 0} \frac{\|e_h^D\|_{\ell^2(G_h)}}{\|e_h^A\|_{\ell^2(G_h)}},$$

with $\|e_h^A\|_{\ell^2(G_h)}$ and $\|e_h^D\|_{\ell^2(G_h)}$ the discrete ℓ^2-norm of the initial error and the error after one full multigrid cycle, respectively. Using (4.41), (4.57) or (4.90) we can also express the cycle convergence factor as

$$\lambda = \sup_{e_h^A \neq 0} \frac{\|M_h^{ng} e_h^A\|_{\ell^2(G_h)}}{\|e_h^A\|_{\ell^2(G_h)}} =: \|M_h^{ng}\|_{\ell^2(G_h)},$$

with $\|M_h^{ng}\|_{\ell^2(G_h)}$ the discrete ℓ_2 operator norm of M_h^{ng} and n the number of multigrid grid levels. On an infinite mesh G_h this expression can be further evaluated using discrete Fourier analysis. Parseval's identity (4.9) implies that

$$\|e_h^A\|^2_{\ell^2(G_h)} = (2\pi)^d \int_{\theta \in \Pi_{(1,1)}} |\widehat{e_h^A}(\theta)|^2 d\theta = (2\pi)^d \|\widehat{e_h^A}\|^2_{L^2((-\pi,\pi)^d)},$$
$$\|M_h^{ng} e_h^A\|^2_{\ell^2(G_h)} = (2\pi)^d \int_{\theta \in \Pi_{(1,1)}} |\widehat{M_h^{ng}}(\theta)\widehat{e_h^A}(\theta)|^2 d\theta = (2\pi)^d \|\widehat{M_h^{ng}}\widehat{e_h^A}\|^2_{L^2((-\pi,\pi)^d)},$$

with $\|\cdot\|_{L^2}$ the L^2-norm. The discrete ℓ_2 operator norm thus satisfies $\|M_h^{ng}\|_{\ell^2(G_h)} = \|\widehat{M_h^{ng}}\|_{L^2((-\pi,\pi)^d)}$.

The discrete ℓ_2 operator norm of a matrix A also satisfies $\|A\|_{\ell^2(G_h)} = \sqrt{\rho(AA^*)}$, see e.g. Golub and Van Loan [5], Theorem 2.3.1. Here A^* refers to the conjugate transposed of A and ρ is the spectral radius, which is defined as

$$\rho(A) = \max\{|\lambda| \mid \lambda \in \sigma(A)\}$$

where

$$\sigma(A) = \{\lambda \in \mathbb{C} \,|\lambda \text{ is an eigenvalue of } A\}.$$

On an infinite mesh G_h the Fourier modes are eigenvectors of the matrix M_h^{ng}, and also of $M_h^{ng}\,(M_h^{ng})^*$, hence

$$\rho\left(M_h^{ng}\,(M_h^{ng})^*\right) = \sup_{\theta\in\Pi_{(1,1)}\setminus\Psi} \rho\left(\widehat{M}_h^{ng}(\theta)\left(\widehat{M}_h^{ng}(\theta)\right)^*\right),$$

with for two-level multigrid $\Psi = \Psi_n$, given by (4.40), and for three-level multigrid $\Psi = \Psi_{n,m}$, given by (4.56). The cycle convergence factor for a multigrid algorithm using n meshes then can be expressed as

$$\lambda = \sup_{\theta\in\Pi_{(1,1)}\setminus\Psi} \sqrt{\rho\Big(\widehat{M}_h^{ng}(\theta)\big(\widehat{M}_h^{ng}(\theta)\big)^*\Big)}.$$

