Chap.9 Fixed points and exponents

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The fixed point and its neighborhood

- A point in the parameter space μ^* which is invariant under R_s will be called a fixed point

$$R_s \mu^* = \mu^*$$

- We should carefully choose the value a in $\lambda_s = s^a$ or the equation will not have a solution
- There is no general theorem to tell us whether there is a discrete or a continuous set of fixed points, or any fixed point at all, let's assume now there is at least one fixed point

The fixed point and its neighborhood

- Critical surface
  - All points μ in this subspace have the property

$$\lim_{s\to\infty} R_s \mu = \mu^*$$

  - R_s drives μ to a different point
  - As s increase, all points in the critical surface are eventually driven to μ^*
  - For sufficiently large s, $R_s \mu$ will be in the immediate neighborhood of μ^*

The fixed point and its neighborhood

- Neighborhood of fixed points
  - For a point μ near μ^* ,we write

$$\mu = \mu^* + \delta\mu$$

  - The equation $\mu' = R_s \mu$ can be written as

$$\delta\mu' = R_s{}^L \delta\mu + O((\delta\mu)^2)$$

  - Define a matrix

$$(R_s{}^L)_{\alpha\beta} = \left(\frac{\partial \mu'_\alpha}{\partial \mu_\beta} \right)_{\mu=\mu^*}$$

The fixed point and its neighborhood

- Neighborhood of fixed points
  - Then we can write

$$\delta\mu' = \sum_{\beta} \delta\mu_{\beta} \left(\frac{\partial \mu'_{\alpha}}{\partial \mu_{\beta}} \right)_{\mu=\mu^*}$$

  - Next to determine the eigenvalues and eigenvectors of $R_s^{\;L}$
  - Suppose the eigenvalues are found to be $\rho_j(s)$ and the corresponding eigenvectors to be e_j
  - Since $R_s R_{s'} e_j = R_{ss'} e_j$

$$\rho_j(s)\rho_j(s') = \rho_j(ss') \quad \text{and} \quad \rho_j(s) = s^{y_j}$$

The fixed point and its neighborhood

- Neighborhood of fixed points
  - Then we use the eigenvectors as a set of basis vectors and write

$$\delta\mu = \sum_j t_j e_j, \delta\mu' = \sum_j t_j' e_j, t_j' = t_j s^{y_j}$$

  - If $y_j > 0$, t_j' will grow as s increases
  - If $y_j < 0$, t_j' will diminish
  - If $y_j = 0$, t_j' will not change
  - The existence of one or more zero y_j implies that there is a continuous set of fixed points.

Large s behavior of R_s and critical exponents

- RG: some transformation of probability distributions
- Interpret $P \propto e^{-H[\sigma]}$ in parameter space u
- In the language of Ising $\mu = (T, h)$
- The critical point of the ferromagnet is given by

$$T = T_c, h = 0$$

- Fundamental hypothesis linking RG to critical phenomena
  - $\mu(T_c, 0)$ is a point on the critical surface of a fixed point

$$\lim_{s \to \infty} R_s \mu(T_c, 0) = \mu^*$$

Large s behavior of R_s and critical exponents

- This hypothesis states that if we decrease the magnification by a sufficiently large amount, we shall not see any change if it is decreased further
- Restrict our discussion to h=0, that means there will be no odd power of σ_x
- Let $\mu(T)$ denote $\mu(T,0)$
- For sufficient small $T-T_c$, $\mu(T)$ must be very close to the critical surface and $R_s\mu(T)$ will move toward μ^* as s increases
- However as $s\rightarrow\infty$, $R_s\mu(T)$ will go away from μ^* since $\mu(T)$ is not quite on the critical surface

Large s behavior of R_s and critical exponents

- The manner $R_s\mu(T)$ goes away from μ^* depends on the positive y_j 's ,we call one of them y_1 , then

$$\delta\mu(T) = \mu(t) - \mu^* = \sum_j t_j(T)e_j$$

- For very large s, we have

$$R_s(T) \approx \mu^* + R_s^L \delta\mu(T)$$
$$= \mu^* + t_1 s^{y_1} e_1 + O(s^{y_2})$$

- Expand $t_1(T)$

$$t_1(T) = A(T - T_c) + B(T - T_c)^2 + \cdots$$

Large s behavior of R_s and critical exponents

- Then for very small $T-T_c$

$$R_s(T) \approx \mu^* + A(T-T_c)s^{y_1}e_1 + O(s^{y_2})$$
$$= \mu^* + (s/\xi)^{1/\upsilon} + O(s^{y_2})$$

- Where we have defined

$$1/\upsilon = y_1, \xi = |A(T-T_c)|^{-\upsilon}$$

- We apply this to examine the temperature and k dependence of $G(k,\mu(T))$ using the previous relation $\lambda_s = s^a$ and $G(k,\mu) = \lambda_s{}^2 s^d G(sk,\mu')$

Large s behavior of R_s and critical exponents

- We get

$$G(k,\mu(T)) = s^{2a+d} G(sk, R_s\mu(T))$$
$$= s^{2a+d} G(sk, \mu^* \pm (s/\xi)^{1/\upsilon} e_1 + O(s^{y_2}))$$

- Since this is true for any value of s provided s is very large, we can set $s=\xi$ to obtain

$$G(k,\mu(t)) = \xi^{2a+d} G(\xi k, \mu^* \pm e_1 + O(\xi^{y_2}))$$
$$\approx \xi^{2a+d} g(\xi k)$$

Large s behavior of R_s and critical exponents

- By the definition of η ,we see that

$$2-\eta = 2a + d$$

$$a = (2-\eta-d)/2$$

- Setting k=0 for sufficiently small $|T-T_c|$

$$G(0,\mu(T)) \propto \xi^{2-\eta} G(0,\mu^* \pm e_1 + O(\xi^{y_2})) \propto |T-T_c|^{-\nu(2-\eta)}$$

- From the definition of γ we get a previously obtained scaling law

$$\gamma = \nu(2-\eta)$$

Large s behavior of R_s and critical exponents

- Now we generalize the above arguments to $h \neq 0$
  The field h appears as an entry in μ representing the term
  $$hb^d \sum_x \sigma_x$$
- The transformation of h under R_s can be easily worked out
  $$h(sb)^d \sum_{x'} \lambda_s \sigma_{x'} = hs^{+a} b \sum_{x'} \sigma_{x'}$$
  $$h' = hs^{\frac{1}{2}(d-\eta)+1}$$

Large s behavior of R_s and critical exponents

- While for free energy, it's not that straightforward to get the conclusion

$$F(\mu(T,h)) = s^{-d} F(R_s \mu(T,h)) + A(\mu(T,h),s)T + B(s)T$$

- The reason is that $G(k) = \langle |\sigma_k|^2 \rangle$ average values of long wavelength (small k) Fourier components of the spin configuration , On the other hand the free energy involves all Fourier components directly.

