Circumference of 3-connected claw-free graphs and large Eulerian subgraphs of 3-edge-connected graphs

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Abstract

The circumference of a graph is the length of its longest cycles. Results of Jackson, and Jackson and Wormald, imply that the circumference of a 3-connected cubic n-vertex graph is $\Omega(n^{0.694})$, and the circumference of a 3-connected claw-free graph is $\Omega(n^{0.121})$. We generalise and improve the first result by showing that every 3-edge-connected graph with m edges has an Eulerian subgraph with $\Omega(m^{0.753})$ edges. We use this result together with the Ryjáček closure operation to improve the lower bound on the circumference of a 3-connected claw-free graph to $\Omega(n^{0.753})$. Our proofs imply polynomial time algorithms for finding large Eulerian subgraphs of 3-edge-connected graphs and long cycles in 3-connected claw-free graphs.

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1 Introduction

Motivated by the Four Color Problem, Tait [43] conjectured in 1880 that every 3-connected cubic planar graph contains a Hamilton cycle. His conjecture remained open until a counterexample was constructed by Tutte [45] in 1946. There has since been much interest and extensive research concerning longest cycles in (special families of) graphs. We use $|G|$ to denote the number of vertices in a graph G and refer to the length of a longest cycle in G as the *circumference* of G. We will be concerned with bounds on the circumference of 3-connected graphs which are either cubic or claw-free.

Barnette [3] showed that every 3-connected cubic n-vertex graph has circumference $\Omega(\log n)$. Bondy and Simonovits [9] improved this lower bound to $\exp(\Omega(\sqrt{\log n}))$, and conjectured that it can be improved further to $\Omega(n^c)$ for some constant $0 < c < 1$. This conjecture was established by Jackson [29], with $c = \log_2(1 + \sqrt{5}) - 1 \approx 0.694$. A construction given by Bondy and Simonovits in [9] gives an infinite family of 3-connected cubic graphs with circumference $\Theta(n^{\log_9 8})$, where $\log_9 8 \approx 0.946$. Our first theorem improves the exponent in the lower bound on circumference given in [29], and also generalises the result to graphs which are not necessarily cubic. We use K_2^3 to denote the graph with two vertices joined by three parallel edges.

Theorem 1.1 *Let G be a 3-edge-connected graph, $e, f \in E(G)$, and assume $G \neq K_2^3$. Then G contains an Eulerian subgraph H such that $e, f \in E(H)$ and $|E(H)| \geq (|E(G)|/6)^\alpha + 2$, where $\alpha \approx 0.753$ is the real root of $4^{1/x} - 3^{1/x} = 2$.*

Given graphs G, H, we say that G is *H-free* if G has no induced subgraph isomorphic to H. In the special case when $H = K_{1,3}$ we say that G is *claw-free*. Jackson and Wormald [30] proved a general lower bound on the circumference of 3-connected $K_{1,d}$-free graphs, which reduces to $\frac{1}{2}|G|^c$, where $c = \log_{150} 2 \approx 0.121$, when G is claw-free. We will obtain the following stronger result.

Theorem 1.2 *If G is a 3-connected claw-free graph, then the circumference of G is at least $(|G|/12)^\alpha + 2$, where $\alpha \approx 0.753$ is the real root of $4^{1/x} - 3^{1/x} = 2$.*

Note that if G is a cubic graph then blowing up each vertex of G to a triangle in an obvious way we obtain a claw-free cubic graph H; and it is easy to see that the circumference of G is $\Theta(|G|^c)$ if and only if the circumference of H is $\Theta(|H|^c)$. Thus the above mentioned construction of Bondy and Simonovits implies that the exponent α in Theorem 1.2 cannot exceed $\log_9 8$.

We prove Theorem 1.2 by reducing the problem to line graphs using the closure result of Ryjáček [40]. For x a vertex in a graph G we use $N_G(x)$ (or simply $N(x)$ if there is no confusion) to denote the neighborhood of x; and for each $S \subseteq V(G)$ we use $G[S]$ to denote the subgraph of G induced by S. Let $G_0, \ldots, G_k$ be a maximal sequence of graphs such that $G_0 = G$ and for each $1 \leq i \leq k$, G_i is obtained from G_{i-1} by taking some $x \in V(G)$ for which $G_{i-1}[N_{G_{i-1}}(x)]$ is connected and adding edges between all pairs of nonadjacent vertices in $N_{G_{i-1}}(x)$. Then G_k is said to be a *Ryjáček closure* of G.

Theorem 1.3 *[40] The Ryjáček closure of a claw-free simple graph G is uniquely determined, and is equal to the line graph $L(H)$ of a triange-free simple graph H. Furthermore, for every cycle C' of $L(H)$ there exists a cycle C of G with $V(C') \subseteq V(C)$.*

The final conclusion of this theorem is a slightly stronger statement than that given by Ryjáček in [40, Theorem 3] (that the circumferences of G and $L(H)$ are the same), but it follows from his proof, see [14, Proposition G] and [13, Lemma 8].

It is clear that in a graph H any Eulerian subgraph with m edges gives rise to a cycle with m vertices in $L(H)$. In addition we will see that $L(H)$ is 3-connected if and only if the removal of all degree one vertices from H results in a graph obtained from a 3-edge-connected graph by subdividing each edge at most once. Thus Theorem 1.2 will follow from Theorem 1.3 and an edge-weighted version of Theorem 1.1.

Bounds on the circumference of order $|G|^c$ have also been obtained for other families of 3-connected graphs G. For graphs embedded on a fixed surface, Chen and Yu [17] proved that every 3-connected n-vertex graph embeddable in the torus or Klein bottle has circumference at least $n^{\log_3 2}$, establishing a conjecture of Moon and Moser [38] and Grünbaum and Walther [26]. This was generalized in [41] to locally planar graphs on orientable surfaces. Infinite families of 3-connected cubic planar graphs G with circumference $\Theta(|G|^c)$ have been constructed (for various constants $0 < c < 1$), see for example [25, 26, 47, 48].

For graphs of bounded maximum degree, Jackson and Wormald [30] proved that every 3-connected n-vertex graph with maximum degree at most d has circumference $\Omega(n^{\log_b 2})$, with $b = 6d^2$. This result was improved to $b = 2(d-1)^2 + 1$ by Chen, Xu and Yu [16], and further improved to $b = 4d+1$ by Chen, Gao, Zang and Yu [15]. When $d \geq 4$, Jackson and Wormald conjecture that the correct value for b is $d-1$, and construct an infinite family of 3-connected n-vertex graphs with maximum degree d and circumference $\Theta(n^{\log_{d-1} 2})$ in [30].

One may also consider families of graphs of connectivity other than three. Bounds on the circumference of families of 2-connected n-vertex graphs tend to be of order $\log n$. In particular Bondy and Entringer [8] showed that that every 2-connected graph with maximum degree at most d has circumference at least $\log_{d-1} n$, and construct an infinite family of such graphs with circumferences of the same order of magnitude. Broersma et al [12] showed that the circumference of a 2-connected claw-free n-vertex graph is also $\Omega(\log n)$. (Note that there can be no analogous result for 2-connected graphs embeddable on a fixed surface since $K_{2,n-2}$ is 2-connected and planar, and has circumference four.)

On the other hand, bounds on the circumference of families of n-vertex graphs of connectivity greater than three may be of order n. Bondy, see [29, Conjecture 1], conjectured that if G is a 3-connected cubic graph and every 3-edge-cut of G is trivial, then G has circumference $\Omega(n)$. A stronger conjecture due to Fleischner, see [29, Conjecture 2], is that every such graph G has a cycle C such that $G - C$ is an independent set of vertices. Both conjectures are true for planar cubic graphs by Tutte's bridge theorem [46]. Fleischner and Jackson [22] showed that Fleischner's conjecture is equivalent to a conjecture of Thomassen [44] that every 4-connected line graph is Hamiltonian. Ryjáček [40] used Theorem 1.3 to show that Thomassen's conjecture is in turn equivalent to the conjecture of Mathews and Sumner [36] that every 4-connected claw-free graph is Hamiltonian. Zhang [49] has verified Thomassen's conjecture for the special case of 7-connected line graphs. This result was extended to 7-connected claw-free graphs by Ryjáček in [40].

An outline of the paper is as follows. Section 2 contains some preliminary results. We introduce a reduction technique called 'edge-splitting' in Subsection 2.1 and characterize when it can be used to split away two edges from a vertex in such a way that 3-edge-connectivity is preserved. In Subsection 2.2, we characterize when a 3-edge-connected graph has an Eulerian subgraph which contains two given edges and four given vertices. In Subsection 2.3, we prove some inequalities based on the concavity of the function $n \to n^c$ when $0 < c < 1$ which we will use in our induction. We prove the aforementioned edge-weighted version of Theorem 1.1 in Section 3 by applying the edge-splitting lemmas to reduce to the case when each of the endvertices of e and f has degree three, and then extending the proof technique for cubic graphs given in [29]. Theorem 1.2 is derived in Section 4. Our proofs of Theorems 1.1 and 1.2 are constructive and give rise to polynomial algorithms. These will be outlined in Section 5.

2 Definitions and preliminary results

Unless specified otherwise all graphs considered may contain loops and multiple edges. We will refer to graphs without loops and multiple edges as *simple graphs*. For any edge e in a graph G, we use $V(e)$ to denote the set of vertices of G that are incident with e. For $S \subseteq E(G)$ we use $G - S$ to denote the graph obtained from G by deleting S. For H and L subgraphs of G, we use $H - L$ to denote the graph obtained from H by deleting $V(H) \cap V(L)$ and all edges of H incident with vertices in $V(H) \cap V(L)$. If L consists of one vertex, say v, then we also write $H - v$ for $H - L$.

2.1 Edge splitting

Let G be a graph, $v \in V(G)$, and e, f be distinct edges of G with $V(e) = \{u, v\}$ and $V(f) = \{v, w\}$. When $d(v) = 2$, the operation of *suppressing v in G* deletes v (and hence also e, f) and adds a new edge between u and w (which may be a loop if $u = w$). When $d(v) \geq 4$ the operation of *splitting e, f at v* deletes e, f from G, adds a new edge between u and w, and suppresses v if v has degree 2 in $G - \{e, f\}$. We use $G_v^{e,f}$ to denote the graph obtained from G by splitting e, f at v. Note that if e is a loop at v then $G_v^{e,f}$ is isomorphic to $G - e$ when $d(v) > 4$, and to the graph obtained from $G - e$ by suppressing v when $d(v) = 4$. When G is k-edge-connected, we say that e, f form a *k-splittable pair at v* if $G_v^{e,f}$ is also k-edge-connected. (Note that loops have no effect on edge-connectivity so a pair containing a loop will always be k-splittable.) If there is no confusion, we will simply say that e, f is a *splittable pair at v*. We need the following consequence of a more general result of Frank (Theorem B, [23]).

Lemma 2.1 *Let G be a 3-edge-connected graph and $v \in V(G)$ such that $d(v) \geq 4$. If $d(v)$ is even then each edge incident with v belongs to a splittable pair at v. If $d(v)$ is odd then there is at most one edge incident with v that does not belong to any splittable pair at v.*

For our purpose, we also need to describe the structure when an edge is not contained in any splittable pair. This structure is illustrated in Figure 1. To describe it more precisely we need some more notation. Given a graph G and disjoint subsets X, Y of $V(G)$, we use $E(X, Y)$ to denote the set, and $\delta(X, Y)$ the number, of edges of G incident with both X and Y. When $X = \{x\}$ or $Y = \{y\}$, we write $\delta(x, Y)$ or $\delta(X, y)$. We also put $\delta(X) = \delta(X, V(G) - X)$. We write $\delta_G(X)$ when the underlying graph G is not clear from the context.

The lemma below is similar to a result for local edge-connectivity due to Szigeti (Theorem 1.6, [42]). We will need the $k = 3$ case (see Figure 1) but we state it for general k as it may be of independent interest.

Lemma 2.2 *Let G be a k-edge-connected graph ($k \geq 3$) and $e \in E(G)$ with $V(e) = \{u, v\}$. Suppose that $d(v) \geq k+2$, and e belongs to no splittable pair at v. Then k is odd, $d(v) = k+2$,*

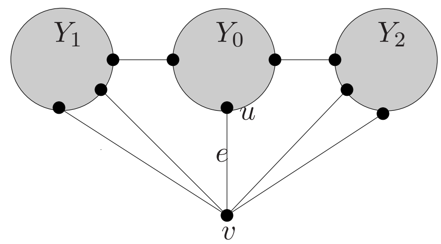

Figure 1: $k = 3$ and the edge e belongs to no splittable pair at v.

and there exists a partition Y_0, Y_1, Y_2 of $V(G) - \{v\}$ such that $u \in Y_0$, $\delta(v, Y_0) = 1$, $\delta(v, Y_1) = \delta(v, Y_2) = (k+1)/2$, $\delta(Y_0, Y_1) = \delta(Y_0, Y_2) = (k-1)/2$, and $\delta(Y_1, Y_2) = 0$.

Proof. Since e is contained in no splittable pair, v is incident with no loops and there exists a family of sets $\mathcal{F} = \{X_1, \ldots, X_t\}$ such that $N(v) \subseteq \bigcup_{i=1}^t X_i$ and, for $1 \le i \le t$, $u \in X_i \subseteq V - \{v\}$ and $\delta(X_i) \le k+1$. We choose $\mathcal{F}$ such that

(1) t is minimum, and

(2) subject to (1), $\sum_{i=1}^t |X_i|$ is maximum.