2. The *asymptotic convergence factor per cycle*, which is defined as

$$\mu = \lim_{m\to\infty}\left(\sup_{e_h^{(0)}\neq 0}\frac{\|e_h^{(m)}\|_{\ell^2(G_h)}}{\|e_h^{(0)}\|_{\ell^2(G_h)}}\right)^{\frac{1}{m}}$$

where $e_h^{(m)}$ is the error after m applications of the multigrid cycle, hence $e_h^{(0)} = e_h^A$ and $e_h^{(1)} = e_h^D$. The asymptotic convergence factor can be further evaluated using (4.41) or (4.57) as

$$\begin{aligned}\mu &= \lim_{m\to\infty}\left(\sup_{e_h^{(0)}\neq 0}\frac{\|\big(M_h^{ng}\big)^m e_h^{(0)}\|_{\ell^2(G_h)}}{\|e_h^{(0)}\|_{\ell^2(G_h)}}\right)^{\frac{1}{m}}\\ &= \lim_{m\to\infty}\left(\|\big(M_h^{ng}\big)^m\|_{\ell^2(G_h)}\right)^{\frac{1}{m}}. \end{aligned} \tag{5.1}$$

Next, we use the following result, Theorem 3.3 from Varga [12]. Let A be an $n\times n$ complex matrix with spectral radius $\rho(A) > 0$ then

$$\|A^m\|_{\ell^2(G_h)} \sim c\begin{pmatrix} m \\ p-1\end{pmatrix}\Big(\rho(A)\Big)^{m-p+1} \quad \text{as } m\to\infty, \tag{5.2}$$

with p the largest order of all Jordan submatrices J_r of the Jordan normal form of A with $\rho(J_r) = \rho(A)$, and c a positive constant. If we use relation (5.2) in (5.1) then we obtain that the asymptotic convergence factor is equal to

$$\mu = \rho(M_h^{ng}).$$

On an infinite mesh G_h the Fourier modes are eigenvectors of the matrix M_h^{ng} and also of $M_h^{ng}\,(M_h^{ng})^*$, hence

$$\rho(M_h^{ng}) = \sup_{\theta\in\Pi_{(1,1)}\setminus\Psi} \rho\left(\widehat{M}^{ng}(\theta)\right).$$

The asymptotic convergence rate then can be expressed as

$$\mu = \sup_{\theta\in\Pi_{(1,1)}\setminus\Psi} \rho\left(\widehat{M}^{ng}(\theta)\right).$$

A requirement for convergence is therefore that the spectral radius satisfies the condition

$$\sup_{\theta \in \Pi_{(1,1)} \setminus \Psi} \rho\left(\widehat{M}^{ng}(\theta)\right) < 1.$$

Since $\|M_h^{ng}\|_{\ell^2(G_h)} \geq \rho(M_h^{ng})$ it may happen that $\|M_h^{ng}\|_{\ell^2(G_h)} > 1$, even though $\rho(M_h^{ng}) < 1$. The error $e_h^{(m)}$ will then increase during the initial iterations, but eventually $e_h^{(m)}$ will decrease to zero because $\lim_{m\to\infty} \|\left(M_h^{ng}\right)^m\|_{\ell^2(G_h)} \to 0$ when $\rho(M_h^{ng}) < 1$.

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Appendix A

Auxilary Results

A.1 Orthonormality of Fourier modes

The Fourier modes $\phi_{nh}(n\theta, x) = e^{\imath n\theta \cdot x/(nh)}$, with $\theta \in \mathbb{R}^d$ and $x \in G_{nh}$, are orthonormal with respect to the scaled Euclidian inner product on G_{nh}, given by (4.3), viz.

$$(\phi_{nh}(n\theta_l, \cdot), \phi_{nh}(n\theta_m, \cdot))_{G_{nh}} = \begin{cases} 1 & \text{if } \theta_l = \theta_m, \\ 0 & \text{otherwise.} \end{cases}$$

