INTRODUCTION TO MODULAR FORMS

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Contents

1. Elliptic functions

1.1. **Periodic functions.** A function $f : \mathbb{R} \to \mathbb{C}$ is *periodic of period* 1 if

$$f(x+n) = f(x), \quad \text{for all } x \in \mathbb{R} \text{ and } n \in \mathbb{Z}. \tag{1.1}$$

There are two natural ways constructing periodic functions. One can simply take any function on the segment $[0,1)$ and extend its values uniquely to $\mathbb{R}$ by requiring periodicity. Another construction uses the averaging method. Let $g : \mathbb{R} \to \mathbb{C}$ be any function of rapid decay at $\pm\infty$ so that the series

$$f(x) := \sum_{n\in\mathbb{Z}} g(x+n) \tag{1.2}$$

converges absolutely. Then f defines a periodic function of period one.

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By classical Fourier analysis any periodic and piecewise continuous function $f : \mathbb{R} \to \mathbb{C}$ has the Fourier series representation

$$f(x) = \sum_{n\in\mathbb{Z}} a_n e(nx) \tag{1.3}$$

with coefficients given by

$$a_n := \int_0^1 f(x)e(-nx)\ dx. \tag{1.4}$$

For f as in (1.2) we note

$$a_n = \int_0^\infty \sum_{n\in\mathbb{Z}} g(x+n)e(-nx)dx = \int_{-\infty}^\infty g(x)e(-nx)dx = \widehat{g}(n),$$

where

$$\widehat{g}(y) := \int_{\mathbb{R}} g(x)e^{-2\pi ixy}\ dx$$

is the Fourier transform of g. Therefor the Fourier expansion (1.3) becomes

$$\sum_{n\in\mathbb{Z}} g(x+n) = \sum_{n\in\mathbb{Z}} \widehat{g}(n)e(nx).$$

Taking $x = 0$ we get the *Poisson summation formula*

$$\sum_{n\in\mathbb{Z}} g(n) = \sum_{n\in\mathbb{Z}} \widehat{g}(n).$$

More generally, for $g : \mathbb{R}^n \to \mathbb{C}$ a "nice" function of rapid decay, we have the higher dimensional Poisson summation formula

$$\sum_{\boldsymbol{n}\in\mathbb{Z}} g(\boldsymbol{n}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^n} \widehat{g}(\boldsymbol{n}).$$

Remark 1.5. Poisson summation formula is a key input in proving the analytic continuation and functional equation of the Riemann zeta function

$$\zeta(s) = \frac{1}{n^s} \quad \text{for } \mathfrak{Re}(s) > 1.$$

1.2. **Elliptic functions.** The next generalization of periodic functions are "periodic" functions on the complex plane $\mathbb{C}$ viewed not just as $\mathbb{R}^2$, but as a Riemannian manifold with a complex structure.

Note that $\mathbb{Z}^n$ is a *lattice* in $\mathbb{R}^n$, i.e. it is a discrete free abelian subgroup of $\mathbb{R}^n$ of (full) rank n. We also need a lattice in $\mathbb{C}$. Let ω_1, ω_2 be two complex numbers which are linearly independent over $\mathbb{R}$, that is

$$\mathbb{C} = \omega_1\mathbb{R} + \omega_2\mathbb{R}.$$

Let

$$\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$$

be the lattice generated by ω_1 and ω_2.

Definition 1.1. A function $f : \mathbb{C} \to \mathbb{C}$ is *elliptic with respect to* Λ iff

(1) f is meromorphic on $\mathbb{C}$,
(2) f is periodic with periods Λ, i.e.

$$f(u+w) = f(u) \quad \text{for all } u \in \mathbb{C} \text{ and } w \in \Lambda. \tag{1.6}$$

Choose a fundamental parallelogram for the lattice Λ,

$$P = t\omega_1 + t_2\omega_2 + \mu, \quad 0 \le t_1, t_2 < 1,$$

with $\mu \in \mathbb{C}$ such that f has no poles or zeros on ∂P.

Some easy observations:

(1) f is completely determined by its values on P, thus can be viewed as a function on the tours $\mathbb{C}/\Lambda$.
(2) If f is holomorphic, then f is a constant: Since P is precompact, f is bounded on P. Then periodicity implies that it is also bounded on $\mathbb{C}$. Then by Liouville's theorem (see e.g. [SS03, p. 50, Corollary 4.5]), f is a constant.