Since $d(v) \ge k+2$ and v is not incident with any loop, we have $t \ge 2$. Let $Y_0 = X_1 \cap X_2$, $Y_1 = X_1 - X_2$ and $Y_2 = X_2 - X_1$. By (1), $Y_i \ne \emptyset$ for $i = 1, 2$. Note that $\delta(Y_0) \ge k$ since $u \in Y_0$ and G is k-edge-connected. Also $\delta(X_1 \cup X_2) \ge k+2$, for otherwise $\delta(X_1 \cup X_2) \le k+1$ and $(\mathcal{F} - \{X_1, X_2\}) \cup \{X_1 \cup X_2\}$ contradicts the choice of $\mathcal{F}$ (via (1)). So

$$(k+1) + (k+1) \ge \delta(X_1) + \delta(X_2) = \delta(Y_0) + \delta(X_1 \cup X_2) + 2\delta(Y_1, Y_2) \ge k + (k+2).$$

Therefore, equality must hold throughout; so $\delta(X_1) = \delta(X_2) = k+1$, $\delta(Y_0) = k$, $\delta(Y_1, Y_2) = 0$, and $\delta(X_1 \cup X_2) = k+2$.

Since $u \in Y_0$ and $v \in V(G) - (X_1 \cup X_2)$, $\delta(Y_0, V(G) - (X_1 \cup X_2)) \ge 1$. Because G is k-edge-connected, $\delta(Y_i) \ge k$ for $i = 1, 2$; and hence

$$(k+1) + (k+1) = \delta(X_1) + \delta(X_2) = \delta(Y_1) + \delta(Y_2) + 2\delta(Y_0, V(G) - (X_1 \cup X_2)) \ge k + k + 2.$$

Equality holds throughout; so $\delta(Y_1) = \delta(Y_2) = k$ and $\delta(v, Y_0) = \delta(Y_0, V(G) - (X_1 \cup X_2)) = 1$.

Since G is k-edge-connected and $\delta(X_1) = k+1$, $G[X_1]$ is $\lceil (k-1)/2 \rceil$-edge-connected. Hence $\delta(Y_0, Y_1) \ge \lceil (k-1)/2 \rceil$. Similarly, $\delta(Y_0, Y_2) \ge \lceil (k-1)/2 \rceil$. Because $\delta(Y_0) = k$, $v \notin X_1 \cup X_2$, and $\delta(v, Y_0) = 1$, we must have $\delta(Y_0, Y_1) = \delta(Y_0, Y_2) = (k-1)/2$. In particular, k is odd.

We may assume $t \ge 3$. For, suppose $t = 2$. Then $N(v) \subseteq Y_0 \cup Y_1 \cup Y_2$. Since $\delta(Y_1, Y_2) = 0$, $\delta(Y_1) = \delta(Y_2) = k$, and $\delta(Y_0, Y_1) = \delta(Y_0, Y_2) = (k-1)/2$, we have $\delta(v, Y_1) = \delta(v, Y_2) = (k+1)/2$. Hence $d(v) = k+2$. Therefore, there are no edges of G leaving $Y_0 \cup Y_1 \cup Y_2 \cup \{v\}$; so $\{Y_0, Y_1, Y_2\}$ is a partition of $V(G) - \{v\}$, and the assertion of the lemma holds.

Suppose $Y_0 \not\subseteq X_3$. Note that $\delta(X_3 \cup Y_0) \ge k+2$ as otherwise $(\mathcal{F} - \{X_3\}) \cup \{X_3 \cup Y_0\}$ contradicts the choice of $\mathcal{F}$ (via (2)). Since $u \in X_3 \cap Y_0$ and G is k-edge-connected, $\delta(X_3 \cap Y_0) \ge k$. Therefore, we have the following contradiction

$$(k+1) + k \ge \delta(X_3) + \delta(Y_0) \ge \delta(X_3 \cup Y_0) + \delta(X_3 \cap Y_0) \ge (k+2) + k.$$

So $Y_0 \subseteq X_3$, i.e., $X_1 \cap X_2 \subseteq X_3$. Hence by symmetry among X_1, X_2, X_3, we also have $X_2 \cap X_3 \subseteq X_1$ and $X_1 \cap X_3 \subseteq X_2$. So $X_1 \cap X_2 = X_1 \cap X_3 = X_2 \cap X_3 = Y_0$ and $\delta(Y_0, X_1 - Y_0) = \delta(Y_0, X_2 - Y_0) = \delta(Y_0, X_3 - Y_0) = (k-1)/2$. This is impossible since we also have $\delta(v, Y_0) = 1$ and $\delta(Y_0) = k$. ∎

We also need to know when an edge is contained in a unique splittable pair at a vertex of degree four in a 3-edge-connected graph, see Figure 2. This follows from a more general result of Jordán [32, Theorem 3.6].

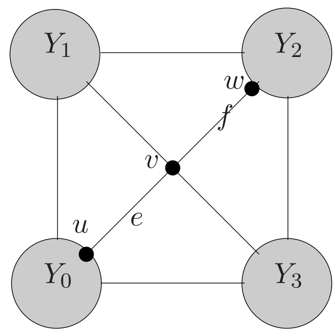

Figure 2: The edge e belongs to a unique splittable pair at v.

Lemma 2.3 *Let G be a 3-edge-connected graph and $e, f \in E(G)$ with $V(e) = \{u, v\}$ and $V(f) = \{v, w\}$. Suppose that $d(v) = 4$ and that e, f is the unique splittable pair at v which contains e. Then there exists a partition Y_0, Y_1, Y_2, Y_3 of $V(G) - \{v\}$ such that $u \in Y_0$, $w \in Y_2$, $\delta(v, Y_i) = 1$ for all $0 \leq i \leq 3$, $\delta(Y_0, Y_1) = \delta(Y_1, Y_2) = \delta(Y_2, Y_3) = \delta(Y_3, Y_0) = 1$, and $\delta(Y_0, Y_2) = \delta(Y_1, Y_3) = 0$.*

2.2 Cyclability

Let G be a graph and $e \in E(G)$ with $V(e) = \{u, v\}$. Then the graph G/e obtained from G by *contracting e to a single vertex z* (where $z \notin V(G)$) is the graph obtained from $G - \{u, v\}$ by adding the new vertex z and replacing each edge f in $G - e$ with at least one end in $\{u, v\}$ by an edge in which the corresponding end vertex/vertices are equal to z. We denote the edge of G/e corresponding to f by the same label f. Note that an edge f of $G - e$ with $V(f) = \{u, v\}$ will be replaced by a loop at z in G/e. More generally, if H is a subgraph of G, then graph G/H obtained from G by *contracting H to a single vertex z* (where $z \notin V(G)$) is the graph obtained from $G - H$ by adding the new vertex z and replacing each edge f in $G - E(H)$ with at least one end in $V(H)$ by an edge in which the corresponding end vertex/vertices are equal to z. We again denote the edge of G/e corresponding to f by the same label f. Note that: contracting a subgraph cannot reduce the edge-connectivity of G; contracting a subgraph of an Eulerian graph results in another Eulerian graph; and, when H is connected, G/H can be obtained from G by successively contracting each edge of H.

Ellingham, Holton and Little obtained the following characterization of 3-connected cubic graphs G with the property that no cycle of G contains a given set of two edges and at most four vertices of G.

Lemma 2.4 *[18] Let G be a 3-connected cubic graph, $X \subseteq V(G)$ with $|X| \leq 4$ and $F = \{e, f\} \subseteq E(G)$. Then no cycle of G contains $X \cup F$ if and only if $|X| = 4$ and G has pairwise disjoint subgraphs $Z_1, Z_2, \dots, Z_m$ such that $V(Z_1), V(Z_2), \dots, V(Z_m)$ partitions $V(G)$, $|X \cap Z_i| = 1$ for $1 \leq i \leq 4$, $e \in E(Z_5, Z_6)$, $f \in E(Z_7, Z_8)$, $\delta(Z_i) = 3$ for all $1 \leq i \leq m$, and either:*
(a) $m = 8$, the graph obtained by contracting each Z_i to a single vertex is the Wagner graph, and G has the structure illustrated in Figure 3(a), or
(b) $m = 10$, the graph obtained by contracting each Z_i to a single vertex is the Petersen graph, and G has the structure illustrated in Figure 3(b).

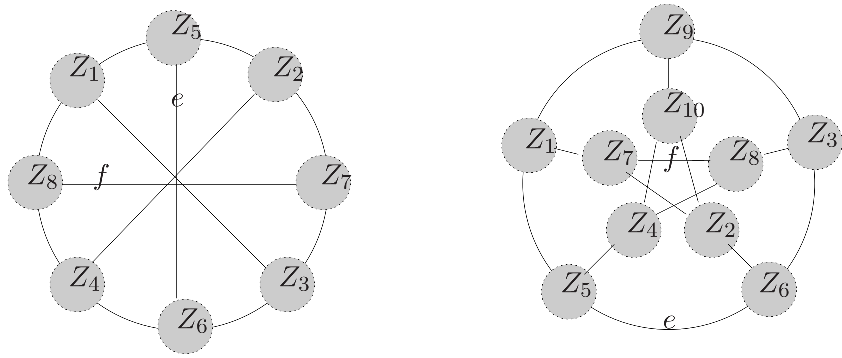

Figure 3: Graphs in which no Eulerian subgraph contains e, f and any four given vertices in Z_1, Z_2, Z_3 and Z_4.

We will need the following extension of Theorem 2.4 to 3-edge-connected graphs which are not necessarily cubic. We use the term *trail* to mean a walk between two vertices in a graph which may repeat vertices but not edges. A *closed trail* is a trail which begins and ends at the same vertex.

Lemma 2.5 *Let G be a 3-edge-connected graph, $X \subseteq V(G)$ with $|X| \leq 4$ and $F = \{e, f\} \subseteq E(G)$. Then no Eulerian subgraph of G contains $X \cup F$ if and only if $|X| = 4$ and G has pairwise disjoint subgraphs $Z_1, Z_2, \dots, Z_m$ such that $V(Z_1), V(Z_2), \dots, V(Z_m)$ partitions $V(G)$, $|X \cap Z_i| = 1$ for $1 \leq i \leq 4$, $e \in E(Z_5, Z_6)$, $f \in E(Z_7, Z_8)$, $\delta(Z_i) = 3$ for all $1 \leq i \leq m$, and either:*
(a) $m = 8$, the graph obtained by contracting each Z_i to a single vertex is the Wagner graph, and G has the structure illustrated in Figure 3(a), or
(b) $m = 10$, the graph obtained by contracting each Z_i to a single vertex is the Petersen graph, and G has the structure illustrated in Figure 3(b).

Proof. It is not difficult to check that if G has the specified subgraphs $Z_1, Z_2, \dots, Z_m$ then no Eulerian subgraph of G can contain $X \cup F$. Hence suppose that no Eulerian subgraph of G contains $X \cup F$. We use induction on $a(G) := \sum_{v \in V(G)}(d(v) - 3)$ to show that the specified subgraphs $Z_1, Z_2, \dots, Z_m$ exist. If $a(G) = 0$ then G is cubic and the assertion follows immediately from Lemma 2.4. Hence suppose $a(G) > 0$ and choose $v \in V(G)$ with $d(v) \geq 4$. By Lemma 2.1, we may choose edges e_1, e_2 incident to v such that the graph $G_v^{e_1,e_2}$ obtained by splitting e_1, e_2 at v in G is 3-edge-connected. Let $V(e_i) = \{v, u_i\}$ for $i = 1, 2$ and let G' be obtained from $G - \{e_1, e_2\}$ by adding a new vertex z and three new edges e_1, e_2, e_3 where $V(e_i) = \{z, u_i\}$ for $i = 1, 2$, and $V(e_3) = \{z, v\}$. We give two of the new edges the same labels as the deleted edges so that we have $E(G) \subseteq E(G')$. Note that $G' = G/e_3$, $e_3 \notin \{e, f\}$, and, if e_1 is a loop in G, then e_1 is an edge between z and v in G'.

The 3-edge-connectivity of $G_v^{e_1,e_2}$ implies that G' is 3-edge-connected, and we have $X \subseteq V(G) \subseteq V(G')$ and $F \subseteq E(G) \subseteq E(G')$. Since no Eulerian subgraph of G can contain $X \cup F$, no Eulerian subgraph of G' can contain $X \cup F$. Since $a(G') < a(G)$ we may use induction to deduce that the specified subgraphs $Z'_1, Z'_2, \ldots, Z'_m$ exist for G'. If $e_3 \in E(Z'_i)$ for some $1 \leq i \leq m$ then we may construct the required subgraphs $Z_1, Z_2, \ldots, Z_m$ for G by putting $Z_i = Z'_i/e_3$ and $Z_j = Z'_j$ for all $i \neq j$. Thus we may assume that $e_3 \notin E(Z'_i)$ for all $1 \leq i \leq m$. We will show that this case cannot occur by constructing an Eulerian subgraph H of G which contains $X \cup F$. Let $\tilde{G}$ be the graph obtained from G' by contracting each subgraph Z'_i to a single vertex z_i.

Suppose $m = 8$. Then $\tilde{G}$ is isomorphic to the Wagner graph and we may assume by symmetry that e_3 is incident to either z_1 and z_5, or z_1 and z_3. Consider the cycle $C = z_5z_6z_3z_7z_8z_4z_2z_5$ of $\tilde{G}$. We may extend $E(C)$ to the Eulerian subgraph H of $G = G'/e_3$ which contains $X \cup F$ as follows. We first assume that $V(e_3) = \{z_1, z_5\}$. For $i \neq 1, 5$ we construct a trail P_i in Z'_i joining the two vertices incident to C and passing through any vertex in $X \cap V(Z_i)$. For $i = 5$ we construct a trail P_5 in Z'_5 joining the vertices incident to C and passing through the vertex incident to e_3. For $i = 1$ we construct a closed trail C_1 in Z'_1 containing the vertex incident to e_3 and the vertex in $X \cap V(Z'_1)$. (These trails exist since G' is 3-edge-connected and hence $Z^*_i = G'/(G' - Z'_i)$ is 3-edge-connected for all $1 \leq i \leq 8$.) We then choose H to be the subgraph of G induced by $\bigcup_{i=2}^{8} E(P_i) \cup E(C_1) \cup E(C)$. We proceed similarly when $V(e_3) = \{z_1, z_3\}$ by interchanging the roles of Z_5 and Z_3 in the construction.