First, consider a finite mesh $G_{nh}^N \subset \mathbb{R}^d$. On this finite mesh only Fourier modes with frequencies $\theta_l = \pi l/N$, with $l \in \mathcal{G}_n^N$ and $N \in \mathbb{N}^d$, can be represented. We also have for $x \in G_{nh}^N$ that $x/(nh) = k$ with $k \in \mathcal{G}_n^N$.
The inner product of the Fourier modes $\phi_{nh}(n\theta_l, x)$ and $\phi_{nh}(n\theta_m, x)$ then is equal to

$$\begin{aligned}
\left(\phi_{nh}(n\theta_l, \cdot), \phi_{nh}(n\theta_m, \cdot)\right)_{G_{nh}^N} &= \left(\Pi_{j=1}^d \frac{n_j}{2N_j}\right) \sum_{x \in G_{nh}} e^{\imath n\theta_l \cdot x/(nh)} e^{-\imath n\theta_m \cdot x/(nh)} \\
&= \left(\Pi_{j=1}^d \frac{n_j}{2N_j}\right) \sum_{k_1=-N_1/n_1}^{N_1/n_1-1} \cdots \sum_{k_d=-N_d/n_d}^{N_d/n_d-1} e^{\imath(n_1\theta_{l_1}k_1 + \cdots + n_d\theta_{l_d}k_d)} \\
&\quad e^{-\imath(n_1\theta_{m_1}k_1 + \cdots + n_d\theta_{m_d}k_d)} \\
&= \sum_{k_1=-N_1/n_1}^{N_1/n_1-1} \cdots \sum_{k_d=-N_d/n_d}^{N_d/n_d-1} \Pi_{j=1}^d \left(\frac{n_j}{2N_j} e^{\imath n_j \theta_{l_j} k_j} e^{-\imath n_j \theta_{m_j} k_j}\right) \\
&= \Pi_{j=1}^d \left(\frac{n_j}{2N_j} \sum_{k_j=-N_j/n_j}^{N_j/n_j-1} e^{\imath \pi n_j l_j k_j/N_j} e^{-\imath \pi n_j m_j k_j/N_j}\right). \qquad \text{(A.1)}
\end{aligned}$$

For the evaluation of the summations on the righthand side of (A.1) we now consider

$$\begin{aligned}\frac{n_j}{2N_j}\sum_{k_j=-N_j/n_j}^{N_j/n_j-1} e^{\imath\pi n_j l_j k_j/N_j} e^{-\imath\pi n_j m_j k_j/N_j} &= \frac{n_j}{2N_j}\sum_{k_j=-N_j/n_j}^{N_j/n_j-1} e^{\imath\pi n_j (l_j-m_j) k_j/N_j} \\ &= \frac{n_j}{2N_j}\sum_{p=0}^{2N_j/n_j-1} e^{\imath\pi n_j (l_j-m_j)(p-N_j/n_j)/N_j} \\ &= e^{-\imath\pi(l_j-m_j)}\frac{n_j}{2N_j}\sum_{p=0}^{2N_j/n_j-1}\left(e^{\imath\pi n_j(l_j-m_j)/N_j}\right)^p.\end{aligned}$$

For the summation, we need to consider two cases:

1. $l-m\neq 0$, then

$$\begin{aligned}e^{-\imath\pi(l_j-m_j)}\frac{n_j}{2N_j}\sum_{p=0}^{2N_j/n_j-1}&\left(e^{\imath\pi n_j(l_j-m_j)/N_j}\right)^p \\ &= e^{-\imath\pi(l_j-m_j)}\frac{n_j}{2N_j}\frac{\left(e^{\imath\pi n_j(l_j-m_j)/N_j}\right)^{2N_j/n_j}-1}{e^{\imath\pi n_j(l_j-m_j)/N_j}-1} \\ &= e^{-\imath\pi(l_j-m_j)}\frac{n_j}{2N_j}\frac{e^{2\pi\imath(l_j-m_j)}-1}{e^{\imath\pi n_j(l_j-m_j)/N_j}-1} \\ &= 0,\end{aligned}$$

   since $e^{2\pi\imath(l_j-m_j)}=1$ if $l_j-m_j\in\mathbb{Z}$.