Chap.9 Gaussian fixed point and fixed point in 4- ε dimensions

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The Gaussian fixed point

- Ginzburg-Landau model
  - it is more convenient to use the wave vector representation.

$$H=\frac{1}{2}\int d^d x(r_0\sigma^2+\frac{1}{4}u\sigma^4+c\Delta\sigma)$$

$$=\frac{1}{2}\sum_k (r_0+ck^2)|\sigma_k|^2+\frac{u}{8}L^{-d}\sum_{k_1,k_2,k_3}\sigma_{k1}\sigma_{k2}\sigma_{k3}\sigma_{-k1-k2-k3}$$

  - Parameter space

$$\mu=(r_0,u,c)$$

The Gaussian fixed point

- Ginzburg-Landau model
  - RG transformation $\mu' = R_s \mu$

$$e^{-H'-AL^d} = \left[\int \delta\vartheta e^{-H} \right]_{\sigma_k \to s^{1-\eta/2}\sigma_{sk}}$$

$$2a + d = 2 - \eta \quad \delta\vartheta = \prod_{\Lambda/s<k<\Lambda} d\sigma_k$$

  - AL^d : additive constant generated by $\int d\sigma_k$
  - This integration is easy when $u = 0$ while $u \neq 0$ it becomes very difficult

The Gaussian fixed point

- Three fixed points for $u = 0$
  - When $u = 0$ the integration will be

$$H' = \frac{1}{2}\sum_{k<\Lambda/s} s^{2-\eta}(ck^2 + r_0)|\sigma_{sk}|^2$$

$$= \frac{1}{2}\sum_{k'<\Lambda} (r_0 s^{2-\eta} + cs^{-\eta}k'^2)|\sigma_{k'}|^2$$

$$AL^d = \frac{1}{2}\sum_{\Lambda/s<k<\Lambda} \ln\left(\frac{2\pi}{r_0 + ck^2}\right)$$

The Gaussian fixed point

- Three fixed points for $u = 0$
  - We can see H' is still a Ginzburg-Landau form with
  $$\mu' = (r_0', 0, c') = (r_0 s^{2-\eta}, 0, cs^{-\eta})$$
  - Set $\eta = 0$,we get Gaussian fixed point
  $$\mu^* = (0, 0, c)$$
    - The value of c is arbitrary

$$H_0{}^* = \frac{c}{2}\int d^d x(\Delta\varphi) = \frac{c}{2}\sum_{k<\Lambda} k^2 |\sigma_k|^2$$

$$e^{-H_0{}^*} = \prod_k e^{-\frac{c}{2}k^2|\sigma_k|^2}$$

The Gaussian fixed point

- Three fixed points for $u = 0$
  - Set $\eta = 2$ we obtain another fixed point

$$\mu_\infty * = (r_0, 0, 0)$$

  - The value of r_0 if arbitrary as long as $r_0 > 0$

$$H_\infty * = \frac{r_0}{2}\int d^d x \sigma_x{}^2 = \frac{r_0}{2}\sum_{k<\Lambda} |\sigma_k|^2$$

$$e^{-H_\infty *} = \prod_x e^{-\frac{r_0}{2}\sigma_x{}^2} = \prod_k e^{-\frac{r_0}{2}|\sigma_k|^2}$$

- For $r_0 < 0$,we get $\mu_{-\infty} * = (-|r_0|, 0^+, 0)$

$$e^{-H_{-\infty} *} = \prod_x e^{-\frac{|r_0|}{2}\sigma_x{}^2 - u\sigma^4(x)}$$

The Gaussian fixed point

- Physical meaning of the three fixed points
  - μ^*_{∞}:
    - Each block spin is independent of all others
    - Spin values tend to be zero because of the Gaussian potential
    - This is the infinite temperature fixed points
  - $\mu^*_{-\infty}$:
    - Each block spin is independent of all others
    - Because of small u and negative r_0 ,spins tend to take a large value
    - This is the zero temperature fixed points

The Gaussian fixed point

- Physical meaning of the three fixed points

  - μ^*_0 :

    - $c\Delta\varphi$ accounts for coupling between neighboring block spins
    - Gaussian distributions for σ_k

The linearized RG near the Gaussian fixed point

- Perturbation near fixed points

$$\begin{cases} H = H^* + \Delta H \\ H' + AL^d = H^* + A^* L^d + \Delta H' + \Delta A L^d \end{cases}$$

  - Then we have

$$e^{-H'-AL^d} = \left[\int \delta\vartheta e^{-H}\right]_{\sigma_k \to s^{1-\eta/2}\sigma_{sk}}$$

$$e^{-H^*+AL^d}(1-\Delta H' - \Delta A L^d) = \left[\int \delta\vartheta e^{-H^*}(1-\Delta H)\right]_{\sigma_k \to s^{1-\eta/2}\sigma_{sk}}$$

The linearized RG near the Gaussian fixed point

- Perturbation near fixed points
  - Before preceding, we remark that in general one can break H into two pieces in *any* manner

$$H = H_0 + H_1$$

  - And obtain

$$H' + AL^d = -\ln\left(\int \delta\vartheta e^{-H}\right)_{\sigma_k \to s^{1-\eta/2}\sigma_{sk}}$$

$$= \left[-\ln\int \delta\vartheta e^{-H_0} - \sum_{m=1}^{\infty}\frac{(-1)^m}{m!} < H_1^m >_c\right]_{\sigma_k \to s^{1-\eta/2}\sigma_{sk}}$$

The linearized RG near the Gaussian fixed point

- Perturbation near fixed points
  - Note that the subscript c means taking the cumulant

$$< H_1^m >_c = < (H_1 - < H_1 >)^m >$$

$$< H_1^m > = \frac{\int \delta\vartheta e^{-H_0} H_1^m}{\int \delta\vartheta e^{-H_0}}$$

  - First two cumulants are

$$< H_1 >_c = < H_1 >$$

$$< H_1^2 >_c = < H_1^2 > - < H_1 >^2$$

The linearized RG near the Gaussian fixed point

- Perturbation near fixed points

$$\Delta H' - \Delta A L^d = \left\langle \Delta H \right\rangle_{\sigma_k \to s^{1-\eta/2}\sigma_{sk}}$$

$$\Delta H = \frac{\int \delta\vartheta e^{-H^*}\Delta H}{\int \delta\vartheta e^{-H^*}}$$

- The average is take over ϑ ,i.e. over σ_k with $\Lambda / s < k < \Lambda$
- One can take high-order powers of expansions

The linearized RG near the Gaussian fixed point

- GL model near Gaussian fixed point
  - The Gaussian fixed point

$$e^{-H^*} = e^{-\frac{1}{2}\int d^d x \Delta\varphi} = \prod_k e^{-\frac{c}{2}k^2|\sigma_k|^2}$$

  - For GL model

$$\langle \Delta H \rangle = \frac{1}{2}\int d^d x (r_0 < \sigma^2 > + \frac{1}{4} u < \sigma^4 >)$$