Recall that around every w, f has a power series expansion

$$f(u) = \sum_{k=m}^{\infty} a_k (u-w)^k$$

with coefficients $a_k \in \mathbb{C}$ and $a_m \neq 0$. Then $m = \mathrm{ord}_w(f)$ is the *order* of f at w and $a_{-1} = \mathrm{res}_w(f)$ is the *residue* of f at w.

Proposition 1.1. *Let f be an elliptic function with respect to Λ. Then we have*

$$\sum_{w\in P} \mathrm{res}_w(f) = 0, \tag{1.7}$$

$$\sum_{w\in P} \mathrm{ord}_w(f) = 0, \tag{1.8}$$

and

$$\sum_{w\in P} \mathrm{ord}_w(f) w \equiv 0 \pmod{\Lambda}. \tag{1.9}$$

Here the last equation means that the left side is always an element in Λ, independent of the choice of P.

Proof. Integrating along ∂P and noting that the integrals along opposite sides cancel out by periodicity we have by Cauchy's theorem (see e.g. [SS03, p. 77, Corollary 2.2])

$$0 = \frac{1}{2\pi i}\int_{\partial P} f(u)du = \sum_{w\in P} \mathrm{res}_w(f),$$

finishing the proof of (1.7). Similarly, (1.8) follows by applying (1.7) to the elliptic functions $\frac{f'}{f}$ and noting that $\mathrm{ord}_w(f) = \mathrm{res}_w(f'/f)$ which follows from an easy computation. We leave the proof of (1.9) as an exercise. $\square$

Exercise 1. *Prove case (3) in Proposition 1.1. (Hint: Consider the function uf'/f. Note that it is no longer elliptic, but you can still integrate it along ∂P. The integral may depend on the choice of P, but you only need to show it lies in Λ.)*

Definition 1.10. Define the *order* of f to be the sum of orders of zeros in P or the negative of the sum of orders of poles in P, i.e.

$$\mathrm{ord}(f) := \sum_{w\in P} \max\{\mathrm{ord}_w(f), 0\} = -\sum_{w\in P} \min\{\mathrm{ord}_w(f), 0\}.$$

Note that relation (1.8) guarantees that this definition is well-defined.

Corollary 1.2. (1) *There is no elliptic function of order* 1*, i.e. there is no elliptic functions with one simple zero (or one simple pole).*
(2) *For any $c \in \mathbb{C}$, f takes the value c for exactly ord(f) points in P counted with multiplicity*

Proof. For (1), suppose an elliptic function f is of order 1, then it has exactly a simple pole in P, say at $w_0 \in P$. Then $\sum_{w\in P} \mathrm{res}_w(f) = \mathrm{res}_{w_0}(f) \neq 0$, contradicting (1.7).

For (2), applying (1.8) for f and $f - c$ we get

$$\begin{aligned}\mathrm{ord}(f) &= -\sum_{w\in P} \min\{\mathrm{ord}_w(f), 0\}\\ &= -\sum_{w\in P} \min\{\mathrm{ord}_w(f-c), 0\}\\ &= \sum_{w\in P} \max\{\mathrm{ord}_w(f-c), 0\},\end{aligned}$$

where the last sum exactly counts the number of times (with multiplicity) when $f(u) = c$ for $u \in P$. □

From (1) above we know that for an function to be elliptic, it must be of order at least 2. One natural way to construct an elliptic function is to start with a function with a double pole and then apply the averaging trick. For example, we can take $g(z) = \frac{1}{z^2}$ and then define the sum

$$\sum_{w\in\Lambda} \frac{1}{(u-w)^2}. \tag{1.11}$$

However there is some convergence issue with this construction. We need to add some "corrected terms" to resolve it.

Definition 1.12. The *Weierstrauss $\wp$ function* is defined by

$$\wp(u) := \frac{1}{u^2} + {\sum_{w\in\Lambda}}' \left(\frac{1}{(u-w)^2} - \frac{1}{w^2}\right), \quad u \notin \Lambda, \tag{1.13}$$

where $'$ means that $w = 0$ is skipped in the summation.

The convergence of the above series will be guaranteed by the following lemma.