2. $l_j-m_j=0$, then

$$e^{-\imath\pi(l_j-m_j)}\frac{n_j}{2N_j}\sum_{p=0}^{2N_j/n_j-1}\left(e^{\imath\pi n_j(l_j-m_j)/N_j}\right)^p=1.$$

Combining both terms then gives

$$\begin{aligned}\frac{n_j}{2N_j}\sum_{k_j=-N_j/n_j}^{N_j/n_j-1} e^{\imath\pi n_j k_j (l_j-m_j)/N_j} &= \delta_{l_j,m_j}, \\ l_j,m_j\in\mathcal{G}_{n_j}^{N_j}, n_j, N_j\in\mathbb{N}, j=1,\cdots,d,&\end{aligned}\tag{A.2}$$

and δ_{l_j,m_j} the Kronecker delta symbol. Combining (A.1) and (A.2) the inner product between two Fourier modes on G_{nh}^N then is equal to

$$\left(\phi_{nh}(n\theta_l,\cdot),\phi_{nh}(n\theta_m,\cdot)\right)_{G_{nh}^N}=\delta_{l,m},\qquad l,m\in\mathcal{G}_n^N.$$

If we take the limit $N_i\to\infty$ using the definition of the scaled Euclidian inner product on G_{nh}, given by (4.3), we obtain

$$\left(\phi_{nh}(n\theta_l,\cdot),\phi_{nh}(n\theta_m,\cdot)\right)_{G_{nh}}=\begin{cases}1 & \text{if } \theta_l=\theta_m,\\ 0 & \text{otherwise.}\end{cases}$$

A.2 Discrete Fourier transform and its inverse on an infinite mesh

Define the discrete Fourier transform of $v_{nh}(x)$ on the mesh G_{nh} as

$$\widehat{v_{nh}}(n\theta) = \Pi_{l=1}^{d}\left(\frac{n_l}{2\pi}\right)\sum_{x\in G_{nh}} v_{nh}(x)e^{-\imath n\theta\cdot x/(nh)}, \qquad \theta\in\Pi_n,$$

with $\Pi_n = [-\frac{\pi}{n_1},\frac{\pi}{n_1})\times\cdots\times[-\frac{\pi}{n_d},\frac{\pi}{n_d})$. The inverse discrete Fourier transform is equal to

$$v_{nh}(x) = \int_{\theta\in\Pi_n}\widehat{v_{nh}}(n\theta)e^{\imath n\theta\cdot x/(nh)}d\theta, \qquad x\in G_{nh}.$$

This relation follows for $x\in G_{nh}$ directly from

$$\begin{aligned} v_{nh}(x) &= \int_{\theta\in\Pi_n}\widehat{v_{nh}}(n\theta)e^{\imath n\theta\cdot x/(nh)}d\theta \\ &= \Pi_{l=1}^{d}\left(\frac{n_l}{2\pi}\right)\sum_{y\in G_{nh}} v_{nh}(y)\int_{\theta\in\Pi_n} e^{\imath n\theta\cdot(x-y)/(nh)}d\theta. \end{aligned}$$

Use $x = jnh$ and $y = knh$ with $j,k\in\mathbb{Z}^d$, then

$$\begin{aligned} v_{nh}(x) &= \Pi_{l=1}^{d}\left(\frac{n_l}{2\pi}\right)\sum_{k\in\mathbb{Z}^d} v_{nh}(knh)\int_{\theta\in\Pi_n} e^{\imath n\theta\cdot(j-k)}d\theta \\ &= \sum_{k\in\mathbb{Z}^d} v_{nh}(knh)\ \Pi_{l=1}^{d}\left(\frac{n_l}{2\pi}\int_{\theta_l=-\frac{\pi}{n_l}}^{\frac{\pi}{n_l}} e^{\imath n_l\theta_l\cdot(j_l-k_l)}d\theta_l\right). \end{aligned}$$