  - Separate ϑ from σ by defining

$$\sigma = \sigma' + \vartheta$$

The linearized RG near the Gaussian fixed point

- GL model near Gaussian fixed point
  - Where

$$\begin{cases} \sigma' = L^{-d/2} \sum\limits_{k<\Lambda/s} \sigma_k e^{ik\cdot x} \\ \vartheta = L^{-d/2} \sum\limits_{\Lambda/s<k<\Lambda} \sigma_k e^{ik\cdot x} \end{cases}$$

$$< \sigma^2 > = < \sigma'^2 + 2\sigma'\cdot\vartheta + \vartheta^2 >$$

$$= \sigma'^2 + < \vartheta^2 >$$

$$< \sigma^4 > = < (\sigma'^2 + 2\sigma'\cdot\vartheta + \vartheta^2)^2 >$$

$$= \sigma'^4 + 2\sigma'\cdot < \vartheta^2 > + 4 < (\vartheta\cdot\sigma')^2 > + < \vartheta^4 >$$

$$= \sigma'^4 + 6\sigma'^2 < \vartheta^2 > + < \vartheta^4 >$$

The linearized RG near the Gaussian fixed point

- GL model near Gaussian fixed point
  - $$< \vartheta^2 > = L^{-d} \sum_{\Lambda/s<k<\Lambda} <|\sigma_k|^2>$$

$$= (2\pi)^{-d} \int_{\Lambda/s}^{\Lambda} d^d k (1/ck^2)$$

$$= \frac{K_d}{c} \int_{\Lambda/s}^{\Lambda} dk\ k^{d-3}$$

$$= n_c (1 - s^{2-d})$$

The linearized RG near the Gaussian fixed point

- GL model near Gaussian fixed point
  - Where

$$n_c = K_d \Lambda^{d-2} / [c(d-2)]$$

$$K_d = 2^{-d+1} \pi^{-d/2} / \Gamma(\frac{1}{2} d)$$

  - K_d is the surface area of a unit sphere in d-dimensional space divided by $(2\pi)^d$

The linearized RG near the Gaussian fixed point

- GL model near Gaussian fixed point
  - $$<\vartheta^4> = \sum_{k_1,k_2,k_3} <\sigma_{k_1}\sigma_{k_2}\sigma_{k_3}\sigma_{-k_1-k_2-k_3}>$$

$$let\ k_1 = -k_2$$

$$= 3\sum_{k_1,k_3} <\sigma_{k_1}\sigma_{-k_1}\sigma_{k_3}\sigma_{-k_3}>$$

$$= 3\sum_{k_1,k_3} <\sigma_{k_1}\sigma_{-k_1}><\sigma_{k_3}\sigma_{-k_3}>$$

$$= 3n_c{}^2(1-s^{2-d})^2$$

The linearized RG near the Gaussian fixed point

- GL model near Gaussian fixed point
  - We get $<\sigma^2>=\sigma'^2+n_c(1-s^{2-d})$

    $<\sigma^4>=\sigma'^4+6n_c(1-s^{2-d})+3n_c{}^2(1-s^{2-d})^2$
  - Then

$$\Delta H=\frac{1}{2}\int d^d x(r_0<\sigma^2>+\frac{1}{4}u<\sigma^4>)$$

$$=\frac{1}{2}\int d^d x(r_0+\frac{3u}{2}n_c(1-s^{2-d})\sigma'^2+\frac{1}{4}u\sigma'^4+\Delta AL^d$$

$$\Delta AL^d=\frac{1}{2}[r_0 n_c(1-s^{2-d})+\frac{3}{4}un_c{}^2(1-s^{2-d})^2]$$

The linearized RG near the Gaussian fixed point

- GL model near Gaussian fixed point
  - Recall that the rescaling transformation

$$\sigma_x' \to \lambda \sigma_{x'} = s^y \sigma_{x'}$$

$$\sigma_k \to \lambda s^{d/2} \sigma_{sk} = s^{y+d/2} \sigma_{sk}$$

  - For Gaussian fixed point, we require

$$\sigma_k \to s^{1-\eta/2} \sigma_{sk} \quad \text{with} \quad \eta = 0$$

$$\text{so} \quad y = 1 - d/2$$

$$\begin{cases} \sigma_x' \to \sigma_{x'} s^{1-d/2} \\ x' = x/s \end{cases}$$

The linearized RG near the Gaussian fixed point

- GL model near Gaussian fixed point
  - Near Gaussian fixed point

$$\langle \Delta H \rangle_{\sigma_k \to s\sigma_{sk}} = \frac{1}{2}\int d^d x'(r_0'\sigma^2 + \frac{1}{4}u'\sigma^4) + \Delta A L^d$$

$$= \Delta H' + \Delta A L^d$$

with

$$\begin{cases} r_0' = s^2(r_0 + \frac{3}{2}u n_c(1 - s^{2-d})) \\ u' = s^{4-d} u \end{cases}$$

  - This is the formula for R_s^L within the parameter space $\mu(r_0, u, c)$ near the Gaussian fixed point

$\mu^*(0,0,c)$

The linearized RG near the Gaussian fixed point

- Renormalization exponents
  - Matrix representation of $R_s^{\,L}$

$$\begin{pmatrix} r_0{}' \\ u' \end{pmatrix} = R_s^{\,L} \begin{pmatrix} r_0 \\ u \end{pmatrix}$$

$$R_s^{\,L} = \begin{pmatrix} s^2 & \frac{3}{2} u n_c (s^2 - s^{4-d}) \\ 0 & s^{4-d} \end{pmatrix}$$

The linearized RG near the Gaussian fixed point

- Renormalization exponents
  - Matrix representation of R_s^L
    - Let $B=\frac{3}{2}un_c$,the eigenvectors are

$$\begin{cases} e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = r_0 \\ e_2 = \begin{pmatrix} -B \\ 1 \end{pmatrix} = -Br_0 + u \end{cases}$$

    - With eigenvalues

$$\begin{cases} t_1 = s^{y_1} = s^2 \\ t_2 = s^{y_2} = s^{4-d} \end{cases} \Rightarrow \begin{cases} y_1 = 1/\upsilon = 2 \\ y_2 = 4-d \end{cases}$$

The linearized RG near the Gaussian fixed point

- Renormalization exponents
  - The conclusions concerning critical behavior are valid if $y_2 < 0$ i.e. $d > d_{upper} = 4$
  - We say