Lemma 1.3. *Let Λ be a lattice in $\mathbb{C}$. For any $R > 0$ and $u \in \mathbb{C}$ with $|u| < R$, the series*

$$\sum_{\substack{w\in\Lambda\\ |w|\geq 2R}} \frac{1}{|u+w|^c} \ll_{R,\Lambda,c} 1$$

for any $c > 2$.

Remark 1.14. In this course for any two quantities A, B, we will use the notation $A \ll_\lambda B$ to mean that there exists some constant $C > 0$ such that $A \leq CB$, and here the subscript means that the bounding constant C may depend on the parameter λ.

We now give the proof of this lemma[1].

Proof. For any $t > 0$ consider the set

$$P(t) = \{aw_1 + bw_2 : a, b \in \mathbb{R},\ \max\{|a|, |b|\} = t\}\,.$$

Note that $\Lambda \setminus \{0\} = \bigsqcup_{n\geq 1}(P(n) \cap \Lambda)$ and it is easy to check $|P(n) \cap \Lambda| = 8n$ and $P(n) = nP(1)$. There exist constants $B > 0$ and $N > 1$ sufficiently large such that $|w| \geq B + \frac{R}{N}$ for any $w \in P(1)$. Since $P(n) = nP(1)$, for any integer $n \geq N$ and for any $w \in P(n)$

$$|w| \geq n\left(B + \tfrac{R}{N}\right) \geq nB + R.$$

In particular, for any $|u| < R$, $|u + w| \geq |w| - |u| \geq nB$. Hence the series $\sum_{\substack{w\in\Lambda\\|w|\geq 2R}} \frac{1}{|u+w|^c}$ is bounded from above by

$$\begin{aligned}\sum_{\substack{w\in\Lambda\\|w|\geq 2R}} \frac{1}{|u+w|^c} &\leq \sum_{n<N} |P(n)\cap\Lambda|R^{-c} + \sum_{n\geq N} |P(n)\cap\Lambda|(nB)^{-c}\\ &\leq 8N^2R^{-c} + 8B^{-c}\sum_{n\geq N}\frac{1}{n^{c-1}} \ll_{R,\Lambda,c} 1,\end{aligned}$$

finishing the proof. $\square$

Exercise 2. *Let Λ be a lattice in $\mathbb{C}$. For any $R > 0$ and $u \in \mathbb{C}$ with $|u| < R$. Show that the series*

$$\sum_{\substack{w\in\Lambda\\|u|\geq 2R}} \frac{1}{|u+w|^2}$$

diverges.

Proposition 1.4. *The Weierstrauss $\wp$ function is a meromorphic function with double poles only at lattice points. Similarly, its derivative*

$$\wp'(u) = -2\sum_{w\in\Lambda}\frac{1}{(u-w)^3}, \quad u \notin \Lambda \tag{1.15}$$

is a meromorphic function with triples poles only at lattice points.

Proof. We only prove the statement for $\wp$; the statement for $\wp'$ follows from similar arguments. It suffices to show for any $R > 0$, $\wp$ is meromorphic for $|u| < R$. For this rewrite

$$\wp(u) = \frac{1}{u^2} + \sum_{|w|<2R}\left(\frac{1}{(u-w)^2} - \frac{1}{w^2}\right) + \sum_{|w|\geq 2R}\left(\frac{1}{(u-w)^2} - \frac{1}{w^2}\right).$$

Note that for the terms in the second sum, using $|u| < R \leq \frac{1}{2}|w|$ we have

$$\left|\frac{1}{(u-w)^2} - \frac{1}{w^2}\right| = \left|\frac{(2w-u)u}{(u-w)^2w^2}\right| \leq \frac{12R}{|w|^3}.$$

Then by Lemma 1.3 we see that the second sum is absolutely convergent, hence defines a holomorphic function in $|u| < R$. We are thus left with a finite sum which

[1] Note that the proof given here differs slightly from the one I gave in class, but they use the same ideas of grouping the summation terms.

clearly is meromorphic with double poles at all the lattice points inside the ball $\{u \in \mathbb{C} : |u| < R\}$. □

Remark 1.16. Clearly $\wp'$ is elliptic with respect to Λ.

