Space-Time Discontinuous Galerkin Methods

Scalar Conservation Equations

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Part 4.

USTC Summer School, September 4 – September 8, 2023

Motivation:

Many problems are defined on time-dependent domains, e.g.

- Fluid-structure interaction
- Free surface problems, such as water waves and multiphase flows with free surfaces

These problems can be efficiently computed using a space-time approach, in which time and space are simultaneously discretized.

Objectives

To develop a numerical scheme for hyperbolic and parabolic conservation laws with the following properties:

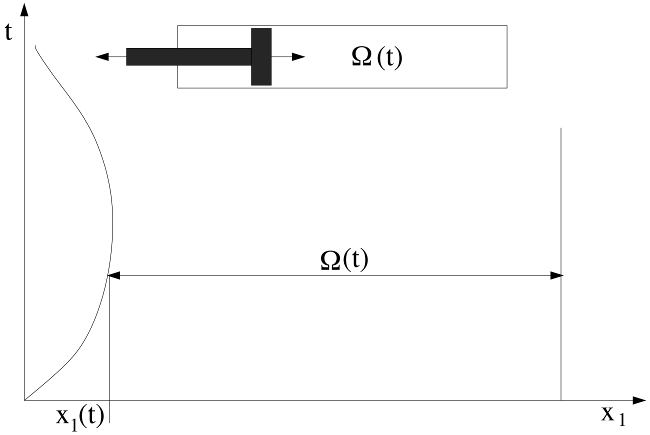
- Conservative numerical discretization on moving and deforming meshes (satisfy geometric conservation law)
- Improve accuracy using *hp*-adaptation
- Maintain accuracy on irregular meshes
- Efficient capturing of discontinuities, interfaces and vortices
- Easy to parallelize

These requirements have been the primary motivation to develop space-time discontinuous Galerkin finite element methods.

Overview

- One-dimensional example: hyperbolic scalar conservation laws
  - space-time formulation
  - discontinuous Galerkin discretization
- Multi-dimensional parabolic scalar conservation laws:
  - space-time discontinous Galerkin discretization
  - ALE formulation

Time-Dependent Flow Domain



Example of a time dependent flow domain $\Omega(t)$.

Scalar Conservation Laws

- Consider the scalar conservation law in the time dependent flow domain $\Omega \subseteq \mathbb{R}$:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x_1} = 0, \qquad x_1 \in \Omega(t),\ t \in (t_0, T),$$

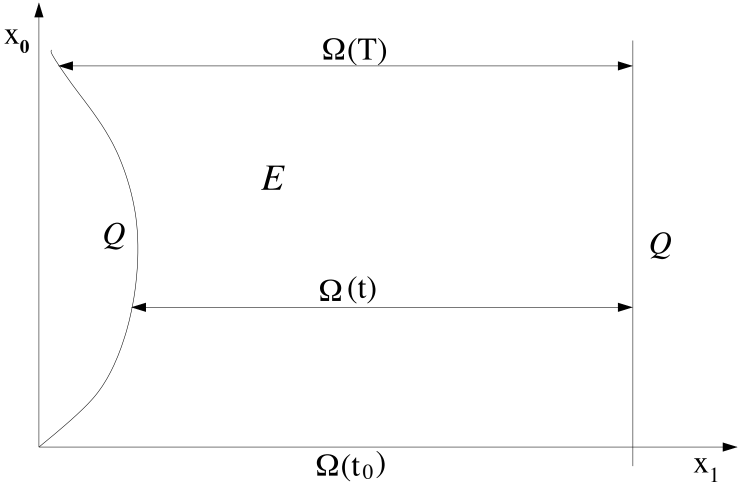
with boundary conditions:

$$u(x_1, t) = \mathcal{B}(u, u_w), \qquad x_1 \in \partial\Omega(t),\ t \in (t_0, T),$$

and initial condition:

$$u(x_1, 0) = u_0(x_1), \qquad x_1 \in \Omega(t_0).$$

Space-Time Domain



Example of a space-time domain $\mathcal{E}$.

Definition of Space-Time Domain

- Let $\mathcal{E} \subset \mathbb{R}^2$ be an open domain.
- A point $x \in \mathbb{R}^2$ has coordinates (x_0, x_1), where x_0 represents time and x_1 the spatial coordinate.
- Define the flow domain Ω at time t as:

$$\Omega(t) := \{x_1 \in \mathbb{R} \,|\, (t, x_1) \in \mathcal{E}\}.$$

- Define the boundary $\mathcal{Q}$ as:

$$\mathcal{Q} := \{x \in \partial\mathcal{E} \,|\, t_0 < x_0 < T\}.$$

- **Note:** The space-time domain boundary $\partial\mathcal{E}$ is equal to:

$$\partial\mathcal{E} = \Omega(t_0) \cup \mathcal{Q} \cup \Omega(T).$$

Space-Time Formulation of Scalar Conservation Laws

- Define the space-time flux vector: $\mathcal{F}(u) := (u, f(u))^T$, then scalar conservation laws can be written as:

$$\operatorname{div} \mathcal{F}(u(x)) = 0, \qquad x \in \mathcal{E}$$

  with boundary conditions:

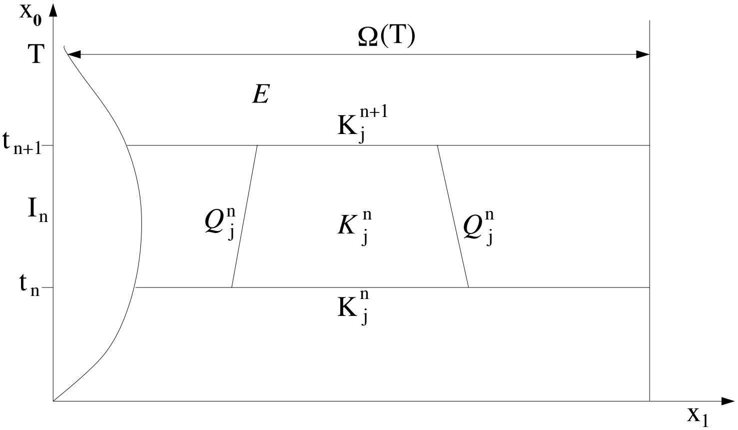
$$u(x) = \mathcal{B}(u, u_w), \qquad x \in \mathcal{Q},$$

  and initial condition:

$$u(x) = u_0(x), \qquad x \in \Omega(t_0).$$

- The div operator is defined as: $\operatorname{div} \mathcal{F} = \frac{\partial \mathcal{F}_i}{\partial x_i}$.