Suppose that $m = 10$. Then $\tilde{G}$ is isomorphic to the Petersen graph and we may assume by symmetry that e_3 is incident to either z_1 and z_7, or z_1 and z_9, or z_9 and z_{10}. In the first two cases we may proceed as in the previous paragraph, using the cycle $C = z_5z_6z_2z_7z_8z_3z_9z_{10}z_4z_5$ of $\tilde{G}$. In the case when $V(e) = \{z_9, z_{10}\}$, we proceed similarly using the two disjoint cycles $C_1 = z_1z_9z_3z_8z_7z_1$ and $C_2 = z_{10}z_2z_6z_5z_4z_{10}$ in $\tilde{G}$. (These cycles give rise to two disjoint Eulerian subgraphs of G' which become one Eulerian subgraph in $G = G'/e_3$.) ∎

2.3 Three inequalities

The purpose of this subsection is to present three inequalities that will be used to estimate the weight of an Eulerian subgraph obtained by combining several smaller Eulerian subgraphs. The first is elementary.

Lemma 2.6 *Let n_1, n_2 be nonnegative reals. Then for any $0 < c \leq 1$,*

$$n_1^c + n_2^c \geq (n_1 + n_2)^c.$$

Lemma 2.7 *Let s be a positive real number and β be the root of $(s+2)^x - s^x = 1$ in $(0, 1)$. Then for any real numbers n_1, n_2, n_3, γ satisfying $n_1 \geq sn_3$, $n_2 \geq n_3 \geq 0$, and $0 < \gamma \leq \beta$ we have*

$$n_1^\gamma + n_2^\gamma \geq (n_1 + n_2 + n_3)^\gamma.$$

Proof. It is not difficult to check that $(s+2)^x - s^x = 1$ has a unique root $\beta \in (0,1)$ and that $(s+2)^\gamma - s^\gamma - 1 \le 0$ for all $0 < \gamma \le \beta$. Let $f(n_1,n_2,n_3) = n_1^\gamma + n_2^\gamma - (n_1+n_2+n_3)^\gamma$. We show that $f(n_1,n_2,n_3) \ge 0$ when $n_1 \ge sn_3$ and $n_2 \ge n_3 \ge 0$. We have $\partial f/\partial n_1 \ge 0$ and $\partial f/\partial n_2 \ge 0$ since $0 < \gamma < 1$, so f is minimised when $n_1 = sn_3$ and $n_2 = n_3$. Thus

$$f(n_1,n_2,n_3) \ge f(sn_3,n_3,n_3) = (s^\gamma + 1 - (s+2)^\gamma)n_3^\gamma \ge 0.$$

∎

Lemma 2.8 *Suppose $n_1, ..., n_k, t, \gamma$ are real numbers with $k \ge 3$, $0 \le n_k \le t \min\{n_1,\dots,n_{k-1}\}$, and $0 < \gamma \le \log_{t+k-1}(k-1)$. Then*

$$\sum_{i=1}^{k-1} n_i^\gamma \ge \left(\sum_{i=1}^{k} n_i\right)^\gamma .$$

Proof. The assertion of this lemma follows from Lemma 2.6 when $n_k = 0$. Thus we may assume $n_k > 0$. Hence $t > 0$ and $\sum_{i=1}^k n_i > 0$. Define $x_i = n_i / \sum_{j=1}^k n_j$, for $i = 1,\dots,k$. Then $x_1, ..., x_k \in [0,1]$, $\sum_{i=1}^k x_i = 1$, and $x_k \le t\min\{x_1,\dots,x_{k-1}\}$. It suffices to show that $\sum_{i=1}^{k-1} x_i^\gamma \ge 1$.

Let $f(x_1,\dots,x_{k-1}) = \sum_{i=1}^{k-1} x_i^\gamma$. We first show that the minimum of $f(x_1,\dots,x_{k-1})$ subject to the constraints that $x_i \ge x_k/t \ge 0$ for all $1 \le i \le k-1$, $\sum_{i=1}^k x_i = 1$, and x_k is fixed, occurs when $x_i = x_k/t$ for all $i = 2,\dots,k$. Let $(a_1,a_2,\dots,a_{k-1})$ be a point at which this minimum occurs and is such that a_1 is as large as possible. By symmetry we have $a_1 \ge a_i$ for all $2 \le i \le k-1$. We may use elementary calculus and the facts that $a_1 \ge a_i$ and $0 \le \gamma < 1$ to deduce that $(a_1+\epsilon)^\gamma + (a_i - \epsilon)^\gamma \le a_1^\gamma + a_i^\gamma$ for all $\epsilon \ge 0$. The choice of $(a_1,a_2,\dots,a_{k-1})$ now implies that $a_i = x_k/t$ for all $2 \le i \le k-1$, $a_1 = 1-(t+k-2)x_k/t$, and $f(a_1,\dots,a_{k-1}) = (1-(t+k-2)x_k/t)^\gamma + (k-2)(x_k/t)^\gamma =: g(x_k)$.

Since $\sum_{i=1}^k x_i = 1$ we have $\sum_{i=1}^k tx_i = t$. We can now use the fact that $x_k \le tx_i$ for all $1 \le i \le k-1$ to deduce that $x_k \le t/(t+k-1)$. We complete the proof by showing that $g(x_k) \ge 1$ for $0 \le x_k \le t/(t+k-1)$. It is not difficult to see that $g'(x_k) = 0$ has a unique solution, and that $g''(x_k) < 0$ for $0 \le x_k \le t/(t+k-1)$. Hence, the minimum of $g(x_k)$ is achieved at $x_k = 0$ or $x_k = t/(t+k-1)$. We have $g(0) = 1$, and

$$g\left(\frac{t}{t+k-1}\right) = (k-1)(t+k-1)^{-\gamma} \ge (k-1)(t+k-1)^{-\log_{t+k-1}(k-1)} = 1.$$

Therefore, $f(x_1,\dots,x_{k-1}) \ge 1$. ∎

We will use the following special cases of Lemmas 2.7 and 2.8.

Corollary 2.9 *Let $\alpha \approx 0.753$ be the real root of $4^{1/x} - 3^{1/x} = 2$. Then:*
(a) for all real numbers n_1, n_2, n_3 satisfying $n_1 \ge 3^{1/\alpha} n_3$ and $n_2 \ge n_3 \ge 0$ we have

$$n_1^\alpha + n_2^\alpha \ge (n_1+n_2+n_3)^\alpha;$$

(b) for all real numbers n_1, n_2, n_3, n_4 satisfying $0 \le n_4 \le \min\{n_1,n_2,n_3\}$ we have

$$n_1^\alpha + n_2^\alpha + n_3^\alpha \ge (n_1+n_2+n_3+n_4)^\alpha;$$

(c) for all real numbers n_1, n_2, n_3, n_4, n_5 *satisfying* $0 \le n_5 \le (4^{1/\alpha} - 4) \min\{n_1, n_2, n_3, n_4\}$ *we have*

$$n_1^\alpha + n_2^\alpha + n_3^\alpha + n_4^\alpha \ge (n_1 + n_2 + n_3 + n_4 + n_5)^\alpha.$$

Proof. Part (a) follows from Lemma 2.7 by taking $s = 3^{1/\alpha}$ and using the fact that $(3^{1/\alpha} + 2)^\alpha - (3^{1/\alpha})^\alpha = 4 - 3 = 1$. Parts (b) and (c) follow from Lemma 2.8 by taking $k = 4$ and $t = 1$, and $k = 5$ and $t = 4^{1/\alpha} - 4$, respectively. ∎

3 Eulerian subgraphs of 3-edge-connected graphs

In this section we prove an edge weighted version of Theorem 1.1. Let G be a graph and let $w : E(G) \to \{1, 2\}$. For any $H \subseteq G$ let $w(H) = \sum_{e \in E(H)} w(e)$, and for any $S \subseteq E(G)$ let $w(S) = \sum_{e \in S} w(e)$. We will show

Theorem 3.1 *Let* G *be a 3-edge-connected graph,* $e, f \in E(G)$*, and* $w : E(G) \to \{1, 2\}$*. Suppose* $G \ne K_2^3$*. Then* G *contains an Eulerian subgraph* H *such that* $e, f \in E(H)$ *and* $w(H) \ge (w(G)/6)^\alpha + 2$*, where* $\alpha \approx 0.753$ *is the real root of* $4^{1/x} - 3^{1/x} = 2$*.*

The multiplicative constant $(1/6)^\alpha$ in Theorem 1.1 is chosen to simplify its proof; it may be improved by considering other exceptional graphs in addition to K_2^3. Note that the conclusion of Theorem 3.1 does not hold for K_2^3 because of the additive constant 2. We need this additive constant for the inductive step in our proof.

We first need to deal with graphs with few edges to provide a basis for our induction.

Lemma 3.2 *Theorem 3.1 holds for graphs with at most 6 edges.*

Proof. The assertion of Theorem 3.1 clearly holds if G is Eulerian. So assume that u, v are vertices of G with odd degree. Since G is 3-edge-connected and $|E(G)| \le 6$, $|G| \le 4$.

If $|G| = 4$ then $G = K_4$. If $|G| = 2$ then, since $G \ne K_2^3$, G is obtained from K_2^3 by adding one, two or three edges, which can be either two more uv-edges and at most one loop, or all loops. In each case it is easy to check that the desired Eulerian subgraph H exists.

Now assume $|G| = 3$. Let w denote the vertex of G other than u, v. Since G is 3-edge-connected and $|E(G)| \le 6$, we see that G has at most two edges between u and v. If there is no edge between u and v, then G is obtained from a path of length 2 by tripling each edge, and it is easy to find the desired H. If there is exactly one edge between u and v, then $d(u) = d(v) = 3$ (as $|E(G)| \le 6$) and $d(w) = 4$ or 6 (if $d(w) = 6$ then there is a loop on w); and the desired H can be found directly. Finally, assume that there are precisely two edges between u and v. Since G is 3-edge-connected and by symmetry, we may assume $d(u) = 5$ (so that there are 3 edges between u and w). Then $d(v) = 3$ (since $|E(G)| \le 6$), and there is just one edge between v and w. Again the desired H exists. ∎

The next lemma will be used to construct the desired Eulerian subgraph of G from an Eulerian subgraph of a graph obtained from G by contracting several disjoint induced subgraphs.

Lemma 3.3 *Let* G *be a 3-edge-connected graph,* $w : E(G) \to \{1, 2\}$*, and let* $C_1, \ldots, C_k$ *be disjoint induced subgraphs of* G *such that* $\delta(C_i) = 3$ *and* $|E(C_i)| < |E(G)| - 3$ *for all* $i = 1, \ldots, k$*. Let* $\tilde{G}$ *denote the graph obtained from* G *by contracting each subgraph* C_i *to a single* *vertex c_i. Suppose Theorem 3.1 holds for all graphs with fewer edges than G, and assume that $\tilde{G}$ contains an Eulerian subgraph $\tilde{H}$ such that $c_i \in V(\tilde{H})$ for all i. Then G contains an Eulerian subgraph H such that the edges of G corresponding to the edges in $\tilde{H}$ are in H and*

$$w(H) \geq \sum_{i=1}^{k} (w(C_i)/6)^\alpha + w(\tilde{H}).$$

Proof. For each i, let e_i, f_i denote the edges of $\tilde{H}$ incident with c_i. Let C_i^* be obtained from G by contracting $G - C_i$ to a single vertex c_i^*. Since $\delta(C_i) = 3$ and G is 3-edge-connected, C_i^* is 3-edge-connected. Assign the edges incident with c_i^* weight 1.

Since $|E(C_i)| < |E(G)| - 3$, we have $|E(C_i^*)| < |E(G)|$. If $C_i^* \neq K_2^3$ then, by assumption, C_i^* contains an Eulerian subgraph H_i such that $e_i, f_i \in E(H_i)$ and $w(H_i) \geq (w(C_i^*)/6)^\alpha + 2 \geq (w(C_i)/6)^\alpha + 2$. On the other hand, if $C_i^* = K_2^3$ then $E(C_i) = \emptyset$, $w(C_i) = 0$ and we may also construct an Eulerian subgraph H_i such that $e_i, f_i \in E(H_i)$ and $w(H_i) = 2 = (w(C_i)/6)^\alpha + 2$.

Since $d(c_i^*) = 3$, we see that H_i uses exactly two edges at c_i^*, namely e_i and f_i. Then $\bigcup_{i=1}^k E(H_i) \cup E(\tilde{H})$ induces an Eulerian subgraph H of G such that

$$w(H) \geq \sum_{i=1}^{k} (w(H_i) - 2) + w(\tilde{H}) \geq \sum_{i=1}^{k} (w(C_i)/6)^\alpha + w(\tilde{H}).$$

Lemma 3.4 *Let L be a 3-edge-connected graph, $w : E(G) \to \{1, 2\}$, and z_1, z_2 be two adjacent vertices of degree three in L. Let L' be obtained from L by deleting the edge joining z_1 and z_2, and then suppressing z_1, z_2 to two edges k_1, k_2, respectively, of weight 1. Suppose Theorem 3.1 holds for all graphs with fewer edges than L. Then L' has an Eulerian subgraph H with $k_1, k_2 \in E(H)$ and $w(H - \{k_1, k_2\}) \geq (w(L')/6)^\alpha$.*

Proof. We use an inner induction on $|E(L)|$. If L' is 3-edge-connected then we may apply Theorem 3.1 to L' to find an Eulerian subgraph H with $k_1, k_2 \in E(H)$ and $w(H') \geq (w(L')/6)^\alpha + 2$. Then $w(H' - \{k_1, k_2\}) \geq (w(L')/6)^\alpha$ as required.