Proposition 1.5. *$\wp$ is even and elliptic with respect to $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$.*

Proof. To see $\wp$ is even, for any $u \notin \Lambda$,

$$\wp(-u) = \frac{1}{u^2} + \sum_{w\in\Lambda}{}' \left(\frac{1}{(u-(-w))^2} - \frac{1}{(-w)^2} \right) = \wp(u),$$

where for the last equality we used that $-(\Lambda \setminus \{0\}) = \Lambda \setminus \{0\}$. To show $\wp$ is elliptic with respect to Λ, it suffices to show

$$\wp(u+\omega_1) = \wp(u) \quad \text{and} \quad \wp(u+\omega_2) = \wp(u), \quad \forall\ u \notin \Lambda.$$

Consider $F_i(u) := \wp(u+\omega_i) - \wp(u)$. Since $\wp'$ is elliptic with respect to Λ,

$$F_i'(u) = \wp'(u+\omega_i) - \wp'(u) = 0.$$

Hence $F_i(u) = c_i$ is a constant. Now take $u = -\frac{w_i}{2}$ to get

$$c_i = \wp\left(\frac{\omega_i}{2}\right) - \wp\left(-\frac{\omega_i}{2}\right) = \wp\left(\frac{\omega_i}{2}\right) - \wp\left(\frac{\omega_i}{2}\right) = 0.$$

This finishes the proof. □

Exercise 3. *Show that $\wp$ and $\wp'$ generate the field of elliptic functions with respect to Λ, that is any elliptic function can be written as a rational function in $\wp$ and $\wp'$.*

Remark 1.17. Let us discuss some of the consequences of this proposition. On the torus $\mathbb{C}/\Lambda$, there are three 2-torsion points (i.e. $u \notin \Lambda, 2u \in \Lambda$), namely $\frac{\omega_1}{2} + \Lambda$, $\frac{\omega_2}{2} + \Lambda$ and $\frac{\omega_3}{2} + \Lambda$ with $\omega_3 = \omega_1 + \omega_2$.

Since $\wp'$ is elliptic, for $i = 1, 2, 3$

$$-\wp'\left(\frac{\omega_i}{2}\right) = \wp'\left(-\frac{\omega_i}{2}\right) = \wp'\left(\omega_i - \frac{\omega_i}{2}\right) = \wp'\left(\frac{\omega_i}{2}\right),$$

implying that $\wp'\left(\frac{\omega_i}{2}\right) = 0$. For $\wp'$, we know it is an order 3 odd elliptic function. It has zeros at the three 2-torsion points in $\mathbb{C}/\Lambda$, Thus these are exactly the three simple zeros modulo Λ of $\wp'$.

For $\wp$, we know it is an order 2 even elliptic function. Thus it takes any value exactly twice counted with multiplicity. For any $c \in \mathbb{C}$, there exists some $w \notin \Lambda$ such that $\wp(w) = c$. If $w \not\equiv -w \pmod{\Lambda}$, (i.e. $w + \Lambda$ is not a 2-torsion point), since $\wp$ is even, then $\pm w \pmod{\Lambda}$ are the two simple zeros of $\wp(u) - c$. If $w + \Lambda$ is one of the three 2-torsion points, then the function $\wp - c$ has a double zero modulo Λ at w, since its derivative, being $\wp'$, also vanishes at w.

This analysis also implies that the three values $\wp(\frac{\omega_1}{2}), \wp(\frac{\omega_2}{2}), \wp(\frac{\omega_3}{2})$ are distinct since otherwise we have e.g. $\wp(\frac{\omega_1}{2}) = \wp(\frac{\omega_2}{2}) = c$, then the function $\wp(u) - c$ has at least four zeros (counted with multiplicity) which is a contradiction.

Proposition 1.6. *Let $\wp$ be the Weierstrauss function with respect to $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$. Then*

(1) *The Laurent expansion of $\wp$ is*

$$\wp(u) = \frac{1}{u^2} + \sum_{\substack{k=1 \\ k \text{ even}}}^{\infty} (k+1)G_{k+2}(\Lambda)u^k,$$

for all u such that $0 < |u| < \inf\{|w| : w \in \Lambda \setminus \{0\}\}$. Here

$$G_k(\Lambda) := \sum_{w\in\Lambda}{}' \frac{1}{w^k}$$

is the weight k Eisenstein series.