Space-Time Slab

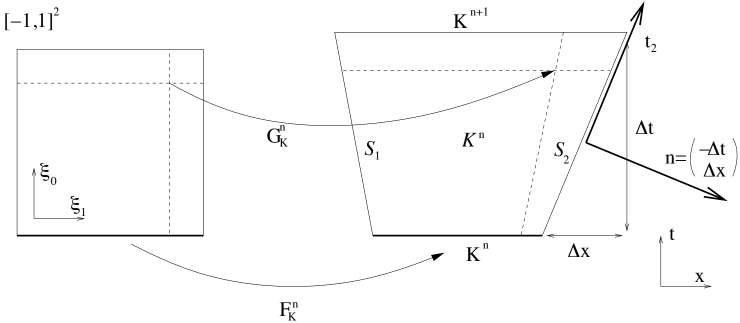


Space-time slab in space-time domain $\mathcal{E}$.

Definition of Space-Time Slab

- Consider a partitioning of the time interval (t_0, T): $\{t_n\}_{n=0}^N$, and set $I_n = (t_n, t_{n+1})$.
- Define a space-time slab as: $\mathcal{I}_n := \{x \in \mathcal{E} \mid x_0 \in I_n\}$
- Split the space-time slab into non-overlapping elements: $\mathcal{K}_j^n$.
- We will also use the notation: $K_j^n = \mathcal{K}_j^n \cap \{t_n\}$ and $K_j^{n+1} = \mathcal{K}_j^n \cap \{t_{n+1}\}$

Geometry of Space-Time Element



Geometry of 2D space-time element in both computational and physical space.

Element Mappings

Definition of the mapping $G^n_{\mathcal{K}}$ which the connects the space-time element $\mathcal{K}^n$ to the reference element $\hat{\mathcal{K}} = (-1, 1)^2$:

- Define a smooth, orientation preserving and invertible mapping Φ^n_t in the interval I_n as:

$$\Phi^n_t : \Omega(t_n) \to \Omega(t) : x_1 \mapsto \Phi^n_t(x_1), \quad t \in I_n.$$

  Note, for many problems, e.g. free surface problems, the mapping Φ^n_t is not given and is part of the equations that need to be solved.

- Split $\Omega(t_n)$ into the tessellation $\bar{\mathcal{T}}^n_h$ with non-overlapping elements K_j.

- Define $\chi_i(\xi_1), \xi_1 \in (-1, 1)$ as the standard linear finite element shape functions:

$$\chi_1(\xi_1) = \tfrac{1}{2}(1 - \xi_1),$$
$$\chi_2(\xi_1) = \tfrac{1}{2}(1 + \xi_1).$$

Element Mappings

- The mapping F_K^n is defined as:

$$F_K^n : (-1, 1) \to K^n : \xi_1 \longmapsto \sum_{i=1}^{2} x_i(K^n)\chi_i(\xi_1),$$

with $x_i(K^n)$ the spatial coordinates of the space-time element at time $t = t_n$.

- Similarly we define the mapping F_K^{n+1}:

$$F_K^{n+1} : (-1, 1) \to K^{n+1} : \xi_1 \longmapsto \sum_{i=1}^{2} \Phi_{t_{n+1}}^n(x_i(K^n))\chi_i(\xi_1).$$

Element Mappings

- The space-time element is defined by linear interpolation in time:

$$G_{\mathcal{K}}^{n} : (-1,1)^2 \to \mathcal{K}^n : (\xi_0, \xi_1) \longmapsto (x_0, x_1),$$

with:

$$(x_0, x_1) = \Big(\tfrac{1}{2}(t_n + t_{n+1}) - \tfrac{1}{2}(t_n - t_{n+1})\xi_0, \\ \tfrac{1}{2}(1-\xi_0)F_K^n(\xi_1) + \tfrac{1}{2}(1+\xi_0)F_K^{n+1}(\xi_1) \Big).$$

- The space-time tessellation is now defined as:

$$\mathcal{T}_h^n := \{\mathcal{K} = G_{\mathcal{K}}^n(\hat{\mathcal{K}}) \mid K \in \bar{\mathcal{T}}_h^n\}.$$

Basis Functions

- Define the basis functions $\hat{\phi}_m, (m = 1, \cdots, (p+1)^2)$, in the master element $\hat{\mathcal{K}}$ as:

$$\hat{\phi}_m(\xi_0, \xi_1) = \xi_0^{i_0} \xi_1^{i_1}.$$

Remark: In practice the best option is to use orthogonal basis functions, e.g. Legendre polynomials or (generalized) Jacobi polynomials.

- Define the basis functions ϕ_m in an element $\mathcal{K}$ as:

$$\phi_m(x) = \hat{\phi}_m \circ G_{\mathcal{K}}^{-1}(x).$$

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Finite Element Space

- Define the finite element space $V_h^p(\mathcal{T}_h^n)$ as:

$$V_h^p(\mathcal{T}_h^n) := \left\{ v_h \,\middle|\, v_h|_{\mathcal{K}} \in \mathcal{Q}^p(\mathcal{K}),\, \forall \mathcal{K} \in \mathcal{T}_h^n \right\},$$

with $\mathcal{Q}^p(\mathcal{K}) = \mathrm{span}\{\phi_m, m = 1, \cdots, (p+1)^2\}$ a tensor product basis.

- The trial functions $u_h : \mathcal{T}_h^n \to \mathbb{R}^2$ are defined in each element $\mathcal{K} \in \mathcal{T}_h^n$ as:

$$u_h(x) = \sum_{m=1}^{(p+1)^2} \hat{U}_m(\mathcal{K}) \psi_m(x), \quad x \in \mathcal{K},$$

with $\hat{U}_m$ the DG expansion coefficients.

Weak Formulation for STDG Method

The scalar conservation laws can be transformed into a weak formulation:

- Find a $u_h \in V_h^p$, such that for all $w_h \in V_h^p$, we have:

$$\sum_{n=0}^{N_T} \sum_{j=1}^{N_n} \Big(\int_{\mathcal{K}_j^n} w_h \,\mathrm{div}\mathcal{F}(u_h) d\mathcal{K} + \int_{\mathcal{K}_j^n} (\mathrm{grad}\, w_h)^T \,\mathfrak{D}(u_h)\, \mathrm{grad}\, u_h d\mathcal{K} \Big) = 0.$$

- The second integral with $\mathfrak{D}(u_h) \in \mathbb{R}^2$ is the stabilization operator necessary to obtain monotone solutions near discontinuities.