$$\mu_0^* \quad \text{is stable for } d > 4$$

$$\mu_0^* \quad \text{is unstable for } d < 4$$

$$\mu_0^* \quad \text{is marginal for } d = 4$$

The linearized RG near the Gaussian fixed point

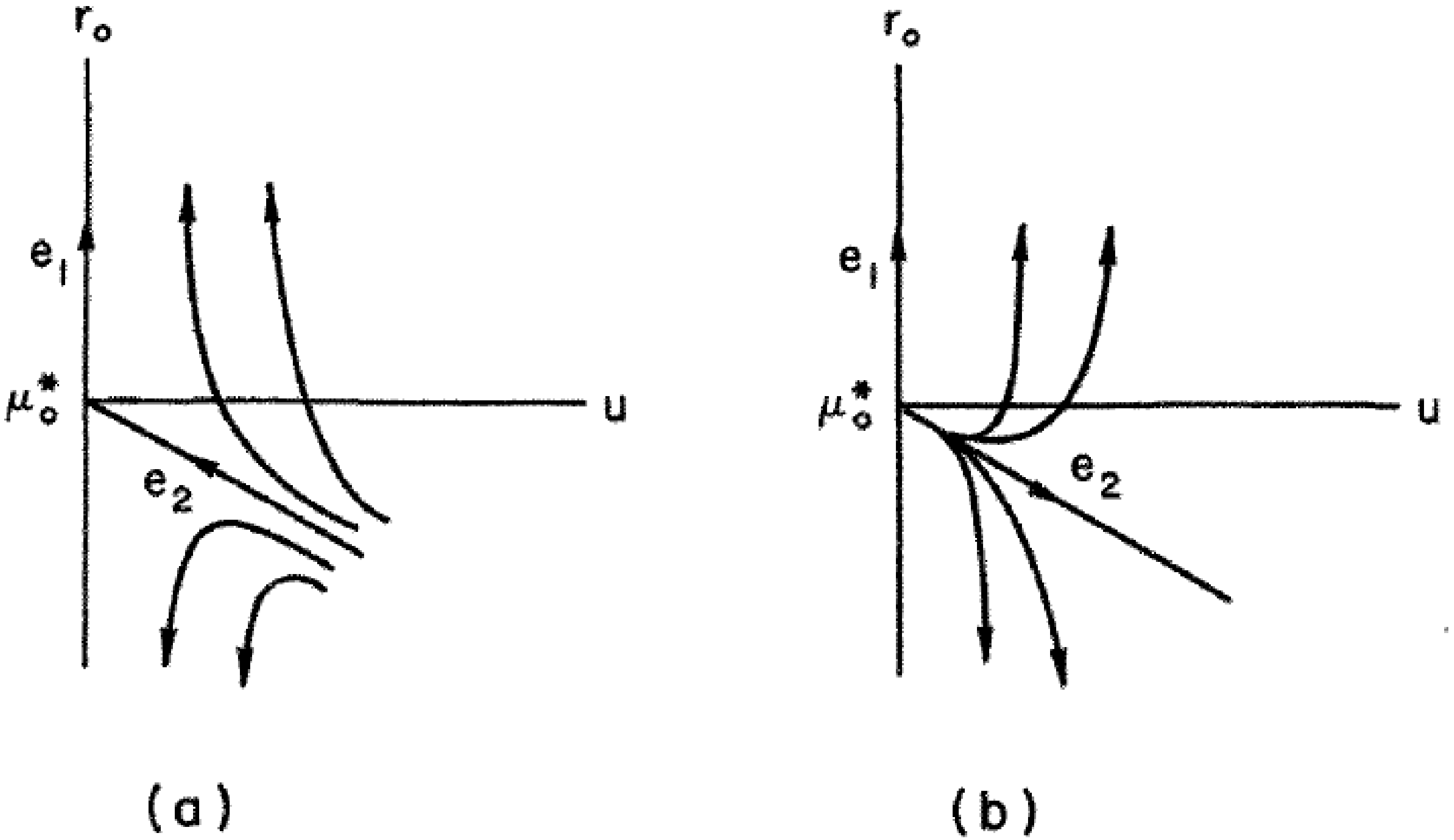


Figure 7.1. The Gaussian fixed point $r_o^* = 0$, $u^* = 0$, and the eigenvectors e_1 and e_2 of R_s^L. The flow lines and arrows show how $R_s\mu$ moves as s increases. (a) $d < 4$. (b) $2 < d < 4$.

Relevant, irrelevant and marginal parameters, scaling fields and crossover

- The parameters $t' = ts^{y_t}, h' = hs^{y_h}; y_t, y_h > 0$
  t, h are relevant scaling fields

$$u' = us^{4-d} \begin{cases} d > 4 & \text{irrelevant} \\ d = 4 & \text{marginal} \\ d < 4 & \text{relevant} \end{cases}$$

- Generally speaking, only the relevant scaling fields are relevant to critical behavior (not always right)

Relevant, irrelevant and marginal parameters, scaling fields and crossover

- The Hamiltonian near the fixed points can be expanded as

$$H = H^* + \sum_i t_i \theta_i + O(t_i^2)$$

- Scaling parameter $\theta_i = \frac{\partial H}{\partial t_i}$

$$\theta_t = \int d^d x \sigma^2$$

$$\theta_h = -\int d^d x \sigma$$

Relevant, irrelevant and marginal parameters, scaling fields and crossover

- A larger parameter space for the GL model

$$\int d^d x[u_6\sigma^6 + v_1(\nabla\sigma^2)^2 + v_2\sigma^2(\nabla\sigma^2)^2]$$

  - Here we have to extend the parameter space

$$\mu = (r_{0,} u, c, u_6, v_1, v_2)$$

  - The Gaussian fixed point remains the same and

$$\mu_0{}^* = (0, 0, c, 0, 0, 0)$$

  - By applying the same approximation procedure, the exponents of the linearized RG $R_s{}^L$ are

$$(2, 4-d, 0, 6-2d, 2-d, 2-d)$$

Relevant, irrelevant and marginal parameters, scaling fields and crossover

- A simple formula for term $\int t_i D_i d^d x$

$$y_i = (1-\frac{d}{2})\times(\text{power of } \sigma \text{ in } D_i) + (-1)\times(\text{power of } \nabla \text{ in } D_i) + d$$

- The more powers of σ and ∇, the more negative y_i and therefore the more irrelevant the corresponding parameter

Relevant, irrelevant and marginal parameters, scaling fields and crossover

- A simple formula for term $\int t_i D_i d^d x$

$$\begin{cases} d > 4 & \text{only } r_0 \text{ is relevant} \\ 3 > d > 2 & r_0, u_6 \text{ are relevant} \\ d < 2 & r_0, u_6, v_1, v_2 \cdots \text{are relevant} \end{cases}$$

> A fixed point is called unstable if there are one or more relevant parameters other than $t \propto (T - T_c)$

Critical exponents for $d > 4$

- Near Gaussian fixed point, we have $\upsilon = 1/2, \eta = 0$ and $y_2 < 0$ for $d > 4$
- Scaling relations

$$
\begin{cases}
\alpha = 2 - \nu d = 2 - d/2 \\
\beta = \dfrac{\nu}{2}(d - 2 + \eta) = \dfrac{1}{4}(d - 2) \\
\gamma = \nu(2 - \eta) = 1 \\
\delta = \dfrac{d + 2 - \eta}{d - 2 + \eta} = \dfrac{d + 2}{d - 2}
\end{cases}
$$

Critical exponents for $d > 4$

- And from the Gaussian approximation we get

$$\begin{cases} \alpha = 2 - d/2 \\ \beta = 1/2 \\ \gamma = 1 \\ \delta = 3 \end{cases}$$

- We expect that the Gaussian approximation is OK. Since the basic assumption is that u is small. Near the Gaussian fixed point u_0^*, u is indeed small

Critical exponents for $d > 4$

- What's the problem with the RG prediction?
  - The RG arguments

$$m(t,h,u) = s^{1-d/2} m(ts^2, hs^{1+d/2}, us^{4-d})$$

  - Set $s = |t_1|^{-1/2}$ and $h = 0$

$$m(t,0,u) = |t_1|^{\frac{1}{4}(d-2)} m(\pm 1, 0, u|t_1|^{\frac{1}{2}(d-4)})$$

  - The RG predictions implicitly assume $m(\pm 1, 0, u|t_1|^{\frac{1}{2}(d-4)})$ while the face is that $m(\pm 1, 0, u)$ for small u is $m \propto u^{-1/2}$ thus