Hence suppose that L' is not 3-edge-connected. Since L is 3-edge-connected, L' is 2-edge-connected and every 2-edge-cut of L' separates k_1 and k_2. Choose a 2-edge-cut $\{g, h\}$ of L' and let L_1^*, L_2^* be the components of $L' - \{g, h\}$ with $k_1 \in E(L_1^*)$ and $k_2 \in E(L_2^*)$. For $i = 1, 2$, construct L_i' from L_i^* by adding a new edge f_i of weight 1 between the endvertices of g and h in L_i^*. Let L_i be obtained from L_i' by subdividing k_i and f_i with two new vertices z_1' and z_2' and then adding an edge between z_1' and z_2'. Then L_i is 3-edge-connected since it can be obtained from L by contracting $L_{3-i} \cup \{z_{3-i}\}$ to a single vertex. We may apply the inner induction to L_i to deduce that L_i' has an Eulerian subgraph H_i with $k_i, f_i \in E(H_i)$ and $w(H_i - \{k_i, f_i\}) \geq (w(L_i')/6)^\alpha$. Then $E(H_1 - f_1) \cup E(H_2 - f_2) \cup \{g, h\}$ induces an Eulerian subgraph H of L' with $k_1, k_2 \in E(H)$ and

$$w(H - \{k_1, k_2\}) \geq (w(L_1')/6)^\alpha + (w(L_2')/6)^\alpha + w(g) + w(h) \geq (w(L')/6)^\alpha$$

by Lemma 2.6. ∎

Proof of Theorem 3.1. We use induction on $|E(G)|$. By Lemma 3.2, we may assume:

$$|E(G)| \geq 7. \tag{3.1}$$

As induction hypothesis, we assume that:

$$\text{the theorem holds for all graphs with fewer than } |E(G)| \text{ edges.} \tag{3.2}$$

We may also assume that:

$$\text{neither } e \text{ nor } f \text{ belongs to a splittable pair in } G. \tag{3.3}$$

For, suppose by symmetry that $\{e, g\}$ is a splittable pair in G, with $V(e) = \{v, u\}$ and $V(g) = \{v, x\}$. Let $G' := G_v^{e,g}$ be obtained from G by splitting $\{e, g\}$ at v, and assign weight 1 to the new edge e' which corresponds to e and g, and also to the other new edge e'' when $d(v) = 4$. Let $f' = f$ if none of $V(f)$ is suppressed; otherwise let $f' = e'$ (if $f = g$) or $f' = e''$ (if $f \neq g$).

Note that $w(G') \geq w(G) - 6$. Also note that $|E(G')| \geq |E(G)| - 2 \geq 5$ (by (3.1)); so $G' \neq K_2^3$. Hence, by (3.2), G' contains an Eulerian subgraph H' such that $e', f' \in E(H')$ and $w(H') \geq (w(G')/6)^\alpha + 2$. Let H be obtained from H' by replacing e' with e and g and replacing e'' (if it exists in H') with the corresponding edges in G. Then by Lemma 2.6,

$$w(H) \geq w(H') + 1 \geq (w(G')/6)^\alpha + 2 + 1 \geq (w(G)/6)^\alpha + 2.$$

□

Assumption (3.3) implies in particular that neither e nor f is a loop or is adjacent to a loop.

We may further assume that:

$$e \text{ and } f \text{ are not adjacent.} \tag{3.4}$$

Suppose on the contrary that $V(e) = \{v, u\}$ and $V(f) = \{v, x\}$. Then $d(v) = 3$ by (3.3) and Lemma 2.1. So $x \neq u$; for otherwise, by (3.3) and Lemma 2.1 we would also have $d(u) = 3$, and (since $G \ncong K_2^3$) G would not be 3-edge-connected.

Let g denote the edge incident with v other than e and f, and let y be the end of g other than v. Note that $y \neq v$ as $d(v) = 3$. Let G' be obtained from $G - g$ by suppressing degree 2 vertices (namely, v and possibly y) and assign weight 1 to the new edge(s) which resulted from the vertex suppression(s). So $w(G') \geq w(G) - 6$ if both e and f have weight 1 in G; otherwise $w(G') \geq w(G) - 8$ and e or f has weight 2 in G. By (3.1), $|E(G')| \geq 4$, and hence $G' \neq K_2^3$. Let e' denote the edge of G' obtained by suppressing v, and if $d(y) = 3$ let e'' denote the edge of G' obtained by supressing y.

First, consider the case when G' is 3-edge-connected. Let f' be an arbitrary edge of G' that is adjacent to e'. By (3.2), G' contains an Eulerian subgraph H' such that $e', f' \in E(H')$ and $w(H') \geq (w(G')/6)^\alpha + 2$. Let H be obtained from H' by replacing e' with e and f and by replacing e'' (if e'' exists and belongs to H') with the edges of $G - g$ incident with y. Now H is an Eulerian subgraph of G and $e, f \in H$. If both e and f have weight 1 in G then $w(H) \geq w(H') + 1 \geq ((w(G) - 6)/6)^\alpha + 2 + 1 \geq (w(G)/6)^\alpha + 2$ (by Lemma 2.6). Otherwise, $w(H) \geq w(H') + 2 \geq ((w(G) - 8)/6)^\alpha + 2 + 2 \geq (w(G)/6)^\alpha + 2$ (by Lemma 2.6).

Thus we may assume that G' is not 3-edge-connected. Then G' has a 2-edge-cut $S = \{g_1, g_2\}$ such that u, x are contained in the same component of $G' - S$, say G_1. We choose S such that G_1 is minimal (under subgraph containment). Let G_2 denote the other component of $G' - S$, and let $V(g_1) = \{u_1, u_2\}$ and $V(g_2) = \{v_1, v_2\}$ with $u_i, v_i \in G_i$ for $i = 1, 2$.

Let G_1' be obtained from G_1 by adding an edge f' between u_1 and v_1 (which may be a loop) and assign f' weight 1. By the minimality of G_1 we see that G_1' is 3-edge-connected. When $G_1' \neq K_2^3$ we may use (3.2) to deduce that G_1' contains an Eulerian subgraph H_1' such that $e', f' \in H_1'$ and $w(H_1') \geq (w(G_1')/6)^\alpha + 2$. In the case when $G_1' = K_2^3$, we choose H_1' to be the Eulerian subgraph of G_1' with $E(H_1') = \{e', f'\}$ and $w(H_1') = w(e') + w(f') = 2$.

Let G_2' be obtained from G by contracting $G[V(G_1) \cup \{v\}]$ to a single vertex z. Then G_2' is 3-edge-connected. Assign weight 1 to g, g_1, g_2 in G_2'. Since G' is not 3-edge-connected, we see that $|E(G_2)| \geq 1$; so $G_2' \neq K_2^3$. Hence by (3.2), G_2' contains an Eulerian subgraph H_2' such that $g_1, g_2 \in H_2'$ and $w(H_2') \geq (w(G_2')/6)^\alpha + 2$.

Let H be the subgraph of G induced by $E(H_1' - \{e', f'\}) \cup \{e, f\} \cup E(H_2')$. Then H is an Eulerian subgraph of G (as both H_1' and H_2' are 2-edge-connected), $e, f \in E(H)$ and $w(H) \geq w(H_1') + w(H_2')$. If $G_1' = K_2^3$ then $w(G_2') \geq w(G) - 9$ and $w(H) \geq 2 + ((w(G)-9)/6)^\alpha + 2 \geq (w(G)/6)^\alpha + 2$ by Lemma 2.6. So assume $G_1' \neq K_2^3$. Note that $w(G_1') + w(G_2') \geq w(G_1) + 1 + w(G_2) + 3 \geq w(G) - 8$. Then $w(H) \geq (w(G_1')/6)^\alpha + 2 + (w(G_2')/6)^\alpha + 2 \geq (w(G)/6)^\alpha + 2$ again by Lemma 2.6. □

We say that a 3-edge-cut S of G is *trivial* if some component of $G - S$ consists of a single vertex and no edge. Otherwise we say that S is *non-trivial*. We may assume that:

$$\text{neither } e \text{ nor } f \text{ is contained in a non-trivial 3-edge-cut of } G. \tag{3.5}$$

For, suppose $S = \{e, g_1, g_2\}$ is a 3-edge-cut of G and let G_1, G_2 be the components of $G - S$ such that $|E(G_i)| \geq 1$ for $i = 1, 2$. Let $V(e) = \{u_1, u_2\}$, $V(g_1) = \{x_1, x_2\}$ and $V(g_2) = \{y_1, y_2\}$ with $u_i, x_i, y_i \in V(G_i)$, $i = 1, 2$. Let G_i' be obtained from G by contracting G_{3-i}, for $i = 1, 2$. By symmetry, assume $f \in E(G_1) \cup S$. Assign weight 1 to e, g_1, g_2 in both G_1' and G_2'. Then $w(G_1') + w(G_2') \geq w(G)$ as the weight of every edge of G is at most 2.

Note that for $i = 1, 2$, G_i' is 3-edge-connected, and $G_i' \neq K_2^3$ (since $|E(G_i)| \geq 1$). So by (3.2), G_1' contains an Eulerian subgraph H_1' such that $e, f \in H_1'$ and $w(H_1') \geq (w(G_1')/6)^\alpha + 2$. Without loss of generality, we may assume that $g_1 \in H_1'$. By (3.2), G_2' contains an Eulerian subgraph H_2' such that $e, g_1 \in H_2'$ and $w(H_2') \geq (w(G_2')/6)^\alpha + 2$.

Let H be the subgraph of G induced by $E(H_1') \cup E(H_2')$. Then H is an Eulerian subgraph of G containing e, f and $w(H) \geq w(H_1') + w(H_2') - 2 \geq (w(G_1')/6)^\alpha + (w(G_2')/6)^\alpha + 2 \geq (w(G)/6)^\alpha + 2$ by Lemma 2.6. □

We may also assume that:

$$\text{for any 3-edge-cut } S \text{ of } G,\ e \text{ and } f \text{ are contained in the same component of } G - S. \tag{3.6}$$

Suppose on the contrary that $S = \{g_1, g_2, g_3\}$ is a 3-edge-cut of G such that $e \in G_1$ and $f \in G_2$, where G_1, G_2 are the components of $G - S$. Let $V(g_1) = \{x_1, x_2\}$, $V(g_2) = \{y_1, y_2\}$, and $V(g_3) = \{z_1, z_2\}$ such that $x_i, y_i, z_i \in G_i$ for $i = 1, 2$.

Let G_i' be obtained from G by contracting G_{3-i}, for $i = 1, 2$. In both G_1' and G_2', assign weight 1 to g_1, g_2 and g_3. Then $w(G_i) = w(G_i') - 3$; so $w(G_1') + w(G_2') = w(G_1) + w(G_2) + 6 \geq w(G)$.

Note that G_i' is 3-edge-connected and, since $|E(G_i)| \geq 1$, $G_i' \neq K_2^3$. By symmetry, we may assume $|G_1'| \leq |G_2'|$.[1] By (3.2), G_1' contains an Eulerian subgraph H_1' such that $e, g_1 \in H_1'$

[1]This assumption will not be used in the proof of (3.6) but will be important when we convert the proof into a polynomial time algorithm in Section 5.

and $w(H_1') \geq (w(G_1')/6)^\alpha + 2$. Without loss of generality, we may assume $g_2 \in H_1'$ (so $g_3 \notin H_1'$). By (3.2) again, G_1' contains an Eulerian subgraph H_1'' such that $e, g_3 \in H_1''$ and $w(H_1'') \geq (w(G_1')/6)^\alpha + 2$. We now have a symmetry between g_1 and g_2, and we may thus assume that $g_1 \in H_1''$.

In G_2' we find an Eulerian subgraph H_2' such that $f, g_1 \in H_2'$ and $w(H_2') \geq (w(G_2')/6)^\alpha + 2$. If $g_2 \in H_2'$, let H be the subgraph of G induced by $E(H_1') \cup E(H_2')$; otherwise we have $g_3 \in H_2'$ and we let H the subgraph of G induced by $E(H_1'') \cup E(H_2')$. Then H is an Eulerian subgraph of G such that $e, f \in H$, and $w(H) = w(H_1') + w(H_2') - 2$ or $w(H) = w(H_1'') + w(H_2') - 2$. Hence $w(H) \geq (w(G_1')/6)^\alpha + (w(G_2')/6)^\alpha + 2 \geq (w(G)/6)^\alpha + 2$ by Lemma 2.6. □

We may further assume that:

$$\text{the vertices incident to } e \text{ and } f \text{ all have degree 3 in } G. \tag{3.7}$$

Suppose on the contrary that $V(e) = \{u, v\}$ and $d(v) \geq 4$. By (3.3), e is not in any splittable pair of G. Lemmas 2.1 and 2.2 now imply that $d(v) = 5$, and $V(G) - \{v\}$ has a partition Y_0, Y_1, Y_2 such that $u \in Y_0$, $\delta(v, Y_0) = 1$, $\delta(v, Y_1) = \delta(v, Y_2) = 2$, $\delta(Y_0, Y_1) = \delta(Y_0, Y_2) = 1$, and $\delta(Y_1, Y_2) = 0$. See Figure 1. By (3.5), $|Y_0| = 1$. So by (3.4), $f \in Y_1$ or $f \in Y_2$. By symmetry, we may assume $f \in Y_1$. But then the edges from Y_1 to $\{u, v\}$ form a non-trivial 3-edge-cut in G which separates e from f, contradicting (3.6). □

Let $V(e) = \{u, v\}$. By (3.7), $d(u) = d(v) = 3$. Let g_i, $i = 1, 2$, denote the other two edges incident with u with $V(g_i) = \{u, u_i\}$; and let h_i, $i = 1, 2$, denote the other two edges incident with v with $V(h_i) = \{v, v_i\}$. Since G is 3-edge-connected and $d(v) = d(u) = 3$, e is the only edge between u and v. So $v \neq u_i$ and $u \neq v_i$ for $i = 1, 2$.