- Alternatively, one can use a limiter, but one has to be careful to ensure that the limiter does not cause problems in solving the algebraic equations resulting from the DG discretization.

  See, F. Yan, J.J.W. van der Vegt, Y. Xia, Y. Xu, to appear in Commun. Comput. Phys. 2023.

Weak Formulation

After integration by parts we obtain the following weak formulation:

- Find a $u_h \in V_h^p$, such that for all $w_h \in V_h^p$, we have:

$$\sum_{n=0}^{N_T} \sum_{j=1}^{N_n} \Big(- \int_{\mathcal{K}_j^n} \mathrm{grad}\, w_h \cdot \mathcal{F}(u_h) d\mathcal{K} + \int_{\partial \mathcal{K}_j^n} w_h^- n^- \cdot \mathcal{F}(u_h^-) d(\partial \mathcal{K}) \\ + \int_{\mathcal{K}_j^n} (\mathrm{grad}\, w_h)^T \, \mathfrak{D}(u_h) \, \mathrm{grad}\, u_h d\mathcal{K} \Big) = 0.$$

Numerical Fluxes

- We can transform the element boundary integrals into:

$$\sum_{\mathcal{K}} \int_{\partial\mathcal{K}} w_h^- n^- \cdot \mathcal{F}^- d(\partial\mathcal{K}) = \sum_{\mathcal{S}} \int_{\mathcal{S}} \Big((w_h^- n^- + w_h^+ n^+) \cdot \tfrac{1}{2}(\mathcal{F}^- + \mathcal{F}^+) + \tfrac{1}{2}(w_h^- + w_h^+)(\mathcal{F}^- \cdot n^- + \mathcal{F}^+ \cdot n^+) \Big) d\mathcal{S}, \qquad (1)$$

with $\mathcal{F}^\pm = \mathcal{F}(u_h^\pm)$, and n^-, n^+ the normal vectors at each side of the face $\mathcal{S}$, which satisfy $n^+ = -n^-$.

Numerical Fluxes

- The formulation must be conservative, which imposes the condition:

$$\int_{\mathcal{S}} w_h n^- \cdot \mathcal{F}^- d\mathcal{S} = -\int_{\mathcal{S}} w_h n^+ \cdot \mathcal{F}^+ d\mathcal{S}, \qquad \forall w_h \in V_h^p(\mathcal{T}_h^n),$$

  hence the second contribution in (1) must be zero.

- The boundary integrals therefore are equal to:

$$\sum_{\mathcal{K}} \int_{\partial\mathcal{K}} w_h^- n^- \cdot \mathcal{F}^- d(\partial\mathcal{K}) = \sum_{\mathcal{S}} \int_{\mathcal{S}} \tfrac{1}{2}(w_h^- - w_h^+) n^- \cdot (\mathcal{F}^- + \mathcal{F}^+) d\mathcal{S},$$

  using the relation $n^+ = -n^-$.

Numerical Fluxes

- Replace the multi-valued trace of the flux at $\mathcal{S}$ with a numerical flux function:

$$H(u_h^-, u_h^+, n) \cong \frac{1}{2} n \cdot (\mathcal{F}^- + \mathcal{F}^+),$$

then we obtain the relation:

$$\begin{aligned}\sum_{\mathcal{K}} \int_{\partial\mathcal{K}} w_h^- n^- \cdot \mathcal{F}^- d(\partial\mathcal{K}) &= \sum_{\mathcal{S}} \int_{\mathcal{S}} (w_h^- - w_h^+) H(u_h^-, u_h^+, n^-) d\mathcal{S} \\ &= \sum_{\mathcal{K}} \int_{\partial\mathcal{K}} w_h^- H(u_h^-, u_h^+, n^-) d(\partial\mathcal{K}),\end{aligned}$$

using the relation $H(u_h^-, u_h^+, n^-) = -H(u_h^+, u_h^-, n^+)$.

Numerical Fluxes

- The numerical flux at the boundary faces $K(t_n)$ and $K(t_{n+1})$, which have as normal vectors $n^- = (\mp 1, 0)^T$, respectively, is defined as:

$$\begin{aligned} H(u_h^-, u_h^+, n^-) &= u_h^+ \qquad \text{at } K(t_n) \\ &= u_h^- \qquad \text{at } K(t_{n+1}). \end{aligned}$$

- The numerical flux at the boundary faces $\mathcal{Q}^n$ is a monotone Lipschitz $H(u_h^-, u_h^+, n)$, which is consistent:

$$H(u, u, n) = n \cdot \mathcal{F}(u)$$

and conservative:

$$H(u_h^-, u_h^+, n^-) = -H(u_h^+, u_h^-, n^+).$$

Riemann Problem

- The monotone Lipschitz flux $H(u_h^-, u_h^+, n)$ is obtained by (approximately) solving the Riemann problem with initial states u_h^- and u_h^+ at the element faces $\mathcal{Q}^n$.
- This procedure introduces upwinding into the discontinuous Galerkin finite element method.

Upwind Fluxes

Consistent, monotone Lipschitz fluxes are:

- Godunov flux

$$H^G(u_h^-, u_h^+, n) = \begin{cases} \min\limits_{u \in [u_h^-, u_h^+]} \hat{f}(u), & \text{if } u_h^- \leq u_h^+ \\ \max\limits_{u \in [u_h^+, u_h^-]} \hat{f}(u), & \text{otherwise,} \end{cases}$$

  with $\hat{f}(u) = \mathcal{F}(u) \cdot n$.

Upwind Fluxes

- Local Lax-Friedrichs flux

$$H^{LLF}(u_h^-, u_h^+, n) = \frac{1}{2}(\hat{f}(u_h^-) + \hat{f}(u_h^+) - C(u_h^+ - u_h^-)),$$

with

$$C = \max_{\inf(u_h^-, u_h^+) \leq s \leq \sup(u_h^-, u_h^+)} |\hat{f}'(s)|,$$

- Roe flux with entropy fix
- HLLC flux
- The choice which numerical flux should be used depends on many aspects, e.g. accuracy, robustness, computational complexity, and personal preference.

Arbitrary Lagrangian Eulerian Formulation

- The space-time normal vector at $\mathcal{Q}$ can be expressed as:

$$n = (-u_g \cdot \bar{n}, \bar{n}),$$

  with u_g the mesh velocity.