Critical exponents for $d > 4$

- What's the problem with the RG prediction?
  - $$m(t,0,u) \propto |t_1|^{\frac{1}{4}(d-2)} \left(u |t_1|^{\frac{1}{2}(d-4)} \right)^{-1/2}$$
    $$\propto |t_1|^{\frac{1}{2}}$$
  - We see that $\beta = 1/2$
  - If we set $t_1 = 0, s = h^{-(d/2+1)^{-1}}$ applying the same procedure we notice that $m \propto u^{-1/3}$ for $t_1 = 0$ and we find $\delta = 3$

The RG for $d = 4 - \varepsilon$ and fixed points to $O(\varepsilon)$

- The Gaussian fixed point becomes unstable for d<4, we expect another fixed point u^* in addition to the Gaussian point $u_0{}^*$
- For d<4, performing the RG transformation, namely finding fixed points and linearized RG near it, becomes difficult
- Nevertheless, for $d = 4 - \varepsilon$ when ε is very small, we hope that it is sufficient to some lower orders of terms in perturbation expansions

The RG for $d = 4 - \varepsilon$ and fixed points to $O(\varepsilon)$

- $d < 4$: we have tow fixed points u^* and u_0^*
- $d \geq 4$: u^* and u_0^* merge (degeneracy of fixed points)
- The fixed point u^* should not be regarded as the continuation of the Gaussian fixed point.
- We expect that u^* is close to u_0^* ,because the growth $u' = s^{4-d} u = s^{\varepsilon} u$ as s increases can be held back by nonlinear terms so far not included in this linearized equation
- For small ε the growth rate is very small

The RG for $d = 4 - \varepsilon$ and fixed points to $O(\varepsilon)$

- We write a differential equation

$$u' = s^{\varepsilon} u$$

$$\frac{du'}{dl} = \varepsilon u' \qquad \text{with} \;\; l = \ln s$$

- If we have a nonlinear term

$$\frac{du'}{dl} = \varepsilon u' - g u'^2 \qquad \text{g constant}$$

$$\Rightarrow u^* = \begin{cases} 0 \\ \varepsilon / g \end{cases}$$

- If u^* is small, then $g = \text{constant} > O(\varepsilon)$

The RG for $d = 4 - \varepsilon$ and fixed points to $O(\varepsilon)$

- Method
  - The basic method is to use perturbation expansions up to $O(\varepsilon^2)$
  - The RG transformation

$$e^{-H' - AL^d} = \left[\int \delta\vartheta e^{-H} \right]_{\sigma_k \to s^{1-\eta/2}\sigma_{sk}}$$

$$\sigma = \sigma' + \vartheta \qquad \delta\vartheta = \prod_{\Lambda/s<k<\Lambda} d\sigma_k$$

$$\begin{cases} \sigma' = L^{-d/2} \sum\limits_{k<\Lambda/s} \sigma_k e^{ik\cdot x} \\ \vartheta = L^{-d/2} \sum\limits_{\Lambda/s<k<\Lambda} \sigma_k e^{ik\cdot x} \end{cases}$$

The RG for $d = 4 - \varepsilon$ and fixed points to $O(\varepsilon)$

- Method
  - The Hamiltonian $H = H_0 + H_1$

$$H' + A'L^d = H_0' + A_0'L^d + \left[\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m!} < H_1^m >_c \right]_{\sigma_k \to s^{1-\eta/2}\sigma_{sk}}$$

$$< H_1^m > = \frac{\int \delta\vartheta e^{-H_0} H_1^m}{\int \delta\vartheta e^{-H_0}}$$

  - And the cumulants are

$$< H_1 >_c = < H_1 >$$

$$< H_1^2 >_c = < H_1^2 > - < H_1 >^2 = < (H_1 - < H_1 >)^2 >$$

The RG for $d = 4 - \varepsilon$ and fixed points to $O(\varepsilon)$

- Method
  - The GL model

$$H_0 = \frac{1}{2}\int d^d x (r_0\sigma^2 + c(\nabla\sigma)^2) = \sum_{k<\Lambda} \frac{1}{2}(r_0 + ck^2)\sigma_k{}^2$$

$$H_1 = \frac{1}{2}\int d^d x \frac{u}{4}\sigma^4 = \frac{u}{8} L^{-d} \sum_{k_1,k_2,k_3} \sigma_{k_1}\sigma_{k_2}\sigma_{k_3}\sigma_{-k_1-k_2-k_3}$$

$$H' = \frac{1}{2}\sum_{k<\Lambda/s} s^{2-\eta}(ck^2 + r_0)\,|\,\sigma_{sk}\,|^2$$

$$= \frac{1}{2}\sum_{k'<\Lambda} (r_0{}^{2-\eta} + s^{-\eta} c k'^2)\,|\,\sigma_{k'}\,|^2$$

The RG for $d=4-\varepsilon$ and fixed points to $O(\varepsilon)$

- Method
  - The GL model

$$A_0' = -\frac{1}{2}\sum_{\Lambda/s<k<\Lambda} \ln\left(\frac{2\pi}{r_0+ck^2}\right)$$

$$<H_1>_c = \frac{u}{8}\int d^d x <\sigma^4>$$

$$<H_1^2>_c = -\frac{u^2}{128}\int d^d x\int d^d y \left\langle (<\sigma^4(x)-<\sigma(x)>^2>)(<\sigma^4(y)-<\sigma(y)>^2>)\right\rangle$$

  - Note that r_0 and u are of order $O(\varepsilon)$ and our aim is to evaluate $<H_1>_c$ and $<H_1^2>_c$ to an accuracy of $O(\varepsilon^2)$

The RG for $d=4-\varepsilon$ and fixed points to $O(\varepsilon)$

- Method
  - We need to evaluate $<\sigma^4>$ to order $O(\varepsilon)$
  - And evaluate $\langle (<\sigma^4(x)-<\sigma(x)>^2>)(<\sigma^4(y)-<\sigma(y)>^2>) \rangle$ to order 1. Therefore we set $d=4$ and $r_0=0$
  - Let $\sigma=\sigma'+\vartheta$

$$<\sigma^4>=\sigma'^4+6\sigma'^2<\vartheta^2>+<\vartheta^4>$$

$$<\vartheta^2>=\sum_{\Lambda>k>\Lambda/s}<\sigma_k\cdot\sigma_{-k}>=\sum_{\Lambda>k>\Lambda/s} d\sigma_k|\sigma_k|^2 e^{-\frac{1}{2}(r_0+ck^2)\sigma_k{}^2}$$

$$=\sum_{\Lambda>k>\Lambda/s}(r_0+ck^2)^{-1}=K_d\int_{\Lambda/s}^{\Lambda} dk k^{d-1}(r_0+ck^2)^{-1}$$