Let G_i, $i = 1, 2$, be obtained from $G - g_i$ by suppressing u to e' and, if $d_G(u_i) = 3$, suppressing u_i to e_i. Define $f' = f$ if $u_i \notin V(f)$ or u_i is not suppressed, and otherwise let $f' = e_i$. Similarly, let H_i, $i = 1, 2$, be obtained from $G - h_i$ by suppressing v to e' and, if $d_G(v_i) = 3$, suppressing v_i to f_i. Define $f' = f$ if $v_i \notin V(f)$ or v_i is not suppressed, and otherwise let $f' = f_i$.

We may assume that:

$$G_1,\ G_2,\ H_1, \text{ and } H_2 \text{ are not 3-edge-connected.} \tag{3.8}$$

Suppose on the contrary that G_1 is 3-edge-connected. Assign weight 1 to the edges of G_1 which resulted from vertex suppressions. Note that $w(G_1) \geq w(G) - 6$ if both e and g_2 have weight 1 in G; otherwise $w(G_1) \geq w(G) - 8$. By (3.2), G_1 contains an Eulerian subgraph H' such that $e', f' \in E(H')$ and $w(H') \geq (w(G_1')/6)^\alpha + 2$. Let H be obtained from H' by replacing e' with e and g_1 and, if e_1 exists and belongs to H', replacing it with the suppressed edges at u_1. Then H is an Eulerian subgraph of G such that $e, f \in H$. If e and g_2 both have weight 1 in G then $w(H) \geq w(H') + 1 \geq ((w(G) - 6)/6)^\alpha + 1 + 2 \geq (w(G)/6)^\alpha + 2$ by Lemma 2.6. So assume that e or g_2 has weight 2 in G. Then $w(H) \geq w(H') + 2 \geq ((w(G) - 8)/6)^\alpha + 2 + 2 \geq (w(G)/6)^\alpha + 2$, again by Lemma 2.6. □

Since G is 3-edge-connected, G_i, H_i are all 2-edge-connected. By (3.8), we may choose a 2-edge-cut S_i of G_i. Note that $S_i \cup \{g_i\}$ is a 3-edge-cut in G; so by (3.6), some component C_i of $G - S_i$ satisfies $e, f \notin C_i$. We choose S_i and C_i such that C_i is maximal. Then $|E(C_i)| \geq 1$; as otherwise, G_i would be 3-edge-connected (by the maximality of C_i). Similarly, we choose T_i to be a 2-edge-cut of H_i, D_i to be the component of $H_i - T_i$ such that $e, f \notin D_i$, and suppose that T_i, D_i have been chosen such that D_i is maximal (so $|E(D_i)| \geq 1$). We remark that the argument given below to verify (3.9) does not use the maximality of C_i and D_i; this maximality will be used later to ensure that the graph obtained from G_i, or H_i, by contracting C_i, or D_i, to a single vertex of degree two and then suppressing this vertex, is 3-edge-connected.

We next show that:

$$C_1,\ C_2,\ D_1 \text{ and } D_2 \text{ are pairwise disjoint.} \tag{3.9}$$

First, suppose $C_1 \cap C_2 \neq \emptyset$. Since $u, v \notin V(C_1 \cup C_2)$, $C_1 \cup C_2 \neq V(G)$. Since G is 3-edge-connected, we have

$$3 + 3 = \delta_G(C_1) + \delta_G(C_2) \geq \delta_G(C_1 \cap C_2) + \delta_G(C_1 \cup C_2) \geq 3 + 3.$$

Thus equality must hold throughout and, in particular, $\delta_G(C_1 \cup C_2) = 3$. Since $d_G(u) = 3$ and $\delta_G(u, C_1 \cup C_2) = 2$ we have $\delta_G(C_1 \cup C_2 \cup \{u\}) = 2$. Since $v \notin V(C_1 \cup C_2) \cup \{u\}$, this contradicts the fact that G is 3-edge-connected.

Next, suppose $C_1 \cap D_1 \neq \emptyset$. We may deduce as above that $\delta_G(C_1 \cup D_1) = 3$. Since $d_G(u) = 3 = d_G(v)$ and $\delta_G(\{u, v\}, C_1 \cup D_1) = 2$, we have $\delta_G(C_1 \cup D_1 \cup \{u, v\}) = 3$. This contradicts (3.6) since f is an edge of $G - (C_1 \cup D_1 \cup \{u, v\})$.

Similar arguments apply to all other pairs. □

Our current knowledge on the structure of G is illustrated in Figure 4.

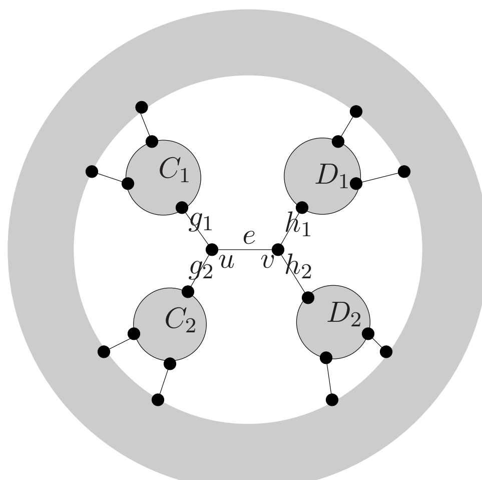

Figure 4: The structure of G around e. Note that for each $i = 1, 2$, one of the edges leaving C_i may be incident to D_j, for some $j = 1, 2$.

Let $V(f) = \{u', v'\}$ and let $g'_i, h'_i, S'_i, C'_i, D'_i$ be defined with respect to f in the same way that g_i, h_i, S_i, C_i, D_i were defined with respect to e. Then $|E(C'_i)| \geq 1$ and $|E(D'_i)| \geq 1$ for $i = 1, 2$, and C'_1, C'_2, D'_1, D'_2 are pairwise disjoint by (3.9) and symmetry. Let $\mathcal{S} = \{C_1, C_2, D_1, D_2\}$, $\mathcal{S}' = \{C'_1, C'_2, D'_1, D'_2\}$, and $K = G - \{u, v, u', v'\} - \bigcup_{X \in \mathcal{S} \cup \mathcal{S}'} X$.

We next show that:

$$\text{for all } X \in \mathcal{S} \text{ and } X' \in \mathcal{S}' \text{ we have either } X = X' \text{ or } X \cap X' = \emptyset. \tag{3.10}$$

Suppose $X \cap X' \neq \emptyset$. Then $\delta_G(X \cap X') \geq 3$ since G is 3-edge-connected. Since $u, v \notin V(X \cup X')$ we also have $\delta_G(X \cup X') \geq 3$. Hence

$$3 + 3 \geq \delta_G(X) + \delta_G(X') \geq \delta_G(X \cap X') + \delta_G(X \cup X') \geq 3 + 3.$$

This implies that $\delta_G(X \cup X') = 3$. The maximality of X and X' now gives $X = X'$. □

We may further assume that:

$$\{C_1, C_2\} \neq \{C_1', C_2'\}. \tag{3.11}$$

Suppose on the contrary that $\{C_1, C_2\} = \{C_1', C_2'\}$. Relabeling if necessary we have $C_1 = C_1'$ and $C_2 = C_2'$. See the first graph in Figure 5. Let k_i be the edge from C_i to $G - (C_i \cup \{u, u'\})$, $i = 1, 2$. Let G^* be obtained from G by contracting $G[C_1 \cup C_2 \cup \{u, u'\}]$ to a single vertex z. Then e, f, k_1, k_2 are the only edges of G^* incident with z. See the second graph in Figure 5. The graph G^* is 3-edge-connected since contraction cannot reduce edge-connectivity.

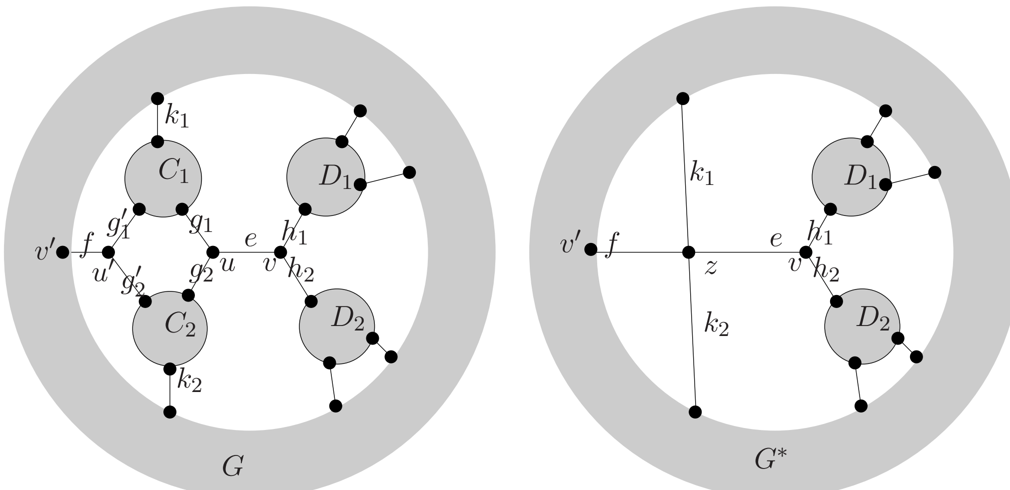

Figure 5: $C_1 = C_1'$ and $C_2 = C_2'$.

We first consider the case when $\{e, k_1\}$ is splittable at z in G^*. Let $\tilde{G}$ be the graph obtained from G^* by splitting e, k_1 at z, and let $\tilde{e}, \tilde{f}$ be the edges of $\tilde{G}$ which correspond to e, k_1 and f, k_2, respectively. Assign weight 1 to $\tilde{e}$ and to $\tilde{f}$. By induction, $\tilde{G}$ has an Eulerian subgraph $\tilde{H}$ containing $\tilde{e}, \tilde{f}$ and with $w(\tilde{H}) \geq (w(\tilde{G})/6)^\alpha + 2$. For $i = 1, 2$ let C_i^* be the 3-edge-connected graph obtained from G by contracting $G - C_i$ to a single vertex z_i. By (3.2), C_1^* has an Eulerian subgraph H_1 containing g_1, k_1 and with $w(H_1) \geq (w(C_1^*)/6)^\alpha + 2$. Similarly, C_2^* has an Eulerian subgraph H_2 containing g_2', k_2 and with $w(H_2) \geq (w(C_2^*)/6)^\alpha + 2$. Let H be the Eulerian subgraph of G with $E(H) = (E(\tilde{H}) - \{\tilde{e}, \tilde{f}\}) \cup E(H_1) \cup E(H_2) \cup \{e, f\}$. Then

$$w(H) \geq w(\tilde{H}) + w(H_1) + w(H_2) \geq (w(\tilde{G})/6)^\alpha + (w(C_1^*)/6)^\alpha + 2 + (w(C_2^*)/6)^\alpha + 2.$$

Since $w(G) \geq (w(\tilde{G}) - 2) + w(C_1^*) + w(C_2^*) + w(\{e, f\})$, we may use Lemma 2.6 to deduce that $w(H) \geq (w(G)/6)^\alpha + 2$.

Hence we may assume that $\{e, k_1\}$ is not splittable at z in G^*, and, by symmetry, $\{e, k_2\}$ is not splittable at z in G^*. Thus $\{e, f\}$ is the only splittable pair at z in G^* that contains e.

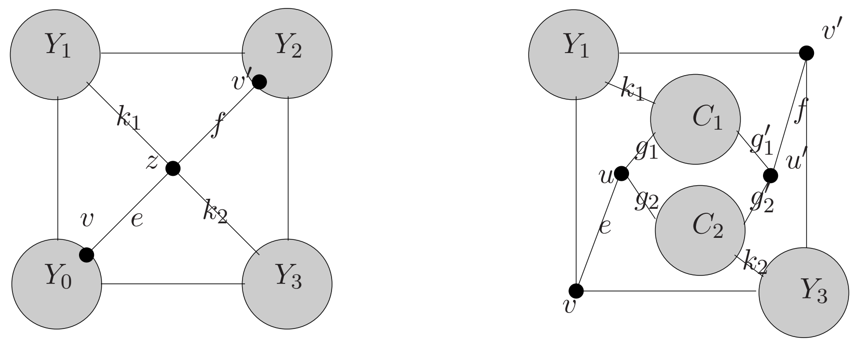

Figure 6: $\{e, f\}$ is the only splittable pair at z.

We may now choose a partition Y_0, Y_1, Y_2, Y_3 of $V(G^*) - z$ satisfying the conclusions of Lemma 2.3, and with $v \in Y_0$ and $v' \in Y_2$. See the first graph in Figure 6. We have $\delta_G(Y_i) = 3$ for all $0 \leq i \leq 3$. Thus (3.5) implies that $Y_0 = \{v\}$ and $Y_2 = \{v'\}$. We may now deduce that G has the structure illustrated in the second graph of Figure 6, and that the graph $\tilde{G}$ obtained from G by contracting Y_1, Y_3, C_1, C_2 to single vertices y_1, y_3, c_1, c_2, respectively, is isomorphic to the cube. We may construct a Hamilton cycle $\tilde{H} = uvy_3c_2u'v'y_1c_1u$ in $\tilde{G}$ which contains e, f. By Lemma 3.3, G has an Eulerian subgraph H such that all edges of $\tilde{H}$ are in H and

$$\begin{aligned}
w(H)) &\geq (w(C_1)/6)^\alpha + (w(C_2)/6)^\alpha + (w(Y_1)/6)^\alpha + (w(Y_3)/6)^\alpha + w(\tilde{H}) \\
&\geq ([w(C_1) + w(C_2) + w(Y_1) + w(Y_3)]/6)^\alpha + 8 \\
&\geq ([w(G) - 24]/6)^\alpha + 8 \\
&\geq (w(G)/6)^\alpha + 2
\end{aligned}$$

by Lemma 2.6. □

We may further assume that:

$$\mathcal{S} \neq \mathcal{S}'. \tag{3.12}$$

Suppose on the contrary that $\mathcal{S} = \mathcal{S}'$. By (3.11) and symmetry we may assume that $C'_1 = C_1, C'_2 = D_2, D'_1 = C_2, D'_2 = D_1$ and

$$w(D_2) = \min\{w(C_1), w(C_2), w(D_1), w(D_2)\}.$$

We first consider the case when K is empty. See the first graph in Figure 7. Then (3.5) implies that $\delta_G(C_1, C_2) = \delta_G(D_1, D_2) = \delta_G(C_2, D_1) = \delta_G(C_1, D_2) = 0$. Hence $\delta_G(C_1, D_1) = \delta_G(C_2, D_2) = 1$ and the graph $\tilde{G}$ obtained from G by separately contracting each of C_1, C_2, D_1, D_2 to single vertices c_1, c_2, d_1, d_2, respectively, is isomorphic to the Wagner graph. In $\tilde{G}$ there is a cycle $\tilde{H} = uvd_1c_1u'v'c_2u$ containing e, f. By Lemma 3.3, G has an Eulerian subgraph H such that all edges of $\tilde{H}$ are in H and

$$\begin{aligned}
w(H) &\geq (w(C_1)/6)^\alpha + (w(C_2)/6)^\alpha + (w(D_1)/6)^\alpha + w(\tilde{H}) \\
&\geq ([w(C_1) + w(C_2) + w(D_1) + w(D_2)]/6)^\alpha + 7 \\
&\geq ([w(G) - 24]/6)^\alpha + 7 \\
&\geq (w(G)/6)^\alpha + 2,
\end{aligned}$$

where the second inequality uses the minimality of $w(D_2)$ and Corollary 2.9(b), the third inequality uses the fact that there are 12 edges in G which do not belong to C_1, C_2, D_1 or D_2, and the last inequality uses Lemma 2.6.