- If we introduce this relation into the numerical fluxes then

$$\hat{f}(u) = \mathcal{F}(u) \cdot n = f(u) \cdot \bar{n} - u_g \cdot \bar{n} u,$$

  which is exactly the flux in an ALE formulation.

Weak Formulation for DG Discretization

After introducing the numerical fluxes we can transform the weak formulation into:

- Find a $u_h \in V_h^p(\mathcal{T}_h^n)$, such that for all $w_h \in V_h^p(\mathcal{T}_h^n)$, the following variational equation is satisfied:

$$\sum_{j=1}^{N_n} \Big(-\int_{\mathcal{K}_j^n} (\mathrm{grad}\, w_h) \cdot \mathcal{F}(u_h) d\mathcal{K} + \int_{K_j(t_{n+1})} w_h^- u_h^- dK -$$

$$\int_{K_j(t_n)} w_h^- u_h^+ dK + \int_{\mathcal{Q}_j^n} w_h^- H(u_h^-, u_h^+; u_g, n^-) d\mathcal{Q} +$$

$$\int_{\mathcal{K}_j^n} (\mathrm{grad}\ w_h)^T \mathfrak{D}(u_h)\, \mathrm{grad}\, u_h\, d\mathcal{K} \Big) = 0.$$

- **Note:** Due to the causality of the time-flux the solution in a space-time slab only depends explicitly on the data from the previous space-time slab.

Parabolic Scalar Conservation Laws

- Parabolic scalar conservation laws on a time-dependent domain $\Omega_t \subset \mathbb{R}^d$:

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{d} \frac{\partial}{\partial x_i} f_i(u(t,\bar{x})) - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(D_{ij}(t,\bar{x}) \frac{\partial u}{\partial x_i}\right) = 0, \text{ in } \Omega_t,$$

- with:
  - u a scalar quantity
  - f_i, $i = 1, \cdots, d$ real-valued flux functions
  - $D \in \mathbb{R}^{d \times d}$ a symmetric positive definite matrix of diffusion coefficients

Space-Time Formulation

- Introduce the convective flux $\mathcal{F} \in \mathbb{R}^{d+1}$ and the symmetric matrix $A \in \mathbb{R}^{(d+1)\times(d+1)}$ as:

$$\mathcal{F}(u) = \left(u, f_1(u), \cdots, f_d(u)\right),$$
$$A = \left(\begin{array}{cc} 0 & 0 \\ 0 & D \end{array} \right).$$

- The parabolic scalar conservation law can be transformed into a space-time formulation as:

$$-\nabla \cdot (-\mathcal{F}(u) + A\nabla u) = 0 \quad \text{in} \quad \mathcal{E},$$

where $\nabla = (\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d})$ denotes the gradient operator in $\mathbb{R}^{d+1}$.

Boundary Conditions

- The boundary $\partial\mathcal{E}$ is divided into disjoint boundary subsets Γ_S, Γ_-, and Γ_+, where each subset is defined as follows:

$$\Gamma_S := \{x \in \partial\mathcal{E} : \bar{n}^T D \bar{n} > 0\},$$
$$\Gamma_- := \{x \in \partial\mathcal{E} \setminus \Gamma_S : \lambda(u) < 0\},$$
$$\Gamma_+ := \{x \in \partial\mathcal{E} \setminus \Gamma_S : \lambda(u) \geq 0\},$$

with:

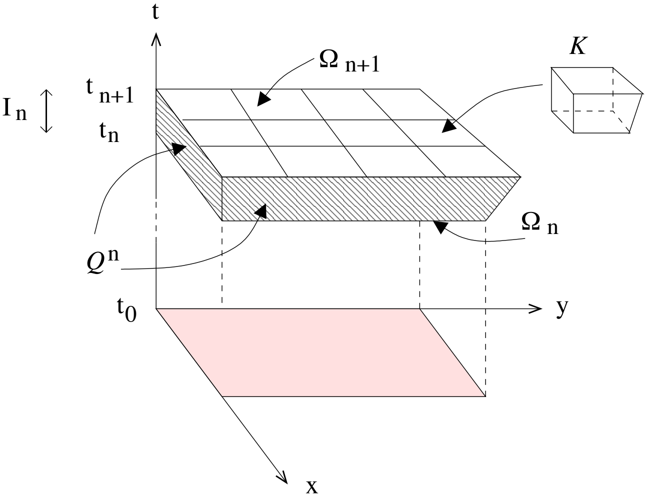
  - n the space-time normal vector at $\partial\mathcal{E}$
  - $\bar{n}$ the spatial part of the space-time normal vector n
  - $\lambda(u) = \frac{d}{du}(\mathcal{F}(u) \cdot n)$

Boundary Conditions

- The boundary conditions on different parts of $\partial\mathcal{E}$ are written as

$$
\begin{aligned}
u &= u_0 \quad \text{on } \Omega_0, \\
u &= g_D \quad \text{on } \Gamma_D, \\
\alpha u + n \cdot (A \nabla u) &= g_M \quad \text{on } \Gamma_M,
\end{aligned}
$$

- $\alpha \geq 0$ and u_0, g_D, g_M given functions defined on the boundary.
- There is no boundary condition imposed on Γ_+.



Space-time slab $\mathcal{E}^n$ with space-time element $\mathcal{K}$.

Finite Element Spaces

- To each element $\mathcal{K}$ we assign a pair of nonnegative integers $p_{\mathcal{K}} = (p_{t,\mathcal{K}}, p_{s,\mathcal{K}})$ as local polynomial degrees
- Define $\mathcal{Q}_{p_{t,\mathcal{K}},p_{s,\mathcal{K}}}(\hat{\mathcal{K}})$ as the set of tensor-product polynomials on $\hat{\mathcal{K}}$ of degree $p_{t,\mathcal{K}}$ in the time direction and degree $p_{s,\mathcal{K}}$ in each spatial coordinate direction
- Define the finite element spaces of discontinuous piecewise polynomial functions as:

$$V_h^{(p_t,p_s)} := \{v \in L^2(\mathcal{E}) : v|_{\mathcal{K}} \circ G_{\mathcal{K}} \in \mathcal{Q}_{(p_{t,\mathcal{K}},p_{s,\mathcal{K}})}(\hat{\mathcal{K}}), \forall \mathcal{K} \in \mathcal{T}_h\}$$

$$\Sigma_h^{(p_t,p_s)} := \{\tau \in L^2(\mathcal{E})^{d+1} : \tau|_{\mathcal{K}} \circ G_{\mathcal{K}} \in [\mathcal{Q}_{(p_{t,\mathcal{K}},p_{s,\mathcal{K}})}(\hat{\mathcal{K}})]^{d+1}, \forall \mathcal{K} \in \mathcal{T}_h\}$$

Trace Operators

- The so called traces of $v \in V_h^{(p_t,p_s)}$ on $\partial\mathcal{K}$ are defined as:

$$v_{\mathcal{K}}^{\pm} = \lim_{\epsilon \downarrow 0} v(x \pm \epsilon n_{\mathcal{K}})$$

- The traces of $\tau \in \Sigma_h^{(p_t,p_s)}$ are defined similarly.