Set $n_l\theta_l = \alpha_l$ and $d\theta_l = d\alpha_l/n_l$ then

$$\begin{aligned} v_{nh}(x) &= \sum_{k\in\mathbb{Z}^d} v_{nh}(knh)\ \Pi_{l=1}^{d}\left(\frac{1}{2\pi}\int_{\alpha_l=-\pi}^{\pi} e^{\imath\alpha_l\cdot(j_l-k_l)}d\alpha_l\right) \\ &= \sum_{k\in\mathbb{Z}^d} v_{nh}(knh)\ \Pi_{l=1}^{d}\delta_{j_l,k_l} \\ &= v_{nh}(jnh) \\ &= v_{nh}(x). \end{aligned}$$

A.3 Discrete Fourier transform and its inverse on a finite mesh

On a periodic domain with a finite mesh G_{nh}^N the discrete Fourier transform and its inverse are defined as

$$\begin{aligned} \widehat{v_{nh}}(n\theta_k) &= \left(\Pi_{l=1}^{d}\frac{n_l}{2N_l}\right)\sum_{x\in G_{nh}^N} v_{nh}(x)e^{-\imath n\theta_k\cdot x/(nh)} \\ v_{nh}(x) &= \sum_{k\in\mathcal{G}_n^N}\widehat{v_{nh}}(n\theta_k)e^{\imath n\theta_k\cdot x/(nh)}, \end{aligned}$$

with $\theta_k = (\theta_{k_1}, \cdots, \theta_{k_d})$, $\theta_{k_l} = \pi k_l / N_l$, $k_l \in \mathcal{G}_{n_l}^{N_l}$ and $x \in G_{nh}^N$. The relation between $v_{nh}(x)$ and $\widehat{v_{nh}}(n\theta_k)$ follows directly from

$$
\begin{aligned}
v_{nh}(x) &= \sum_{k \in \mathcal{G}_n^N} \widehat{v_{nh}}(n\theta_k) e^{\imath n \theta_k \cdot x/(nh)} \\
&= \left(\Pi_{l=1}^d \frac{n_l}{2N_l} \right) \sum_{k \in \mathcal{G}_n^N} \sum_{y \in G_{nh}^N} v_{nh}(y) e^{-\imath n \theta_k \cdot y/(nh)} e^{\imath n \theta_k \cdot x/(nh)} \\
&= \left(\Pi_{l=1}^d \frac{n_l}{2N_l} \right) \sum_{k \in \mathcal{G}_n^N} \sum_{m \in \mathcal{G}_n^N} v_{nh}(mnh) e^{\imath n \theta_k \cdot (j-m)} \\
&= \sum_{m \in \mathcal{G}_n^N} v_{nh}(mnh) \left(\Pi_{l=1}^d \frac{n_l}{2N_l} \right) \sum_{k_1 = -N_1/n_1}^{N_1/n_1 - 1} \cdots \sum_{k_d = -N_d/n_d}^{N_d/n_d - 1} e^{\imath (n_1 \theta_{k_1}(j_1 - m_1) + \cdots + n_d \theta_{k_d}(j_d - m_d))} \\
&= \sum_{m \in \mathcal{G}_n^N} v_{nh}(mnh) \, \Pi_{l=1}^d \left(\frac{n_l}{2N_l} \sum_{k_l = -N_l/n_l}^{N_l/n_l - 1} e^{\imath n_l \theta_{k_l}(j_l - m_l)} \right) \\
&= \sum_{m \in \mathcal{G}_n^N} v_{nh}(mnh) \, \Pi_{l=1}^d \left(\frac{n_l}{2N_l} \sum_{k_l = -N_l/n_l}^{N_l/n_l - 1} e^{\imath \pi n_l k_l (j_l - m_l)/N_l} \right) \\
&= \sum_{m \in \mathcal{G}_n^N} v_{nh}(mnh) \, \delta_{j,m} \\
&= v_{nh}(jnh) = v_{nh}(x).
\end{aligned}
$$

where we used (A.2) in the seventh step.