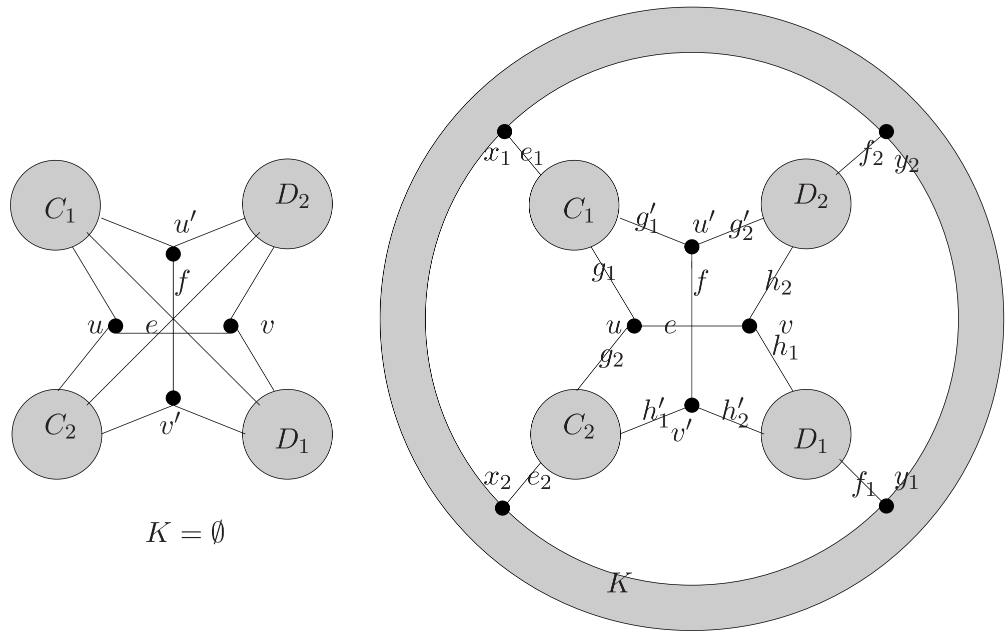

Figure 7: The structure of G when $\mathcal{S} = \mathcal{S}'$.

Hence we may assume that K is not empty. The 3-edge-connectivity of G now implies that $\delta_G(X, K) = 1$ and $\delta_G(X, Y) = 0$ for all $X, Y \in \mathcal{S}$. Let e_1, e_2, f_1, f_2 be the edges from K to C_1, C_2, D_1, D_2, respectively, and let x_1, x_2, y_1, y_2 be the endvertices of e_1, e_2, f_1, f_2 in K, respectively. See the second graph in Figure 7. Since G is 3-edge-connected, K is connected.

Suppose K has a cut edge, say k, separating $\{x_1, y_1\}$ from $\{x_2, y_2\}$. Let J_i denote the component of $K - k$ containing $\{x_i, y_i\}$, for $i = 1, 2$. See Figure 8. Then $\delta(J_i) = 3$ for $i = 1, 2$. Let $\tilde{G}$ be obtained from G by separately contracting $C_1, C_2, D_1, D_2, J_1, J_2$ to single vertices $c_1, c_2, d_1, d_2, j_1, j_2$, respectively. Then $\tilde{G}$ is isomorphic to the Petersen graph and we may construct a cycle $\tilde{H}$ in $\tilde{G}$ which contains e, f and all vertices of $\tilde{G}$ other than d_2. By Lemma 3.3, G has an Eulerian subgraph H such that all edges of $\tilde{H}$ are in H and

$$\begin{aligned} w(H) &\geq (w(C_1)/6)^\alpha + (w(C_2)/6)^\alpha + (w(D_1)/6)^\alpha + (w(J_1)/6)^\alpha + (w(J_2)/6)^\alpha + w(\tilde{H}) \\ &\geq ([w(C_1) + w(C_2) + w(D_1) + w(D_2)]/6)^\alpha + (w(J_1)/6)^\alpha + (w(J_2)/6)^\alpha + 9 \\ &\geq ([w(C_1) + w(C_2) + w(D_1) + w(D_2) + w(J_1) + w(J_2)]/6)^\alpha + 9 \\ &\geq ([w(G) - 30]/6)^\alpha + 9 \\ &\geq (w(G)/6)^\alpha + 2, \end{aligned}$$

where the second inequality uses the minimality of $w(D_2)$ and Corollary 2.9(b), the third and fifth inequalities use Lemma 2.6, and the fourth inequality uses the fact that there are 15 edges in G which do not belong to C_1, C_2, D_1, D_2, J_1 or J_2.

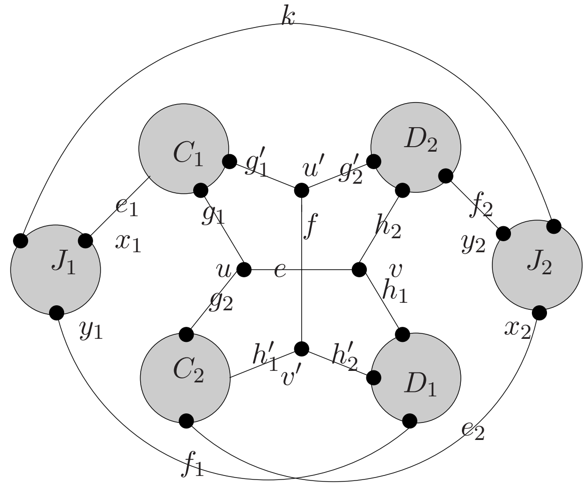

Figure 8: The case when K has a cut edge separating $\{x_1, y_1\}$ from $\{x_2, y_2\}$

Hence we may assume that K has a no cut edge separating $\{x_1, y_1\}$ from $\{x_2, y_2\}$. Let L be the graph obtained from K by adding two new vertices z_1, z_2, an edge g from z_1 to z_2, and four other edges joining z_1 to x_1, y_1 and z_2 to x_2, y_2. Let F be obtained from L by contracting g, and let K^* be obtained from $L - g$ by suppressing z_1 and z_2 to edges k_1 k_2, respectively, of weight 1. Then F is 3-edge-connected since it can be obtained from G by contracting $G - K$ to a single vertex. The fact that K has no cut edge separating $\{x_1, y_1\}$ from $\{x_2, y_2\}$ now implies that L is also 3-edge-connected. We may now apply Lemma 3.4 to L to deduce that K^* has an Eulerian subgraph H^* such that $k_1, k_2 \in E(H^*)$ and $w(H^* - \{k_1, k_2\}) \geq (w(K)/6)^\alpha$.

Let G' be obtained from G by contracting K to a single vertex z, and $\tilde{G}'$ be obtained from G' by contracting C_1, C_2, D_1, D_2 to single vertices c_1, c_2, d_1, d_2, respectively. Then $\tilde{H}' := uvd_1zd_2u'v'c_2zc_1u$ is an Eulerian subgraph of $\tilde{G}'$ which contains e, f. By Lemma 3.3, there is an Eulerian subgraph H' of G' such that all edges of $\tilde{H}'$ are contained in H' and

$$\begin{aligned} w(H') &\geq (w(C_1)/6)^\alpha + (w(C_2)/6)^\alpha + (w(D_1)/6)^\alpha + (w(D_2)/6)^\alpha + w(\tilde{H}') \\ &\geq ([w(C_1) + w(C_2) + w(D_1) + w(D_2)]/6)^\alpha + 10 \\ &\geq ([w(G) - w(K) - 28]/6)^\alpha + 10 \end{aligned}$$

where the second inequality uses Lemma 2.6, and last inequality uses the fact that there are 14 edges in G which do not belong to C_1, C_2, D_1, D_2, or K.

The facts that $k_1, k_2 \in E(H^*)$ and that $E(\tilde{H}') \subseteq E(H')$ imply that $E(H^* - \{k_1, k_2\}) \cup E(H')$ induces an Eulerian subgraph H of G with $e, f \in H$ and

$$w(H) \geq w(H^* - \{k_1, k_2\}) + w(H') \geq (w(K)/6)^\alpha + ([w(G) - w(K) - 28]/6)^\alpha + 10 \geq (w(G)/6)^\alpha + 2$$

by Lemma 2.6. $\square$

Let $\mathcal{S} \cup \mathcal{S}' = \{X_1, X_2, \ldots, X_q\}$ where $w(X_1) \geq w(X_2) \geq \ldots w(X_q)$. By (3.12), $q \geq 5$. Relabeling if necessary we may suppose that $X_q = C_1$. Let $r = 4^{1/\alpha} - q$. We may assume that:

$$w(K) \leq rw(C_1). \tag{3.13}$$

Suppose on the contrary that $w(K) \geq rw(C_1)$. Since

$$w(G) \geq \sum_{i=1}^{q} w(X_i) + w(K) + w(e, f, g_1, g_2, h_1, h_2, g_1', g_2', h_1', h_2')$$

we have

$$w(G) - w(C_1) - w(C_2) - 10 \geq (q - 2 + r)w(C_1) = (4^{1/\alpha} - 2)w(C_1) = 3^{1/\alpha}w(C_1).$$

Recall that, for $i = 1, 2$, S_i is the 2-edge-cut which separates C_i from e in $G - g_i$. See Figure 4. Let $S_1 = \{e_1, e_2\}$, $V(e_i) = \{x_i, y_i\}$ with $x_i \in C_1$ and $y_i \notin C_1$. Similarly let $S_2 = \{l_1, l_2\}$, $V(l_i) = \{a_i, b_i\}$ with $a_i \in C_2$ and $b_i \notin C_2$. Let G' be obtained from G by deleting C_1, suppressing u to e', and adding an edge g with $V(g) = \{y_1, y_2\}$ (which may be a loop). Assign weight 1 to both e' and g in G'. Recall that the maximality of C_1 implies that G' is 3-edge-connected.

Let G'' be obtained from G' by contracting C_2 to a vertex c_2 and assign weight one to e', l_1, l_2 in G''. Since G' is 3-edge-connected, G'' is 3-edge-connected. We also have $f \in E(G'')$ by (3.5) and (3.6). By (3.2), G'' has an Eulerian subgraph H'' such that $e', f \in E(H'')$ and $w(H'') \geq (w(G'')/6)^\alpha + 2 \geq ([w(G) - w(C_1) - w(C_2) - 10]/6)^\alpha + 2$.

Without loss of generality, we may assume $l_1 \in H''$. Let C_2^* be the 3-edge-connected graph obtained from G by contracting $G - C_2$ to the single vertex z. Assign weight 1 to g_2, l_1, l_2 in C_2^*. Recall that $E(C_2) \neq \emptyset$, and hence $C_2^* \neq K_2^3$. So by (3.2), C_2^* contains an Eulerian subgraph H' such that $g_2, l_1 \in E(H')$ and $w(H') \geq (w(C_2^*)/6)^\alpha + 2 \geq ([w(C_2) + 3]/6)^\alpha + 2$.

Let $J = (H'' - c_2) \cup (H' - z) \cup \{l_1, e'\}$. Then $J \subseteq G'$ and $w(J) \geq w(H') + w(H'') - 2$. Let $H = (J - e') \cup \{u, e, g_2\}$ if $g \notin E(J)$, and otherwise let H be the Eulerian subgraph of G obtained from $(J - e') \cup \{u, e, g_2\}$ by replacing g by a path P between y_1 and y_2 and with $E(P) \subseteq E(C_1) \cup S_1$. Then $e, f \in E(H)$ and $w(H) \geq w(J) + 1 \geq w(H') + w(H'') - 1$. Now Corollary 2.9(a) and the facts that $w(C_2) \geq w(C_1)$, and $w(G) - w(C_1) - w(C_2) - 10 \geq 3^{1/\alpha}w(C_1)$, give:

$$\begin{aligned} w(H) &\geq w(H'') + w(H') - 1 \\ &\geq ([w(G) - w(C_1) - w(C_2) - 10]/6)^\alpha + ((w(C_2) + 3)/6)^\alpha + 3 \\ &\geq (w(G)/6)^\alpha + 2. \end{aligned}$$

□

We can now complete the proof of the theorem. Note that $4^{1/\alpha} < 7$ so the fact that $0 \leq w(K) \leq (4^{1/\alpha} - q)w(C_1)$ by (3.13) implies that $q \leq 6$. For all $1 \leq i \leq 4$ we have

$$\sum_{j=5}^{q} w(X_j) + w(K) \leq (q - 4 + r)w(X_i) = (4^{1/\alpha} - 4)w(X_i) \tag{3.14}$$

by (3.13). Choose $x_i \in V(X_i)$ for $1 \leq i \leq 4$.