The RG for $d=4-\varepsilon$ and fixed points to $O(\varepsilon)$

- Method
  - Since $r_0 \sim O(\varepsilon)$,we expand

$$(r_0 + ck^2)^{-1} = k^{-2}/c - (k^{-4}/c^2)r_0$$

$$<\vartheta^2> = K_d \int_{\Lambda/s}^{\Lambda} dk(c^{-1}k^{d-3} + c^{-2}k^{d-5})$$

$$= n_c(1-s^{-2+\varepsilon}) - K_4 c^{-2} r_0 \ln s + O(\varepsilon^2)$$

$$<H_1> = \frac{1}{2}\int d^d x (r_0^{(1)}\sigma'^2 + \frac{u}{4}\sigma'^4)$$

$$r_0^{(1)} = \frac{3u}{4c}K_4\Lambda^2(1-s^{-2}) + uc\varepsilon - r_0 \cdot \frac{3u}{2c^2}K_4 \ln s$$

The RG for $d = 4-\varepsilon$ and fixed points to $O(\varepsilon)$

- Method
  - There is no term in $< H_1 >$ proportional to $(\nabla\sigma')^2$
  - That means c is unchanged up to $O(\varepsilon)$
  - Thus $\eta = 0$,in other words

$$\eta = O(\varepsilon^2)$$

  - After tedious calculations, which can become simpler with the help of Feynman diagram, and after scale transformation

$$\sigma_k \to s^{1-\eta/2}\sigma_{sk} = s\sigma_{sk}$$

The RG for $d = 4 - \varepsilon$ and fixed points to $O(\varepsilon)$

- Method

$$\begin{cases} r_0' = s^2[r_0 + \frac{3u}{4c}K_d\Lambda^2(1-s^{-2}) + uc\varepsilon - r_0 \cdot \frac{3u}{4c^2}K_4 \ln s + u^2 D] \\ \qquad +O(\varepsilon^3) \\ c' = c \\ u' = s^\varepsilon[u - \frac{9u^2}{2c^2}K_4 \ln s] + O(\varepsilon^3) \end{cases}$$

The RG for $d = 4 - \varepsilon$ and fixed points to $O(\varepsilon)$

- Method
  - Let $u' = u = u^*$ and $s^{\varepsilon} = 1 + \varepsilon \ln s$

$$\ln s[\varepsilon u^* - \frac{9u^{*2}}{2c^2} K_4] = 0$$

$$u^* = \begin{cases} 0 & \text{Gaussian fixed point} \\ \varepsilon \dfrac{2c^2}{9K_4} & \text{New fixed point} \end{cases}$$

$$r_0^* = -\varepsilon \frac{\Lambda^2 c}{6}$$

$$\Rightarrow \mu^* \equiv (-\varepsilon \frac{\Lambda^2 c}{6}, c, \varepsilon \frac{2c^2}{9K_4})$$

The RG for $d = 4 - \varepsilon$ and fixed points to $O(\varepsilon)$

- Linearized formula near the new fixed point

$$\begin{cases} \delta r_0 = r_0 - r^* \\ \delta u = u - u^* \end{cases}$$

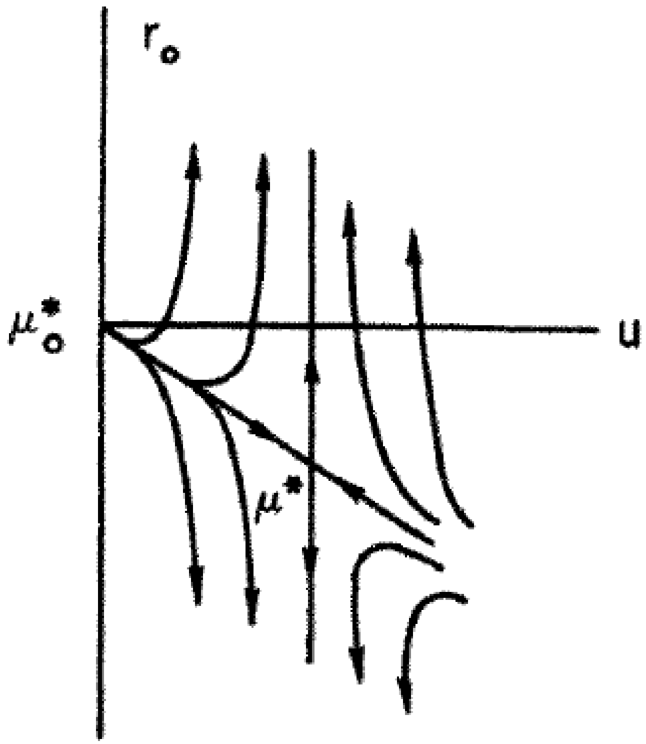
$$\begin{pmatrix} \delta r_0' \\ \delta u' \end{pmatrix} = R_s^L \begin{pmatrix} \delta r_0 \\ \delta u \end{pmatrix}$$

$$R_s^L = \begin{pmatrix} s^{y_1} & B(s^{y_1} - s^{y_2}) \\ 0 & s^{y_2} \end{pmatrix}$$

$$\text{where } \begin{cases} y_1 = 2 - \dfrac{1}{3}\varepsilon \equiv 1/\upsilon \\ y_2 = -\varepsilon \\ B = \dfrac{3}{4}\Lambda^2 K_4 \end{cases}$$

The RG for $d = 4 - \varepsilon$ and fixed points to $O(\varepsilon)$

- $y_2 = -\varepsilon$ is negative means the new fixed point is stable



The Gaussian fixed point μ_o^* and the stable fixed point μ^* for $d = 4 - \epsilon$, $\epsilon > 0$.

The RG for $d=4-\varepsilon$ and fixed points to $O(\varepsilon)$

- Critical exponents up to $O(\varepsilon)$

$$\begin{cases} \alpha=\dfrac{\varepsilon}{6} \\ \beta=\dfrac{1}{2}-\dfrac{\varepsilon}{6} \\ \gamma=1+\dfrac{\varepsilon}{6} \\ \delta=3+\varepsilon \\ \eta=0 \\ \upsilon=\dfrac{1}{2}+\dfrac{\varepsilon}{12} \end{cases} \xrightarrow{d=3\Rightarrow\varepsilon=1} \begin{cases} \alpha=\dfrac{1}{6} \\ \beta=\dfrac{1}{3} \\ \gamma=\dfrac{7}{6} \\ \eta=0 \\ \upsilon=\dfrac{7}{12} \end{cases} \quad \text{Numerical} \begin{cases} \alpha=0.1096 \\ \beta=0.3253 \\ \gamma=1.237 \\ \eta=0.037 \\ \upsilon=0.6301 \end{cases}$$

Chap.9 Renormalization groups in selected models

Youjin Deng

09.12.20

Definitions of the RG for discrete spins

- Some of the main contributors
  - Niemeijer and van Leeuwen
  - Nauenberg and Nienhuis
  - Kadanoff and Haughton
  - Wilson