- Note that functions $v \in V_h^{(p_t,p_s)}$ and $\tau \in \Sigma_h^{(p_t,p_s)}$ are in general multivalued on a face $S \in \mathcal{F}_{\text{int}}$.

Average and Jump Operators

- Introduce the functions $v_i := v|_{\mathcal{K}_i}$, $\tau_i := \tau|_{\mathcal{K}_i}$, $n_i := n|_{\partial\mathcal{K}_i}$
- The average operator on $S \in \mathcal{F}_{\mathrm{int}}$ is defined as:

$$\{\!\{v\}\!\} = \frac{1}{2}(v_i^- + v_j^-), \quad \{\!\{\tau\}\!\} = \frac{1}{2}(\tau_i^- + \tau_j^-), \quad \text{on } S \in \mathcal{F}_{\mathrm{int}},$$

- The jump operator on $S \in \mathcal{F}_{\mathrm{int}}$ is defined as:

$$[\![v]\!] = \; v_i^- \; n_i \; + \; v_j^- \; n_j, \quad [\![\tau]\!] = \tau_i^- \cdot n_i + \tau_j^- \cdot n_j, \text{ on } S \in \mathcal{F}_{\mathrm{int}},$$

with i and j the indices of the elements $\mathcal{K}_i$ and $\mathcal{K}_j$ which connect to the face $S \in \mathcal{F}_{\mathrm{int}}$.

Average and Jump Operators

- On a face $S \in \mathcal{F}_{\text{bnd}}$, the average and jump operators on $S \in \mathcal{F}_{\text{bnd}}$ are defined as:

$$\{\!\{v\}\!\} = v^-, \qquad \{\!\{\tau\}\!\} = \tau^-,$$

$$[\![v]\!] = v^- \, n, \qquad [\![\tau]\!] = \tau^- \cdot n$$

- Note that the jump $[\![v]\!]$ is a vector parallel to the normal vector n and the jump $[\![\tau]\!]$ is a scalar quantity.

- We also need the spatial jump operator $\langle\!\langle\cdot\rangle\!\rangle$ for functions $v \in V_h^{(p_t,p_s)}$, which is defined as:

$$\langle\!\langle v\rangle\!\rangle = v_i^- \bar{n}_i + v_j^- \bar{n}_j, \quad \text{on } S \in \mathcal{F}_{\text{int}}, \qquad \langle\!\langle v\rangle\!\rangle = v^- \, \bar{n}, \quad \text{on } S \in \mathcal{F}_{\text{bnd}}.$$

Space-Time DG Discretization

- Introduce an auxiliary variable $\sigma = A\nabla u$ to obtain the following system of first order equations:

$$\begin{aligned} \sigma &= A\nabla u, \\ -\nabla \cdot (-\mathcal{F}(u) + \sigma) &= 0. \end{aligned}$$

Weak Formulation for Auxiliary Variable

- Multiply the auxiliary equation with an arbitrary test function $\tau \in \Sigma_h^{(p_t, p_s)}$ and integrate over an element $\mathcal{K} \in \mathcal{T}_h$

$$\int_{\mathcal{K}} \sigma \cdot \tau \, \mathrm{d}\mathcal{K} = \int_{\mathcal{K}} A \nabla u \cdot \tau \, \mathrm{d}\mathcal{K}, \quad \forall \tau \in \Sigma_h^{(p_t, p_s)}$$

- Substitute σ and u with their numerical approximation and integrate by parts twice and sum over all elements:

$$\int_{\mathcal{E}} \sigma_h \cdot \tau \, \mathrm{d}\mathcal{E} = \int_{\mathcal{E}} A \nabla_h u_h \cdot \tau \, \mathrm{d}\mathcal{E} + \sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\partial\mathcal{K}} A(\hat{u}_h - u_h^-) n \cdot \tau^- \, \mathrm{d}\partial\mathcal{K}$$

- The variable $\hat{u}_h$ is the *numerical flux* that must be introduced to account for the multivalued trace on $\partial\mathcal{K}$.

Weak Formulation for Auxiliary Variable

- The following relation holds for vectors τ and scalars ϕ, piecewise smooth on $\mathcal{T}_h$:

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}} (\tau \cdot n)\phi \, \mathrm{d}\partial\mathcal{K} = \sum_{S\in\mathcal{F}} \int_S \{\!\{\tau\}\!\} \cdot [\![\phi]\!] \, \mathrm{d}S + \sum_{S\in\mathcal{F}_{\mathrm{int}}} \int_S [\![\tau]\!] \{\!\{\phi\}\!\} \, \mathrm{d}S$$

- Using the symmetry of the matrix A, the last contribution in the auxiliary equation then results in

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}} A(\hat{u}_h - u_h^-)n \cdot \tau^- \, \mathrm{d}\partial\mathcal{K}$$
$$= \sum_{S\in\mathcal{F}} \int_S \{\!\{A\tau\}\!\} \cdot [\![\hat{u}_h - u_h]\!] \, \mathrm{d}S + \sum_{S\in\mathcal{F}_{\mathrm{int}}} \int_S \{\!\{\hat{u}_h - u_h\}\!\} [\![A\tau]\!] \, \mathrm{d}S$$

Numerical Fluxes for Auxiliary Equation

- The following numerical fluxes result in a consistent and conservative scheme with a sparse matrix:

$$
\begin{aligned}
\hat{u}_h &= \{\!\{u_h\}\!\} && \text{on } S \in \mathcal{F}_{\text{int}}, \\
\hat{u}_h &= g_D && \text{on } S \in \cup_n \mathcal{S}_D^n, \\
\hat{u}_h &= u_h^- && \text{elsewhere.}
\end{aligned}
$$

- Note that on faces $S \in \mathcal{S}_S^n$, which are the element boundaries K^n and K^{n+1}, the normal vector n has values $n = (\pm 1, \underbrace{0, \ldots, 0}_{d\ \times})$ and thus $An = (\underbrace{0, \ldots, 0}_{(d+1)\ \times})$. Hence there is no coupling between the space-time slabs.