A.4 Parsevals identity

Using (4.8) and (4.7) we obtain Parsevals identity

$$
\begin{aligned}
\int_{\theta \in \Pi_n} |\widehat{v_{nh}}(n\theta)|^2 d\theta &= \int_{\theta \in \Pi_n} \widehat{v_{nh}}(n\theta) \overline{\widehat{v_{nh}}(n\theta)} d\theta \\
&= \left(\Pi_{l=1}^d \frac{n_l}{2\pi} \right) \sum_{x \in G_{nh}} v_{nh}(x) \int_{\theta \in \Pi_n} \widehat{v_{nh}}(n\theta) e^{\imath n \theta \cdot x/(nh)} d\theta \\
&= \left(\Pi_{l=1}^d \frac{n_l}{2\pi} \right) \sum_{x \in G_{nh}} |v_{nh}(x)|^2 \\
&= \left(\Pi_{l=1}^d \frac{n_l}{2\pi} \right) \| v_{nh}(x) \|^2_{\ell^2(G_{nh})}.
\end{aligned}
$$

A.5 Aliasing modes in 2D

Consider $\hat{\theta} = (\theta_1 \pm 2\pi/n_1, \theta_2 \pm 2\pi/n_2)$, with $\theta \in \Pi_n$ and $x \in G_{nh}$. Then

$$
\begin{aligned}
\phi_{nh}(n\hat{\theta}, x) &= e^{\imath n_1 (\theta_1 \pm 2\pi/n_1) x_1/(n_1 h_1)} e^{\imath n_2 (\theta_2 \pm 2\pi/n_2) x_2/(n_2 h_2)} \\
&= e^{\imath n_1 \theta_1 x_1/(n_1 h_1)} e^{\pm 2\imath \pi k_1} e^{\imath n_2 \theta_2 x_2/(n_2 h_2)} e^{\pm 2\imath \pi k_2} \qquad \text{with } k_i \in \mathbb{Z} \\
&= e^{\imath n \theta \cdot x/(nh)} \\
&= \phi_{nh}(n\theta, x),
\end{aligned}
$$

where we used $x = knh$ if $x \in G_{nh}$. The modes with frequency $\hat{\theta}$, where $\theta = \hat{\theta} \,(\mathrm{mod}\, 2\pi/n)$, therefore coincide with $e^{\imath n\theta\cdot x/(nh)}$.
Assume the following mesh coarsenings $G_{h_1,h_2} \to G_{2h_1,2h_2}$, $G_{h_1,h_2} \to G_{2h_1,h_2}$ and $G_{h_1,h_2} \to G_{h_1,2h_2}$. Given the modes with frequency $\theta_\beta^\alpha \in \Pi_{(1,1)}$, with $\alpha \in \{(0,0),(1,1),(1,0),(0,1)\}$ and $\beta \in \{(0,0),(\frac{1}{2},\frac{1}{2}),(\frac{1}{2},0),(0,\frac{1}{2})\}$, then we have the following relation between modes on G_h and G_{nh}, with $n \in \{(2,2),(2,1),(1,2)\}$

$$\begin{aligned}\phi_h(\theta_\beta^\alpha, x) &= \phi_h(\theta_\beta^{\alpha'}, x)\\ &= \phi_{nh}(n\theta_\beta^{\alpha'}, x), \quad \theta_\beta^{\alpha'} \in \Pi_n,\ x \in G_{nh},\end{aligned}$$

with

$$\alpha' = \begin{cases}(0,0) & \text{if } n = (2,2)\\ (0,\bar{\alpha}_2) & \text{if } n = (2,1)\\ (\bar{\alpha}_1,0) & \text{if } n = (1,2)\end{cases} \tag{A.3}$$