Definitions of the RG for discrete spins

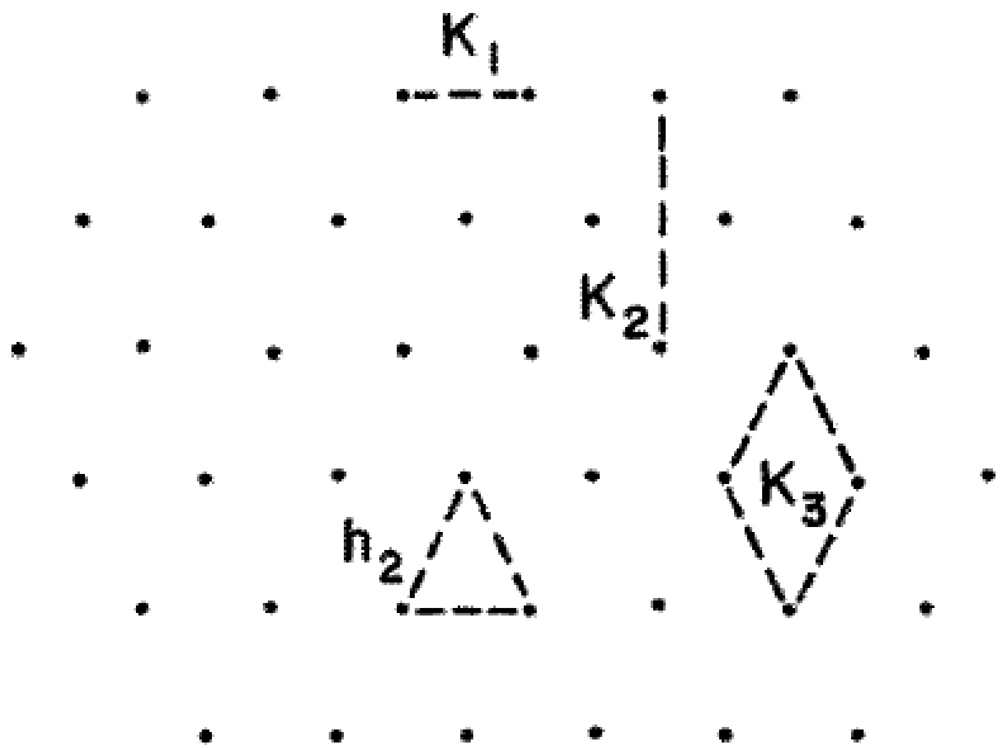
- Ising model on the triangular lattice
  - The Hamiltonian reads

$$
\begin{aligned}
-H = \sum_r \Big(& K_1 \sum_{\delta = n.n} \sigma_r \sigma_{r+\delta} + K_2 \sum_{\delta = n.n.n} \sigma_r \sigma_{r+\delta} \\
& + K_3 \sigma_r \sigma_{r+\delta} \sigma_{r+\delta'} \sigma_{r+\delta''} + \cdots \\
& + h\sigma_r + h_2 \sigma_r \sigma_{r+\delta} \sigma_{r+\delta'} + \cdots \Big)
\end{aligned}
$$

  - Where K_1 is the nearest-neighbor (n.n) coupling parameter, K_2 is the next-nearest-neighbor (n.n.n) coupling parameter, K_3 the "4 spin" coupling parameter, h is the magnetic field, and h_2 .. are higher odd spin coupling parameters.

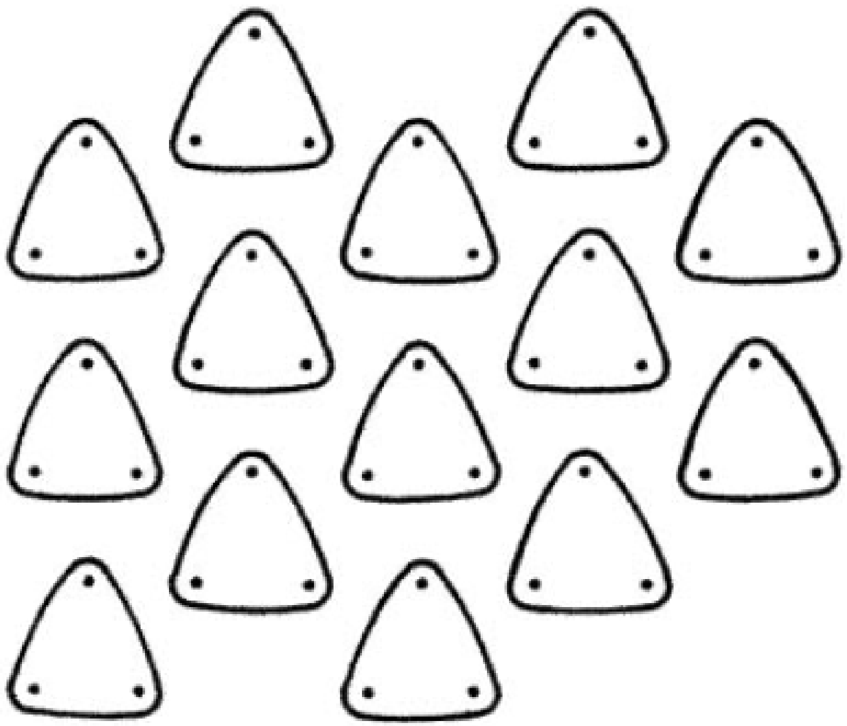
Definitions of the RG for discrete spins

- Ising model on the triangular lattice
  - This figure shows which spins are coupled in these terms.



Definitions of the RG for discrete spins

- Ising model on the triangular lattice
  - A Kadanoff transformation can be defined by forming blocks each of which contain three spins as shown in this figure



Definitions of the RG for discrete spins

- Ising model on the triangular lattice
  - The mean of the three spins $\sigma_{r'}^{1}, \sigma_{r'}^{2}, \sigma_{r'}^{3}$ in the block r' can take four values, namely $\pm 1, \pm 1/3$, instead of just ± 1
  - One can force the four values into two values by defining the new block spin as

$$\sigma_r' = \mathrm{sgn}(\sigma_r^1{}' + \sigma_r^2{}' + \sigma_r^3{}')$$

$$= \frac{1}{2}(\sigma_r^1{}' + \sigma_r^2{}' + \sigma_r^3{}' - \sigma_r^1{}'\sigma_r^2{}'\sigma_r^3{}')$$

  - Which of course is either +1 or -1. This is not quite the same as the coarse graining we are used to

Definitions of the RG for discrete spins

- The RG transformation
  - The RG transformation $R_{\sqrt{3}}\mu=\mu'$ is defined by

$$e^{-H[\sigma']-AL^d}=\sum_{\sigma}\rho[\sigma',\sigma]e^{-H[\sigma]}$$

    - Where $\rho[\sigma',\sigma]$ is a product taken over the new blocks

$$\rho[\sigma',\sigma]=\prod_{r'}p(\sigma_{r'}';\sigma_{r'}^{1},\sigma_{r'}^{2},\sigma_{r'}^{3})$$

    - The function p is a projector, explicitly

$$p=\frac{1}{2}[1+\frac{1}{2}\sigma_{r'}'(\sigma_{r'}^{1}+\sigma_{r'}^{2}+\sigma_{r'}^{3}-\sigma_{r'}^{1}\sigma_{r'}^{2}\sigma_{r'}^{3})]$$