Numerical Fluxes for Auxiliary Equation

- Substitute the numerical flux into the auxiliary equation and use that A contains continuous functions, we obtain for each space-time slab $\mathcal{E}^n$:

$$\sum_{\mathcal{K}\in\mathcal{T}_h^n} \int_{\partial\mathcal{K}} A(\hat{u}_h - u_h^-) n \cdot \tau^- \, \mathrm{d}\partial\mathcal{K}$$
$$= - \sum_{S\in\mathcal{S}_{ID}^n} \int_S [\![u_h]\!] \cdot A\{\!\{\tau\}\!\} \, \mathrm{d}S + \sum_{S\in\mathcal{S}_D^n} \int_S g_D n \cdot A\tau \, \mathrm{d}S.$$

- Summing over all space-time slabs and using the symmetry of matrix A we can introduce the lifting operator to obtain

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}} A(\hat{u}_h - u_h^-) n \cdot \tau^- \, \mathrm{d}\partial\mathcal{K} = \int_{\mathcal{E}} AR_{ID}([\![u_h]\!]) \cdot \tau \, \mathrm{d}\mathcal{E}$$

Lifting Operators

- Define the global lifting operator $R_{ID} : (L^2(\cup_n \mathcal{S}_{ID}^n))^{d+1} \to \Sigma_h^{(p_t,p_s)}$ as:

$$R_{ID}(\phi) = R(\phi) - R(\mathcal{P} g_D n)$$

- Define the global lifting operator $R : (L^2(\cup_n \mathcal{S}_{ID}^n))^{d+1} \to \Sigma_h^{(p_t,p_s)}$ as:

$$\int_{\mathcal{E}} R(\phi) \cdot q \, \mathrm{d}\mathcal{E} = -\sum_{S} \int_{S} \phi \cdot \{\!\{q\}\!\} \, \mathrm{d}S, \quad \forall q \in \Sigma_h^{(p_t,p_s)}, \forall S \in \cup_n \mathcal{S}_{ID}^n.$$

Lifting Operators

- Using the symmetry of the matrix A, the lifting operator R_{ID} satisfies the relation:

$$\int_{\mathcal{E}} AR_{ID}([\![u_h]\!]) \cdot \tau \, \mathrm{d}\mathcal{E}$$

$$= - \sum_{S \in \cup_n \mathcal{S}^n_{ID}} \int_S A[\![u_h]\!] \cdot \{\!\{\tau\}\!\} \, \mathrm{d}S + \sum_{S \in \cup_n \mathcal{S}^n_{D}} \int_S A g_D n \cdot \tau \, \mathrm{d}S$$

Numerical Fluxes for Auxiliary Equation

- Combine all terms, then we obtain for all $\tau \in \Sigma_h^{(p_t,p_s)}$:

$$\int_{\mathcal{E}} \sigma_h \cdot \tau \, \mathrm{d}\mathcal{E} = \int_{\mathcal{E}} A\nabla_h u_h \cdot \tau \, \mathrm{d}\mathcal{E} + \int_{\mathcal{E}} AR_{ID}([\![u_h]\!]) \cdot \tau \, \mathrm{d}\mathcal{E},$$

- This implies that we can express $\sigma_h \in \Sigma_h^{(p_t,p_s)}$ as:

$$\sigma_h = A\nabla_h u_h + AR_{ID}([\![u_h]\!]) \quad \text{a.e. } \forall x \in \mathcal{E}.$$

Weak Formulation for Parabolic Scalar Conservation Laws

- The weak formulation for parabolic scalar conservation laws can be expressed as:

  Find a $u_h \in V_h^{(p_t,p_s)}$, such that $\forall v \in V_h^{(p_t,p_s)}$ the following relation is satisfied:

$$\int_{\mathcal{E}} (-\mathcal{F}(u_h) + \sigma_h) \cdot \nabla_h v \, \mathrm{d}\mathcal{E} - \sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\partial \mathcal{K}} (-\hat{\mathcal{F}}_h + \hat{\sigma}_h) \cdot n v^- \, \mathrm{d}\partial\mathcal{K} = 0.$$

- Here we replaced $\mathcal{F}(u_h)$, σ_h on $\partial\mathcal{K}$ with the numerical fluxes $\hat{\mathcal{F}}_h$, $\hat{\sigma}_h$, to account for the multivalued traces on $\partial\mathcal{K}$.

Numerical Fluxes

- Separate the numerical fluxes into an convective flux $\hat{\mathcal{F}}_h$ and a diffusive flux $\hat{\sigma}_h$.

- For the convective flux, the obvious choice is an upwind flux. Here we use the Local Lax-Friedrichs flux for convenience:

$$\hat{\mathcal{F}}_h(u_h^-, u_h^+) = \{\!\{\mathcal{F}(u_h)\}\!\} + C_S[\![u_h]\!]$$

- The parameter C_S is chosen as:

$$C_S = \max_{u \in [u_h^-, u_h^+]} |\lambda(u)| \quad \text{on } S \in \mathcal{F}_{\text{int}}$$

with $\lambda(u) = \frac{d}{du}(\mathcal{F}(u) \cdot n)$.

Convective Numerical Fluxes

- After summation over all elements we obtain:

$$\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\partial\mathcal{K}}(\{\!\{\mathcal{F}(u_h)\}\!\} + C_S[\![u_h]\!])\cdot nv^- \,\mathrm{d}\partial\mathcal{K}$$

$$= \sum_{S\in\mathcal{F}_{\mathrm{int}}}\int_S(\{\!\{\mathcal{F}(u_h)\}\!\} + C_S[\![u_h]\!])\cdot[\![v]\!]\,\mathrm{d}S + \sum_{S\in\mathcal{F}_{\mathrm{bnd}}}\int_S \mathcal{F}(u_h)\cdot nv\,\mathrm{d}S$$

Numerical Fluxes for Auxiliary Variable

- Introduce, the diffusive flux $\hat{\sigma}_h = \{\!\{\sigma_h\}\!\}$, then after summation over all elements we obtain:

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}} \{\!\{\hat{\sigma}_h\}\!\} \cdot n v^- \, \mathrm{d}\partial\mathcal{K} = \sum_{S\in\mathcal{F}} \int_S \{\!\{\sigma_h\}\!\} \cdot [\![v]\!] \, \mathrm{d}S$$

- Recall also the relation

$$\sigma_h = A\nabla_h u_h + AR_{ID}([\![u_h]\!]) \quad \text{a.e. } \forall x \in \mathcal{E}.$$

DG Discretization for Primal Variable

- Combining all terms and eliminating σ_h, we obtain the DG formulation for u_h:

$$\int_{\mathcal{E}} \big(- \mathcal{F}(u_h) + A\nabla_h u_h + AR_{ID}([\![u_h]\!])\big) \cdot \nabla_h v \, \mathrm{d}\mathcal{E}$$

$$+ \sum_{S\in\mathcal{F}_{\text{int}}} \int_S (\{\!\{\mathcal{F}(u_h)\}\!\} + C_S[\![u_h]\!]) \cdot [\![v]\!] \, \mathrm{d}S + \sum_{S\in\mathcal{F}_{\text{bnd}}} \int_S \mathcal{F}_h(u_h) \cdot nv \, \mathrm{d}S$$

$$- \sum_{S\in\mathcal{F}} \int_S \big(A\{\!\{\nabla_h u_h\}\!\} + A\{\!\{R_{ID}([\![u_h]\!])\}\!\}\big) \cdot [\![v]\!] \, \mathrm{d}S = 0$$

Simplifying the DG Discretization

- The DG discretization can be simplified using the following steps.
- Recall the lifting operator R_{ID} satisfies the relation

$$\int_{\mathcal{E}} AR_{ID}([\![u_h]\!]) \cdot \nabla_h v \, d\mathcal{E}$$
$$= - \sum_{S \in \cup_n \mathcal{S}_{ID}^n} \int_S A[\![u_h]\!] \cdot \{\!\{\nabla_h v\}\!\} \, dS + \sum_{S \in \cup_n \mathcal{S}_{D}^n} \int_S A g_D n \cdot \nabla_h v \, dS$$

- The lifting operator R_{ID} has nonzero values only on faces $S \in \mathcal{S}_{ID}^n$.

Simplifying the DG Discretization

- Using the lifting operators R and R_{ID} we obtain:

$$
\begin{aligned}
&-\sum_{S\in\mathcal{F}}\int_S A\{\!\{R_{ID}([\![u_h]\!])\}\!\}\cdot[\![v]\!]\,\mathrm{d}S\\
&=\int_{\mathcal{E}} AR([\![u_h]\!])\cdot R([\![v]\!])\,\mathrm{d}\mathcal{E}-\int_{\mathcal{E}} AR(\mathcal{P}g_D n)\cdot R([\![v]\!])\,\mathrm{d}\mathcal{E}
\end{aligned}
$$

Lifting Operators

- Define the local lifting operator $r_S : (L^2(S))^{d+1} \rightarrow \Sigma_h^{(p_t,p_s)}$ as:

$$\int_{\mathcal{E}} r_S(\phi) \cdot q \, \mathrm{d}\mathcal{E} = -\int_S \phi \cdot \{\!\{q\}\!\} \, \mathrm{d}S, \quad \forall q \in \Sigma_h^{(p_t,p_s)}, \forall S \in \cup_n \mathcal{S}_{ID}^n.$$

- The support of the operator r_S is limited to the element(s) that share the face S.

Simplifying the DG Discretization

- Following the approach of Brezzi we replace each global lifting operator with the local lifting operators r_S, and make the following simplifications:

$$\int_{\mathcal{E}} AR([\![u_h]\!]) \cdot R([\![v]\!]) \, \mathrm{d}\mathcal{E} \cong \sum_{S \in \cup_n \mathcal{S}_{ID}^n} \sum_{\mathcal{K} \in \mathcal{T}_h} \eta_{\mathcal{K}} \int_{\mathcal{K}} Ar_S([\![u_h]\!]) \cdot r_S([\![v]\!]) \, \mathrm{d}\mathcal{K},$$

$$\int_{\mathcal{E}} AR(\mathcal{P} g_D n) \cdot R([\![v]\!]) \, \mathrm{d}\mathcal{E} \cong \sum_{S \in \cup_n \mathcal{S}_{D}^n} \sum_{\mathcal{K} \in \mathcal{T}_h} \eta_{\mathcal{K}} \int_{\mathcal{K}} Ar_S(\mathcal{P} g_D n) \cdot r_S([\![v]\!]) \, \mathrm{d}\mathcal{K}$$

- A sufficient condition for the constant $\eta_{\mathcal{K}}$ to guarantee a stable and unique solution is $\eta_{\mathcal{K}} > n_f$, with n_f the number of faces of an element.

- The advantage of this replacement is that the discretization matrix is considerably sparser than when the global lifting operators are used.

DG Discretization for Parabolic Scalar Conservation Laws

- Define the form $a_a : V_h^{(p_t,p_s)} \times V_h^{(p_t,p_s)} \to \mathbb{R}$ $a_d : V_h^{(p_t,p_s)} \times V_h^{(p_t,p_s)} \to \mathbb{R}$:

$$a_a(u_h, v) = -\int_{\mathcal{E}} \mathcal{F}(u_h) \cdot \nabla_h v \, \mathrm{d}\mathcal{E} + \sum_{S \in \mathcal{F}_{\text{int}}} \int_S (\{\!\{\mathcal{F}(u_h)\}\!\} + C_S [\![u_h]\!]) \cdot [\![v]\!] \, \mathrm{d}S$$
$$+ \sum_{S \in (\cup_n \mathcal{S}^n_{MDSp} \cup \Gamma_+)} \int_S \mathcal{F}(u_h) \cdot n v \, \mathrm{d}S,$$

- Define the bilinear form $a_d : V_h^{(p_t,p_s)} \times V_h^{(p_t,p_s)} \to \mathbb{R}$:

$$
\begin{aligned}
a_d(u_h, v) = &\int_{\mathcal{E}} D\overline{\nabla}_h u_h \cdot \overline{\nabla}_h v \, \mathrm{d}\mathcal{E} \\
&- \sum_{S \in \cup_n \mathcal{S}_{ID}^n} \int_S \left(D\langle\!\langle u_h \rangle\!\rangle \cdot \{\!\{\overline{\nabla}_h v\}\!\} + D\{\!\{\overline{\nabla}_h u_h\}\!\} \cdot \langle\!\langle v \rangle\!\rangle \right) \mathrm{d}S \\
&+ \sum_{S \in \cup_n \mathcal{S}_{ID}^n} \sum_{\mathcal{K} \in \mathcal{T}_h} \eta_{\mathcal{K}} \int_{\mathcal{K}} D\bar{r}_S([\![u_h]\!]) \cdot \bar{r}_S([\![v]\!]) \, \mathrm{d}\mathcal{K} \\
&+ \sum_{S \in \cup_n \mathcal{S}_M^n} \int_S \alpha u_h v \, \mathrm{d}S,
\end{aligned}
$$