Suppose that no Eulerian subgraph of G contains $\{x_1, x_2, x_3, x_4, e, f\}$. Then, by Lemma 2.5, G has the structure depicted in Figure 3(a) or (b). Since all 3-edge-cuts which contain e or f are trivial by (3.6), we have $|Z_i| = 1$ for $5 \leq i \leq 8$. We may now deduce that $\mathcal{S} = \{Z_1, Z_2, Z_3, Z_4\} = \mathcal{S}'$, which contradicts the fact that $\mathcal{S} \neq \mathcal{S}'$ by (3.12).

Thus $\{x_1, x_2, x_3, x_4, e, f\}$ is contained in an Eulerian subgraph H' of G. Let $\tilde{G}$ be the graph obtained from G by contracting X_i to the single vertex x_i for $1 \le i \le 4$. We may obtain an Eulerian subgraph $\tilde{H}$ of $\tilde{G}$ which contains $\{x_1, x_2, x_3, x_4, e, f\}$ from H' by contracting the edges which belong to Z_i for all $1 \le i \le 4$. By Lemma 3.3, there is an Eulerian subgraph H of G such that all edges of $\tilde{H}$ are contained in H and

$$\begin{aligned} w(H) &\ge (w(X_1)/6)^\alpha + (w(X_2)/6)^\alpha + (w(X_3)/6)^\alpha + (w(X_4)/6)^\alpha + w(\tilde{H}) \\ &\ge \left(\left[w(X_1) + w(X_2) + w(X_3) + w(X_4) + \sum_{j=5}^{q} w(X_j) + w(K)\right]/6\right)^\alpha + 8 \\ &\ge ([w(G) - 40]/6)^\alpha + 8 \\ &\ge ([w(G) - 40]/6 + 6^{1/\alpha})^\alpha + 2 \\ &\ge (w(G)/6)^\alpha + 2 \end{aligned}$$

where the second inequality uses (3.14) and Corollary 2.9(c), the third inequality uses the fact that there are at most 20 edges of G which do not belong to $X_1, X_2, \ldots, X_q$ or K, and the fourth inequality uses Lemma 2.6. ∎

4 Corollaries

It is easy to see that Theorem 1.1 is simply the case of Theorem 3.1 when all edges have the same weight 1. Theorem 1.1 in turn has the following consequence.

Corollary 4.1 *Let G be a 3-edge-connected graph with $|G| \ge 2$, and let $e, f \in E(G)$. Then G contains an Eulerian subgraph H such that $e, f \in H$ and $|H| \ge (|G|/12)^\alpha + 1$, where $\alpha \approx 0.753$ is the real root of $4^{1/x} - 3^{1/x} = 2$.*

Proof. Choose a counterexample G so that $|E(G)|$ is minimum. It is easy to see that the corollary holds if $|G| = 2$ and hence $|G| \ge 3$.

Suppose G has a vertex v of degree at least 5. Then by Lemma 2.1 there exists a splittable pair g, h at v. By splitting g, h at v, we arrive at a 3-edge-connected graph $G' := G_v^{g,h}$. Since $d(v) \ge 5$, $|G'| = |G|$. Let $e' = e$ if $e \notin \{g, h\}$; otherwise let e' denote the edge resulted from suppressing v. Define f' analogously. By the choice of G, G' contains an Eulerian subgraph H' such that $e', f' \in H'$ and $|H'| \ge (|G'|/12)^\alpha + 1$. Then H' gives rise the desired Eulerian subgraph in G.

So we may assume $\Delta(G) \le 4$. By Theorem 1.1, G contains an Eulerian subgraph H such that $e, f \in H$ and $|E(H)| \ge (|E(G)|/6)^\alpha + 2$. Since $\Delta(G) \le 4$, $\Delta(H) \le 4$; and so, $|E(H)| \le 2|H|$. Hence $|H| \ge |E(H)|/2 \ge (|G|/12)^\alpha + 1$, a contradiction. ∎

Theorem 1.2 follows directly from the next result.

Theorem 4.2 *Let G be a 3-connected claw-free graph and let $x, y \in V(G)$. Then G contains a cycle C such that $x, y \in C$ and $|C| \ge (|G|/6)^\alpha + 2$, where $\alpha \approx 0.753$ is the real root of $4^{1/x} - 3^{1/x} = 2$.*

Proof. Choose a counterexample G, x, y so that $|G|$ is minimum and, subject to this condition, $|E(G)|$ is maximum.

We claim that G is the line graph of a simple graph G_1. Let G^* denote the Ryjáček closure of G. Suppose $G^* \neq G$. Then $|E(G^*)| > |E(G)|$ so, by the choice of G, G^* has a cycle C^* such that $x, y \in V(C^*)$ and $|C^*| \geq (|G|/6)^\alpha + 2$. Then by Theorem 1.3, G has a cycle C such that $V(C^*) \subseteq V(C)$, a contradiction. So $G = G^*$, and the claim follows.

Since G is 3-connected, for each edge-cut S in G_1 of size at most 2, $G - S$ has exactly two components, one of which is trivial. Let $U = \{v \in V(G_1) : d_{G_1}(v) \geq 3\}$. Then $U \neq \emptyset$, and for any $v \in V(G_1) - U$, all neighbors of v are contained in G_1 and the edges at v form a 1-edge-cut or 2-edge-cut in G_1.

Let G_2 and $w : E(G_2) \to \{1, 2\}$ be defined as follows. For each 1-edge-cut uv of G_1 with $u \in U$ (hence $d_{G_1}(v) = 1$), we delete v and add a loop at u. For each 2-edge-cut $\{ab, bc\}$ of G_1 (hence $d_{G_1}(b) = 2$), we delete b and add an edge between a and c with weight 2. The loops and all other edges in $G_1[U]$ have weight 1. Then G_2 is 3-edge-connected and $w(G_2) = |G|$.

Since $x, y \in V(G)$ we have $x, y \in E(G_1)$. Let $x' = x$ if $x \in E(G_2)$; otherwise, let x' denote the edge of G_2 used to replace x. Define y' analogously. By Theorem 3.1, G_2 contains an Eulerian subgraph H_2 such that $x', y' \in H_2$ and $w(H_2) \geq (w(G_2)/6)^\alpha + 2$. Then H_2 gives rise to a cycle C in G such that $x, y \in C$ and $|C| \geq (w(G_2)/6)^\alpha + 2 = (|G|/6)^\alpha + 2$. ∎

5 Algorithmic considerations

There is a large gap between best known polynomial algorithms for approximating the longest cycle in a graph and hardness results. The best known polynomial time approximation algorithm, due to Gabow [24], finds a cycle of length at least $\exp(\Omega(\sqrt{\log c(G)/\log\log c(G)})$ in any graph G (which gives a polynomial algorithm for constructing a cycle of length at least $\min\{[\log c(G)]^t, c(G)\}$ for any fixed t). Alon, Yuster and Zwick [1], give a polynomial algorithm for constructing a cycle of length at least $\min\{\log |G|, c(G)\}$. On the other hand, Karger, Motwani, and Ramkumar [33], show that it is NP-hard to find a path of length at least $r\ell(G)$ for any fixed $r > 0$, where $\ell(G)$ demotes the length of a longest path in G. Better approximation algorithms are known for graphs of bounded degree, see [20, 21]. In [21], Feder, Motwani and Subi give a polynomial time algorithm for finding a cycle of length at least $c(G)^{(\log_3 2)/2} > c(G)^{0.315}$ in any graph of maximum degree three. Their algorithm is based on a polynomial-time algorithm for constructing a cycle of weight at least $w(G)^{\log_3 2}$ in any 3-connected cubic graph G equipped with nonnegative edge-weights. On the other hand Bazgan, Santha, and Tuza [4], show that, for any fixed $r > 0$, it is NP-hard to find a path of length at least $r|G|$ in a cubic Hamiltonian graph G.

The situation seems to be just as unclear for exact algorithms. Algorithms for solving the Travelling Salesman Problem, [6, 27, 34, 35], can be used to find a Hamilton cycle in a graph G, or deduce that no such cycle exists, in $O^*(2^{|G|})$ time. (The O^*-notation means that factors which are polynomial in $|G|$ are suppressed.) The time complexity can be improved to $O^*(b^{|G|})$, for various constants b with $1 < b < 2$, when G has bounded maximum degree, see [5, 19, 28]. It is conceivable that these algorithms could be modified to give similar results for constructing longest cycles but the only specific results we know of are an algorithm of Monien [37], subsequently improved by Bodlaender [7] to find a longest cycle in an arbitrary graph G in time $O(c(G)!\, 2^{c(G)}\, |G|)$, and a recent result of Broersma et al [10] which gives an $O^*(1.8878^{|G|})$ algorithm for finding a longest cycle when G is claw-free.

We indicate in Subsection 5.1 below how our proof of Theorem 3.1 can be adapted to give a polynomial time algorithm for finding an Eulerian subgraph H in a $\{1, 2\}$-edge-weighted, 3-edge-connected graph G such that $w(H) \geq (w(G)/6)^\alpha + 2$. In particular, this finds a cycle of length at least $(|G|/4)^\alpha > (|G|/4)^{0.753}$ in any 3-connected cubic graph G. Our algorithm uses a subroutine which finds an Eulerian subgraph containing two given edges and four given vertices in a 3-edge-connected graph (when such a subgraph exists). This will be described in Subsection 5.2. We then use the algorithm from Subsection 5.1 to obtain a polynomial algorithm for finding a cycle of length at least $(|G|/6)^\alpha$ in any 3-connected claw-free graph G in Subsection 5.3.

5.1 Large Eulerian subgraphs containing two given edges

Recall the proof of Theorem 3.1. Let G be a 3-edge-connected graph, $e, f \in E(G)$, and $w : E(G) \to \{1, 2\}$. We outline an algorithm for finding an Eulerian subgraph H in G such that $e, f \in H$ and $w(H) \geq (w(G)/6)^\alpha + 2$. For convenience, we write (G, e, f) for the input, with the understanding that edges are assigned weights 1 or 2. We will use the fact that, given a graph G, two disjoint subsets $X, Y \subseteq V(G)$, and a fixed integer k, we can use maxflow computations to find either k edge-disjoint paths joining X to Y, or a minimal set $X' \subseteq V(G)$ with $X \subseteq X'$, $Y \subseteq V(G) \setminus X'$ and $\delta(X') < k$, in $O(|E(G)|)$ time.

Algorithm EULERIANSUBGRAPH
INPUT: A 3-edge-connected graph G, $e, f \in E(G)$, and $w : E(G) \to \{1, 2\}$.
OUTPUT: An Eulerian subgraph H of G such that $e, f \in H$ and $w(H) \geq (w(G)/6)^\alpha + 2$.
COMPLEXITY: $f(|E(G)|) = O(|E(G)|)^3$.

Step 1. Check if e or f belongs to a splittable pair. If not, go to Step 2. If yes, we apply the argument in (3.3) to reduce the problem to (G', e', f'), with $|E(G')| \leq |E(G)| - 1$. This shows $f(|E(G)|) \leq f(|E(G)| - 1) + O(|E(G)|^2)$, as it takes $O(|E(G)|)$ time to check whether a particular splitting preserves 3-edge-connectivity and there are $O(|E(G)|)$ splittings to check.

Step 2. Check if e and f are adjacent. If not, go to Step 3. If yes, then by the argument in (3.4) we reduce the problem to (G', e', f') with $|E(G')| \leq |E(G)| - 1$ (when G' is 3-edge-connected), or (G_1', e', f') and (G_2', g_1, g_2) (when G' is not 3-edge-connected, with $|E(G_1')| + |E(G_2')| = |E(G)|$. Note that G', G_1', G_2' can be found in $O(|E(G)|)$ time. So $f(|E(G)|) \leq f(|E(G')| - 1) + O(|E(G)|)$ or $f(|E(G)|) \leq f(|E(G_1')|) + f(|E(G_2')|) + O(|E(G)|)$.

Step 3. Check to see if e or f is contained in a non-trivial 3-edge-cut of G. If not, go to Step 4. If yes, we use the argument for (3.5) to reduce the problem to (G_1', e, f) and (G_2', e, g_1), with $|E(G_1')| + |E(G_2')| = |E(G)| + 3$ and $|E(G_i')| < |E(G)|$. Note that G_1', G_2' can be found in $O(|E(G)|)$ time, so $f(|E(G)|) \leq f(|E(G_1')|) + f(|E(G_2')|) + O(|E(G)|)$.

Step 4. Check if there is a 3-edge-cut S such that e and f are contained in different components of $G - S$. If not, go to Step 5. If yes, we use the argument for (3.6) to reduce the problem to (G_1', e, g_1), (G_1', e, g_3) and (G_2', g_1, f), with $|E(G_1')| \leq |E(G_2')| < |E(G)|$ and $|E(G_1')| + |E(G_2')| = |E(G)| + 3$. This implies $f(|E(G)|) \leq 2f(|E(G_1')|) + f(|E(G_2')|) + O(|E(G)|)$. Note that the multiplicative factor of '2' in the first term on the right hand side of this inequality is compensated for by the fact that $|E(G_1')| \leq (|E(G)| + 3)/2$.

Step 5. Construct G_i, H_i with respect to e as in the paragraph above (3.8). Similarly construct G'_i, H'_i with respect to f. Check if there is some $i \in \{1, 2\}$ such that G_i, H_i, G'_i or H'_i is 3-edge-connected. If not, got to Step 6. If yes, say, G_1 is 3-edge-connected, then we use the argument for (3.8) to reduce the problem to (G_1, e', f') with $|E(G_1)| \le |E(G)| - 1$. This implies $f(|E(G)|) \le f(|E(G)| - 1) + O(|E(G)|)$.