Proof. Using (4.10) we obtain for $x \in G_{nh}$ the expression

$$\begin{aligned}\phi_h(\theta_\beta^\alpha, x) &= e^{\imath\theta_\beta^\alpha\cdot x/h}\\ &= e^{\imath(\theta_\beta^{00} - (\bar{\alpha}_1 \mathrm{sign}((\theta_\beta^{00})_1), \bar{\alpha}_2 \mathrm{sign}((\theta_\beta^{00})_2))\pi)\cdot x/h}\\ &= e^{\imath\theta_\beta^{00}\cdot x/h} e^{-\imath\pi(\bar{\alpha}_1 \mathrm{sign}((\theta_\beta^{00})_1), \bar{\alpha}_2 \mathrm{sign}((\theta_\beta^{00})_2))\cdot jn}\end{aligned} \tag{A.4}$$

where we used $x = jnh$ if $x \in G_{nh}$. The second term on the righthand side of (A.4) can be further evaluated as

$$\begin{aligned}e^{-\imath\pi(\bar{\alpha}_1 \mathrm{sign}((\theta_\beta^{00})_1), \bar{\alpha}_2 \mathrm{sign}((\theta_\beta^{00})_2))\cdot jn} &= 1, && \text{if } n = (2,2)\\ &= e^{-\imath\pi\bar{\alpha}_2 \mathrm{sign}((\theta_\beta^{00})_2) j_2 n_2}, && \text{if } n = (2,1)\\ &= e^{-\imath\pi\bar{\alpha}_1 \mathrm{sign}((\theta_\beta^{00})_1) j_1 n_1} && \text{if } n = (1,2)\end{aligned}$$

and we obtain the following expression for $x \in G_{nh}$

$$\begin{aligned}\phi_h(\theta_\beta^\alpha, x) &= e^{\imath\theta_\beta^{00}\cdot x/h}, && \text{if } n = (2,2)\\ &= e^{\imath(\theta_\beta^{00} - \pi(0, \bar{\alpha}_2 \mathrm{sign}((\theta_\beta^{00})_2))\cdot x/h}, && \text{if } n = (2,1)\\ &= e^{\imath(\theta_\beta^{00} - \pi(\bar{\alpha}_1 \mathrm{sign}((\theta_\beta^{00})_1), 0))\cdot x/h} && \text{if } n = (1,2),\end{aligned}$$

which is equivalent with

$$\begin{aligned}\phi_h(\theta_\beta^\alpha, x) &= e^{\imath(\theta_\beta^{00} - (\bar{\alpha}_1' \mathrm{sign}((\theta_\beta^{00})_1), \bar{\alpha}_2' \mathrm{sign}((\theta_\beta^{00})_2))\pi)\cdot x/h}\\ &= e^{\imath\theta_\beta^{\alpha'}\cdot x/h}\\ &= e^{\imath n\theta_\beta^{\alpha'}\cdot x/(nh)}, \qquad \theta_\beta^{\alpha'} \in \Pi_n,\ x \in G_{nh}\end{aligned}$$

with α' given by (A.3).

Assume the following mesh coarsenings $G_{2h_1,2h_2} \to G_{4h_1,4h_2}$, $G_{2h_1,h_2} \to G_{4h_1,h_2}$ and $G_{h_1,2h_2} \to G_{h_1,4h_2}$. Given modes with frequencies $\theta_\beta^{\alpha'} \in \Pi_n$ on the mesh G_{nh}, with $n \in \{(2,2),(2,1),(1,2)\}$, then we have the following aliasing relation between modes on the meshes G_{nh} and G_{mh}, with $m \in \{(4,4),(4,1),(1,4)\}$,