- Define $\ell : V_h^{(p_t,p_s)} \to \mathbb{R}$ as:

$$\begin{aligned}\ell(v) = &- \sum_{S \in \cup_n \mathcal{S}_D^n} \int_S g_D D\bar{n} \cdot \overline{\nabla}_h v \, \mathrm{d}S \\ &+ \sum_{S \in \cup_n \mathcal{S}_D^n} \sum_{\mathcal{K} \in \mathcal{T}_h} \eta_{\mathcal{K}} \int_{\mathcal{K}} D\bar{r}_S(\mathcal{P} g_D n) \cdot \bar{r}_S([\![v]\!]) \, \mathrm{d}\mathcal{K} + \sum_{S \in \cup_n \mathcal{S}_M^n} \int_S g_M v \, \mathrm{d}S \\ &- \sum_{S \in \cup_n \mathcal{S}_{DBSm}^n} \int_S \mathcal{F}(g_D) \cdot n v \, \mathrm{d}S + \int_{\Omega_0} c_0 v \, \mathrm{d}\Omega.\end{aligned}$$

DG Discretization for Parabolic Scalar Conservation Laws

- Note, we introduced the following boundary and initial conditions in the DG discretization:

$$
\begin{aligned}
D\overline{\nabla}_h u_h \cdot \bar{n} = g_M - \alpha u_h \quad & \text{on } S \in \cup_n \mathcal{S}_M^n, \\
u_h = g_D \quad & \text{on } S \in \cup_n \mathcal{S}_{DBSm}^n, \\
u_h = u_0 \quad & \text{on } \Omega_0,
\end{aligned}
$$

- The space-time DG discretization for the parabolic scalar conservation law can now be formulated as:

Find a $u_h \in V_h^{(p_t,p_s)}$, such that $\forall v \in V_h^{(p_t,p_s)}$ the following relation is satisfied:

$$
a(u_h, v) = \ell(v)
$$

ALE DG Formulation

- On faces $S \in \mathcal{S}_S^n$, the space-time normal vector is equal to:

$$n = (\pm 1, \underbrace{0, \ldots, 0}_{d\ \times})$$

  and is not affected by the mesh velocity.

- On the faces $S \in \mathcal{S}_I^n$ the space-time normal vector depends on the mesh velocity u_g:

$$n = (-u_g \cdot \bar{n}, \bar{n}),$$

  which also holds on the boundary faces $S \in \mathcal{F}_{\text{bnd}} \setminus (\Omega_0 \cup \Omega_T)$.

ALE DG Formulation

- On $S \in \cup_n \mathcal{S}_I^n$, the flux can be written in the ALE formulation as:

$$\{\!\{\mathcal{F}(u_h)\}\!\} \cdot [\![v]\!] = \{\!\{f(u_h) - u_g u_h\}\!\} \cdot \langle\!\langle v \rangle\!\rangle,$$

- All other contributions are not affected by the mesh velocity.

- The form $a_a(\cdot,\cdot)$ in the ALE formulation is now equal to:

$$
\begin{aligned}
a_a(u_h, v) = &-\int_{\mathcal{E}} \mathcal{F}(u_h)\cdot\nabla_h v \, \mathrm{d}\mathcal{E} \\
&+ \sum_{S\in\cup_n\mathcal{S}_I^n} \int_S \left(\{\!\{f(u_h) - u_g u_h\}\!\}\cdot\langle\!\langle v\rangle\!\rangle + C_S[\![u_h]\!]\cdot[\![v]\!]\right) \mathrm{d}S \\
&+ \sum_{S\in\cup_n\mathcal{S}_S^n} \int_S \left(\{\!\{\mathcal{F}(u_h)\}\!\} + C_S[\![u_h]\!]\right)\cdot[\![v]\!] \, \mathrm{d}S \\
&+ \sum_{S\in(\cup_n\mathcal{S}_{MDSp}^n\cup\Gamma_+)} \int_S (f(u_h) - u_g u_h)\cdot\bar{n}v \, \mathrm{d}S,
\end{aligned}
$$

ALE DG Formulation

- The linear form $\ell(\cdot)$ in the ALE formulation is now equal to:

$$
\begin{aligned}
\ell(v) = &- \sum_{S \in \cup_n \mathcal{S}_D^n} \int_S g_D D\bar{n} \cdot \overline{\nabla}_h v \,\mathrm{d}S \\
&+ \sum_{S \in \cup_n \mathcal{S}_D^n} \sum_{\mathcal{K} \in \mathcal{T}_h} \eta_{\mathcal{K}} \int_{\mathcal{K}} D\bar{r}_S(\mathcal{P} g_D n) \cdot \bar{r}_S([\![v]\!]) \,\mathrm{d}\mathcal{K} + \sum_{S \in \cup_n \mathcal{S}_M^n} \int_S g_M v \,\mathrm{d}S \\
&- \sum_{S \in \cup_n \mathcal{S}_{DBSm}^n} \int_S (f(g_D) - g_D u_g) \cdot \bar{n} v \,\mathrm{d}S + \int_{\Omega_0} c_0 v \,\mathrm{d}\Omega,
\end{aligned}
$$

- The bilinear form $a_d(\cdot, \cdot)$ is not influenced by the mesh velocity.

Conclusions

The main properties of space-time discontinuous Galerkin finite elements methods can be summarized as:

- The space-time discontinuous Galerkin finite element method results in a very local, element wise discretization, which has as benefits:
  - the space-time discretization automatically satisfies the geometric conservation law for deforming elements
  - efficient grid adaptation using local grid refinement, no complications caused by hanging nodes and gradient reconstruction
  - combines very well with unstructured grids
  - boundary conditions can be easily implemented

Conclusions

- no special numerical treatment is required to achieve higher order accuracy
- no interpolation is necessary after remeshing or local mesh refinement, only time fluxes need to be transferred
- maintains accuracy on irregular grids
- efficient parallelization

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