Step 6. Construct C_i, D_i with respect to e as in the paragraph above (3.9). Similarly construct C'_i, D'_i with respect to f. Check to see if $\{C_1, C_2\} = \{C'_1, C'_2\}$ or $\{C_1, C_2\} = \{D'_1, D'_2\}$ or $\{D_1, D_2\} = \{C'_1, C'_2\}$ or $\{D_1, D_2\} = \{D'_1, D'_2\}$. If not, go to Step 7. If yes, say $\{C_1, C_2\} = \{C'_1, C'_2\}$, apply the argument in (3.11): we either reduce the problem to $(\tilde{G}, \tilde{e}, \tilde{f})$, (C^*_1, g_1, k_1) and (C^*_2, g'_2, k_2); or find a partition Y_0, Y_1, Y_2, Y_3 of $V(G^*) - \{z\}$ given by Lemma 2.3, and reduce the problem to $G/(G - Y_1), G/(G - Y_3), G/(G - C_1)$ and $G/(G-C_2)$ via Lemma 3.3. In the first case, $f(|E(G)|) \le f(|E(\tilde{G})|) + f(|E(C^*_1)|) + f(|E(C^*_2)|) + O(|E(G)|)$ with $|E(\tilde{G})|, |E(C^*_1)|, |E(C^*_2)| < |E(G)|$ and $|E(\tilde{G})| + |E(C^*_1)| + |E(C^*_2)| \le |E(G)| + 6$. In the latter case, $f(|E(G)|) \le f(|E(Y_1)| + 3) + f(|E(Y_3)| + 3) + f(|E(C_1)|+3) + f(|E(C_2)|+3) + O(|E(G)|)$ with $|E(Y_1)| + |E(Y_3)| + |E(C_1)| + |E(C_2)| \le |E(G)| - 8$.

Step 7. Check to see if $\mathcal{S} = \mathcal{S}'$. If not go to Step 8. If yes, we apply the argument in (3.12). We find the member of $\mathcal{S}$ with minimum weight, say D_2. When $K = \emptyset$, we reduce the problem to $G/(G-C_1), G/(G-D_1), G/(G-C_2)$ via Lemma 3.3. We have $f(|E(G)|) \le f(|E(C_1)| + 3) + f(|E(C_2)| + 3) + f(|E(D_1)| + 3) + O(|E(G)|)$ with $|E(C_1)| + |E(C_2)| + |E(D_1)| \le |E(G)| - 8$. When $K \ne \emptyset$, we reduce the problem to either $G/(G-C_1), G/(G-C_2), G/(G-D_1), J_1, J_2$ via Lemma 3.3 (if K has a cut-edge separating $\{x_1, y_1\}$ from $\{x_2, y_2\}$), or to $K^*, G/(G - C_1), G/(G - C_2), G/(G - D_1), G/(G - D_2)$ via Lemmas 3.4 and 3.3 (if K has no cut-edge separating $\{x_1, y_1\}$ from $\{x_2, y_2\}$). In the former case, $f(|E(G)|) \le f(|E(C_1)|+3) + f(|E(C_2)|+3) + f(|E(D_1)|+3) + f(|E(J_1)|+3) + f(|E(J_2)|+3) + O(|E(G)|)$ with $|E(C_1)| + |E(C_2)| + |E(D_1)| + |E(J_1)| + |E(J_2)| = |E(G)| - |E(D_2)| - 15$. In the latter case, $f(|E(G)|) \le f(|E(C_1)| + 3) + f(|E(C_2)| + 3) + f(|E(D_1)| + 3) + f(|E(D_2)| + 3) + f(|E(K^*)|) + O(|E(G)|)$ with $|E(C_1)| + |E(C_2)| + |E(D_1)| + |E(D_2)| + |E(K^*)| = |E(G)| - 12$.

Step 8. Check to see if $w(K) \ge r \min_{X \in \mathcal{S} \cup \mathcal{S}'}\{w(X)\}$, where $r = 4^{1/\alpha} - |\mathcal{S} \cup \mathcal{S}'|$. If not, go to Step 9. If yes, we use the argument in (3.3) to reduce the problem to (G'', e', f'), (C^*_2, g_2, l_1), and possibly finding an x_1x_2-path in C_1. We have $|E(G'')| + |E(C^*_2)| < |E(G)|$ and $f(|E(G)|) \le f(|E(G'')|) + f(|E(C^*_2)|) + O(|E(G)|)$.

Step 9. We proceed as in the last paragraph of the proof of Theorem 3.1. Choose the four heaviest subgraphs $X_1, X_2, X_3, X_4 \in \mathcal{S} \cup \mathcal{S}'$ and let $\tilde{G}$ be obtained from G by contracting each X_i to a single vertex x_i, for $1 \le i \le 4$. We can use the algorithm COVER, given in Subsection 5.2 below, to construct an Eulerain subgraph $\tilde{H}$ of $\tilde{G}$ containing $\{e, f, x_1, x_2, x_3, x_4\}$ in time $O(|E(\tilde{G})|^3)$. This allows us to reduce the problem to that for (X'_i, e_i, f_i), $1 \le i \le 4$, where $X'_i = G/(G - X_i)$ and e_i, f_i are the edges of $\tilde{H}$ incident to x_i. This gives $f(|E(G)|) \le \sum_{i=1}^4 f(|E(X'_i)|) + O(|E(\tilde{G})|^3)$, where $|E(X'_i)| < |E(G)|$ for $1 \le i \le 4$ and $|E(\tilde{G})| + \sum_{i=1}^4 |E(X'_i)| = |E(G)| + 12$.

From Steps 1–9, we see that $f(|E(G)|) = O(|E(G)|^3)$. So given a 3-edge-connected graph

G, $e, f \in E(G)$, and weight function $w : E(G) \to \{1, 2\}$, one can, in $O(|E(G)|^3)$ time, find an Eulerian subgraph H such that $e, f \in H$ and $w(H) \geq (w(G)/6)^\alpha + 2$.

5.2 Eulerian subgraphs containing a given set of four vertices and two edges

Let G be a 3-edge-connected graph. We say that G is *essentially* 4*-edge-connected* if all 3-edge-cuts of G are trivial. Let $F \subseteq E(G)$ and $X \subseteq V(G)$ with $|F| = 2$ and $|X| \leq 4$. We say that (G, F, X) is *admissible* if G has an Eulerian subgraph H which contains $F \cup X$. We will outline an $O(|E(G)|^3)$ algorithm which constructs such a subgraph H given an admissible triple (G, F, X). Note that we can check whether a given triple is admissible, and construct disjoint subgraphs $Z_1, Z_2, \ldots, Z_m$ as in Lemma 2.5 if it is not, in $O(|E(G)|^3)$, as follows. We use maxflow computations to check if G is essentially 4-edge-connected in $O(|E(G)|^2)$ time. If yes then it suffices to check if G is the Wagner graph or the Peterson graph with F, X as indicated in Figure 3. If not then we find a non-trivial 3-edge-cut S of G and construct the components G_1, G_2 of $G - S$. If $|V(G_i) \cap X| \leq 1$ and $E(G_i) \cap F = \emptyset$ for some $1 \leq i \leq 2$, then we reduce the problem to (G', F, X') where $G' = G/G_i$ and X' is the image of X under this contraction. Otherwise we deduce that (G, F, X) is admissible.

We first give a special case of the algorithm for cubic graphs. We use a result of Andersen et al [2] that an essentially 4-edge-connected cubic graph G on at least fourteen vertices has at least $(|E(G)| + 12)/5$ *removable edges* i.e. edges e such that $G - e$ is homeomorphic to an essentially 4-edge-connected cubic graph.

Algorithm Cubic cover
INPUT: An admissible triple (G, F, X) where G is cubic.
OUTPUT: A cycle C of G such that $F \cup X$ is contained in C.
COMPLEXITY: $f_1(|E(G)|) = O(|E(G)|^3)$.

Step 1 Let $F = \{e, f\}$. Put $G^+ = G$ if e, f are adjacent and otherwise let G^+ be obtained from G by subdividing e, f with two new vertices x_5, x_6 and adding a new edge joining them. Check to see if G^+ is essentially 4-edge-connected. If yes, go to Step 3. If not, go to Step 2.

Step 2 Construct a non-trivial 3-edge-cut S^+ in G^+. Then S^+ gives rise to a non-trivial 3-edge-cut S in G such that $G - S$ has two components G_1, G_2 and at least one of them, say G_1, has $E(G_1) \cap F = \emptyset$. Let $G_1' = G/G_2$ and $G_2' = G/G_1$. It is not difficult to see that the problem can be reduced to two admissible triples $(G_1', F_1, X_1), (G_2', F_2, X_2)$ for suitably defined sets F_1, F_2, X_1, X_2. Hence $f_1(|E(G)|) \leq f_1(|E(G_1')|) + f_1(|E(G_2')|) + O(|E(G)|^2)$ where $|E(G_1')|, |E(G_2')| < |E(G)|$ and $|E(G_1')| + |E(G_2')| = |E(G)| + 3$.

Step 3 Check to see if $|G| \geq 16$ and if G^+ has a removable edge h which does not belong to F (when e, f are adjacent) and is not incident with $X \cup \{x_5, x_6\}$ (when e, f are not adjacent). If not, go to Step 4. If yes, let G_1 be the cubic graph which is homeomorphic to $G - h$. Then (G_1, F, X) is admissible since $|G_1| \geq 14$ and G_1^+ is essentially 4-edge-connected. We have $|E(G')| = |E(G)| - 3$ and $f_1(|E(G)|) \leq f_1(|E(G')|) + O(|E(G)|^2)$.

Step 4 By the above mentioned result of [2], we have $|G^+| \leq 48$, so $|G| \leq 46$. We can now find C by exhaustive search.

We next give an algorithm based on the proof of Lemma 2.5 which reduces the general case to that of cubic graphs.

Algorithm REDUCE TO CUBIC
INPUT: An admissible triple (G, F, X).
OUTPUT: Either an Eulerian subgraph H such that $F \cup X$ is contained in H, or an admissible triple (G', F, X) such that G' is cubic, $G = G'/\{e_1, e_2, \ldots, e_s\}$ for some $e_1, e_2, \ldots, e_s \in E(G')$ with $s = \sum_{v \in V(G)} (d_G(v) - 3)$.
COMPLEXITY: $f_2(|E(G)|) = O(|E(G)|^3)$.

Step 1 We construct a sequence of graphs $G = G_0, G_1, \ldots, G_s = G'$ recursively. Given G_i we construct G_{i+1} as follows. Find a vertex $v_i \in V(G_i)$ of degree at least four, and edges $f_i = v_i u_i$, $g_i = v_i w_i$ incident to v_i such that the graph G_{i+1} obtained from $G - \{f_i, g_i\}$ by adding a new vertex z_i and three new edges e_i, f_i, g_i from z_i to v_i, u_i, w_i respectively, is 3-edge-connected. (The edges f_i, g_i exist by the argument given in first paragraph in the proof of Lemma 2.5, with $G, G', v, u_1, u_2, e_1, e_2, e_3$ replaced by $G_i, G_{i+1}, v_i, u_i, w_i, f_i, g_i, e_i$ respectively, and $G_i = G_{i+1}/e_i$.) Each step in this recursion takes $O(|E(G)|^2)$ time so the whole step takes $O(|E(G)|^3)$ time.

Step 2 Check to see if (G', F, X) is admissible. If yes output (G', F, X). If not, go to Step 3.

Step 3 Construct pairwise disjoint subgraphs $Z_1, Z_2, \ldots, Z_m$ of G' as described in Lemma 2.5. Choose $e_i \in \{e_1, e_2, \ldots, e_s\}$ such that $e_i \notin E(Z_j)$ for all $1 \le j \le m$ and i is as large as possible. (The edge e_i exists since (G, F, X) is admissible.) Then $T := \{e_{i+1}, \ldots, e_s\} \subseteq \bigcup_{j=1}^{m} E(Z_j)$ so (G_i, F, X) is not admissible. We may construct pairwise disjoint subgraphs $Z'_1, Z'_2, \ldots, Z'_m$ of G_i as described in Lemma 2.5 by putting $Z'_j = Z_j/(E(Z_j) \cap T)$. We may now use maxflow computations for each Z'_j to construct an Eulerian subgraph H_i of G_i with $F \cup X$ contained in H_i in time $O(|E(G_i)|^2)$ as in the proof of Lemma 2.5. We then construct the required subgraph H from H_i by contracting any edges of $e_1, e_2, \ldots, e_{i-1}$ which belong to $E(H_i)$.

It is straightforward to combine these two algorithms to obtain:

Algorithm COVER
INPUT: An admissible triple (G, F, X).
OUTPUT: An Eulerian subgraph H such that $F \cup X$ is contained in H.
COMPLEXITY: $f_3(|E(G)|) = O(|E(G)|^3)$.

5.3 Long cycles in claw-free graphs

Let G be a 3-connected claw-free graph. It takes $O(|E(G)||V(G)|)$ time to find the Ryjáček closure G^* of G. We can find a graph G_1 such that $L(G_1) = G^*$ in $O(|E(G^*)|)$ time by a result of Roussopolos [39]. From the proof of Theorem 4.2, G_1 is obtained from a 3-edge-connected graph G'_1 by adding some pendant edges and by subdividing certain edges of G'_1 exactly once. By assigning appropriate weights to edges of G'_1 and replacing pendant edges with loops of weight 1, we arrive at a $\{1, 2\}$-edge-weighted 3-edge-connected graph G_2 with $w(G_2) = |E(G_1)| = |G^*| = |G|$. Applying Algorithm EULERIANSUBGRAPH to G_2, we find an Eulerian subgraph H of G_2 such that $w(H) \ge (w(G_2)/6)^\alpha + 2 = (|G|/6)^\alpha + 2$. Now an Euler tour of H can be transformed into a cycle in G of length at least $w(H)$ as in [11]. The complexity of the algorithm is $O(|E(G_2)|^3) = O(|V(G)|^3)$.

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