$$\begin{aligned}\phi_{nh}(n\theta_\beta^{\alpha'}, x) &= \phi_h(\theta_{\beta'}^{\alpha'}, x)\\ &= \phi_{mh}(m\theta_{\beta'}^{\alpha'}, x), \quad \theta_{\beta'}^{\alpha'} \in \Pi_m,\ x \in G_{mh}\end{aligned}$$

with

$$\begin{aligned}
&\alpha' = (0,0), \qquad &&\beta' = (0,0), \qquad &&\text{if } m = (4,4),\\
&\alpha' = (0,\bar{\alpha}_2), \qquad &&\beta' = (0,\bar{\beta}_2) \qquad &&\text{if } m = (4,1),\\
&\alpha' = (\bar{\alpha}_1,0), \qquad &&\beta' = (\bar{\beta}_1,0) \qquad &&\text{if } m = (1,4).
\end{aligned}$$

Proof.

$$\begin{aligned}
\phi_{nh}(n\theta_\beta^{\alpha'}, x) &= e^{\imath n\theta_\beta^{\alpha'}\cdot x/(nh)}\\
&= e^{\imath\theta_{00}^{00}\cdot x/h} e^{-\imath\pi(\bar{\beta}_1 \mathrm{sign}((\theta_{00}^{00})_1),\bar{\beta}_2\mathrm{sign}((\theta_{00}^{00})_2))\cdot x/h}\\
&\quad e^{-\imath\pi(\bar{\alpha}_1'\mathrm{sign}((\theta_\beta^{00})_1),\bar{\alpha}_2'\mathrm{sign}((\theta_\beta^{00})_2))\cdot x/h}.
\end{aligned}$$

Use now for $x \in G_{mh}$ the relation

$$e^{-\imath\pi(\bar{\beta}_1\mathrm{sign}((\theta_{00}^{00})_1),\bar{\beta}_2\mathrm{sign}((\theta_{00}^{00})_2))\cdot x/h} = e^{-\imath\pi(\bar{\beta}_1\mathrm{sign}((\theta_{00}^{00})_1),\bar{\beta}_2\mathrm{sign}((\theta_{00}^{00})_2)\cdot mj}$$

then we obtain

$$\begin{aligned}
e^{-\imath\pi(\bar{\beta}_1\mathrm{sign}((\theta_{00}^{00})_1),\bar{\beta}_2\mathrm{sign}((\theta_{00}^{00})_2))\cdot mj} &= 1 && \text{if } m = (4,4)\\
&= e^{-\imath\pi\bar{\beta}_2\mathrm{sign}((\theta_{00}^{00})_2)j_2m_2} && \text{if } m = (4,1)\\
&= e^{-\imath\pi\bar{\beta}_1\mathrm{sign}((\theta_{00}^{00})_1)j_1m_1} && \text{if } m = (1,4).
\end{aligned}$$

Combining all terms we obtain for $x \in G_{mh}$

$$\begin{aligned}
\phi_{mh}(m\theta_\beta^{\alpha'}, x) &= e^{\imath\theta_\beta^{\alpha'}\cdot x/h}\\
&= e^{\imath\theta_{00}^{00}\cdot x/h} e^{-\imath\pi(\bar{\beta}_1'\mathrm{sign}((\theta_{00}^{00})_1),\bar{\beta}_2'\mathrm{sign}((\theta_{00}^{00})_2))\cdot x/h}\\
&\quad e^{-\imath\pi(\bar{\alpha}_1'\mathrm{sign}((\theta_{\beta'}^{00})_1),\bar{\alpha}_2'\mathrm{sign}((\theta_{\beta'}^{00})_2))\cdot x/h}\\
&= e^{\imath\theta_{\beta'}^{\alpha'}\cdot x/h}\\
&= e^{\imath m\theta_{\beta'}^{\alpha'}\cdot x/(mh)} \qquad \text{with } \theta_{\beta'}^{\alpha'} \in \Pi_m.
\end{aligned}$$