    - We can see

$$p(+1;\sigma^{1},\sigma^{2},\sigma^{3})+p(-1;\sigma^{1},\sigma^{2},\sigma^{3})=1$$

Definitions of the RG for discrete spins

- The RG transformation
  - We need to shrink the new lattice by a factor $\sqrt{3}$ to obtain the same lattice spacing as the old one.
  - The whole RG is defined by

$$R_s = (R_{\sqrt{3}})^l, \quad s = 3^{l/2}$$

$$l = 0, 1, 2 \cdots$$

  - Note that there is no rescaling of the spin variable σ in contrast to the replacement $\sigma \to s^{1-\eta/2-d/2}\sigma$ in the earlier definition of the linear RG. Here the value of σ is always ± 1. No rescaling can be defined, nor is it necessary

Definitions of the RG for discrete spins

- Transformation for various average values
  - The transformation formulas for various average values are more complicated.

$$< e^{\sum_{r'} \lambda_{r'} \sigma'_{r'}} >_{\mu'} = < e^{\sum_{r'} \lambda_{r'} t_{r'}} >_{\mu'}$$

$$t_{r'} = \frac{1}{2}(\sigma_{r'}{}^{1} + \sigma_{r'}{}^{2} + \sigma_{r'}{}^{3} - \sigma_{r'}{}^{1}\sigma_{r'}{}^{2}\sigma_{r'}{}^{3})$$

  - We can obviously see

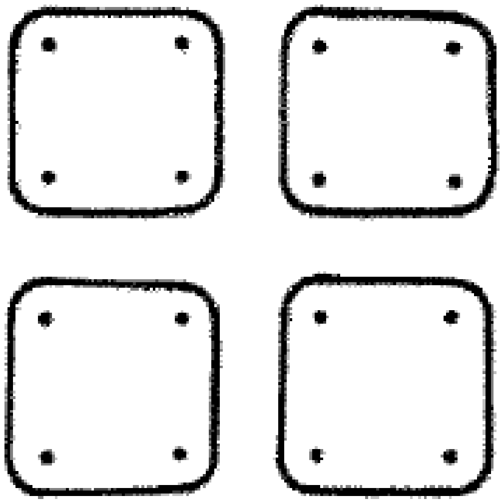
$$m(\mu') = < \sigma'_{r'} >_{\mu'} = < t_{r'} >_{\mu}$$

$$G(r/\sqrt{3}, \mu') = < \sigma'_{r'} \sigma'_{0} >_{\mu} = < t_{r'} t_{0} >_{\mu}$$

etc.

Definitions of the RG for discrete spins

- Two-dimensional Ising on a square lattice
  - This figure shows the formation of new blocks, each of which includes four spins.



Definitions of the RG for discrete spins

- Two-dimensional Ising on a square lattice
  - New block spins can be defined as

$$\sigma_{r'}' = \mathrm{sgn}(\sigma_{r'}^{1} + \sigma_{r'}^{2} + \sigma_{r'}^{3} + \sigma_{r'}^{4})$$

  - Except that special attention has to be paid to the cases where the four spins sum to zero.
  - One can arbitrarily assign $\sigma_{r'}' = 1$ for the configurations shown in the next figure (b), and $\sigma_{r'}' = -1$ for the inverse of these configurations as shown in the next figure (c)

Definitions of the RG for discrete spins

- Two-dimensional Ising on a square lattice
  - Two different cases where the four spins sum to zero.

$\begin{matrix} + & - \\ + & - \end{matrix}$,	$\begin{matrix} - & + \\ - & + \end{matrix}$,
$\begin{matrix} + & - \\ - & + \end{matrix}$,	$\begin{matrix} - & + \\ + & - \end{matrix}$,
$\begin{matrix} + & + \\ - & - \end{matrix}$.	$\begin{matrix} - & - \\ + & + \end{matrix}$.
(b)	(c)

Homeworks

1, A brief survey of percolation. Please include the definition of the percolation model, its applications, exact solution on Cayley tree, exact solution of the percolation threshold on the square lattice.

2,Exact solution of the mean-field Blume-Capel model.

$$H = -\frac{K}{N}\sum_{i \neq j=1}^{N} s_i s_j + D\sum_{k=1}^{N} s_i^2 \quad (s = 0, \pm 1)$$

Please locate the line of the phase transitions, which is consisted of a line of critical point, a tricritical point, and a segment of 1st order transition. Summarize the behavior of the energy density, the specific heat, the magnetization density, the susceptibility, the vacancy density, the compressibility near the transition line, and as a function of temperature 1/K. Derive critical exponents.

Homeworks

3, Ising model is an amazingly beautiful toy model in statistical mechanics. Please describe the exact solution of the Ising model in 1D (including partition sum and all observable quantities), the exact solution of the critical point on the square lattice. Summarize the critical exponent for d>1. Summarize its generalization to the Potts model (ordinary and chiral). Can you locate the critical point of the ordinary Potts model on the square lattice (using the duality relation)?

4, Quantum Ising model plays an important role in quantum statistics. Its Hamiltonian is

$$H = -t\sum_{\langle ij\rangle}\sigma_i^z\sigma_j^z - \sum_k \sigma_k^x$$

where sigma is the Pauli matrix. Please describe its phase transition. Using the path-integral language (Suzuki-Trotter formula), a d-dimensional quantum Ising model can be mapped onto a (d+1)-dimensional quantum Ising model. Please derive this mapping.

Homeworks

5, The classical XY model is frequently used to describe the universal behavior in the phase transition between Mott-insulator and superfluidity (or superconductivity). Please give reasoning why this is possible. Summarize the phase transition of the classical XY model in 1D, 2D, 3D, and more. Give the reasoning why the 2D XY model does not have long-ranged ordering at non-zero temperature (as rigorously as possible)—Mermin-Wagner theorem.

6. In real worlds, we have bosons and fermions. By interchanging two of such particles, the phase of the wave function accumulates 0 and PI, respectively, for bosons and fermions. Their statistical behavior obeys the Bose-Einstein and the Fermi-Dirac statistics, respectively. Please derive them. In some 2D systems with strong interactions, however, the excitons—psedo-particles—do not obey either of the statistics. Interchanging two excitons can lead to a phase change of any value between 0 and 2 Pi. Such pesudo-particles are named “anyons”. They are used explain the quantum Hall effects as well as the recently discovered quantum Spin-Hall effects, and are found to have profound implication in quantum computation. Two beautiful models for anyons are the Kitaev model and the Wen model (arXiv: quant-ph/9707021). Please give a survey of anyons and these two models.

Homeworks

7, Define the renormalization group (RG) in the language of the Ginzburg-Landau model. Explain the universality by RG. Derive the Gaussian the fixed point and the associated critical exponents. Derive the new fixed point in d=4-epsilon dimensions and the associated critical exponents.

8, Monte Carlo method is an important tool in researches, engineering, as well as in industries. In particular, Markovian-chain Monte Carlo (MCMC) method are found extensive applications in statistical physics. Please give a survey of MCMC. Design your own program for the 2D Ising model, calculate energy density, specific heat, magnetization density, and susceptibility. Locate the critical point, and calculate the associated critical exponents.