(2) *The functions $\wp$ and $\wp'$ satisfy the relation*

$$(\wp'(u))^2 = 4(\wp(u))^3 - g_2(\Lambda)\wp(u) - g_3(\Lambda), \tag{1.18}$$

where $g_2(\Lambda) = 60G_4(\Lambda)$ and $g_3(\Lambda) = 140G_6(\Lambda)$.

(3) *Let $e_i = \wp(\omega_i/2)$ for $i = 1, 2, 3$ with ω_i as above. Then the cubic equation satisfied by $\wp$ and $\wp'$ is equivalent to*

$$(\wp'(u))^2 = 4(\wp(u) - e_1)(\wp(u) - e_2)(\wp(u) - e_3).$$

This equation is nonsingular, meaning the right side has distinct roots.

Proof. For (1), we use the geometric series square formula

$$\frac{1}{(1-z)^2} = \sum_{k=0}^{\infty}(k+1)z^k, \quad \forall\ |z| < 1.$$

Thus for $|u| < |w|$ we have

$$\frac{1}{(u-w)^2} - \frac{1}{w^2} = \frac{1}{w^2}\left(\frac{1}{(1-u/w)^2} - 1\right) = \sum_{k=1}^{\infty}\frac{(k+1)u^k}{w^{k+2}}.$$

Hence for $|u| < \inf\{|w| : w \in \Lambda \setminus \{0\}\}$ we have

$$\begin{aligned}\wp(u) &= \frac{1}{u^2} + \sum_{w\in\Lambda}{}' \sum_{k=1}^{\infty}(k+1)\frac{u^k}{w^{k+2}} \\ &= \frac{1}{u^2} + \sum_{\substack{k=1 \\ k \text{ even}}}^{\infty}(k+1)G_{k+2}(\Lambda)u^k,\end{aligned}$$

where for the second line we changed order of summationand used the fact that $G_k(\Lambda) = 0$ whenever k is odd.

For (2) let us define

$$F(u) := (\wp'(u))^2 - 4(\wp(u))^3 + g_2(\Lambda)\wp(u) + g_3(\Lambda)$$

and we wish to show $F(u) = 0$. Since F is meromorphic, it suffices to show $F(u) = 0$ for $|u| < \inf\{|w| : w \in \Lambda\setminus\{0\}\}$. By direct computation we see that for $|u| < \inf\{|w| : w \in \Lambda \setminus \{0\}\}$ the Laurent expansion of $(\wp'(u))^2$ and $4(\wp(u))^3 - g_2(\Lambda)\wp(u) - g_3(\Lambda)$ both equal

$$4u^{-6} - 24G_4(\Lambda)u^{-2} - 80G_6(\Lambda) + O(u^2).$$

In particular, this implies that F is holomorphic with a Laurent expansion $F(u) = O(u^2)$. But since F is elliptic, it must be a constant, which then together with $F(u) = O(u^2)$ implies that $F = 0$ as desired.

For (3), first the fact that e_1, e_2, e_3 are distinct already follows from Remark 1.17. Now factoring the cubic polynomial in the right side of (1.18) we have there exist complex numbers c_1, c_2, c_3 such that

$$(\wp'(u))^2 = 4(\wp(u) - c_1)(\wp(u) - c_2)(\wp(u) - c_3).$$

Plugging in $u = \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}$ respectively and noting that the left side vanishes (cf. Remark 1.17) we get that $\{e_1, e_2, e_3\} \subset \{c_1, c_2, c_3\}$. But since e_1, e_2, e_3 are distinct, this must be an equality, concluding the proof. $\square$

1.3. **The modular discriminant and j-invariant.**

Definition 1.19. The *modular discriminant* $\Delta = \Delta(\Lambda)$ is defined by

$$\Delta := 16 \prod_{1 \le i < j \le 3} (e_i - e_j)^2, \tag{1.20}$$

where e_1, e_2, e_3 are the simple roots of the cubic polynomial in (1.18).

Lemma 1.7. *For any lattice Λ, $\Delta(\Lambda) = g_2(\Lambda)^3 - 27g_3(\Lambda)^2$ and $\Delta(\Lambda) \neq 0$.*

Proof. The non-vanishing of Δ follows from the definition (1.20) and the fact that e_1, e_2, e_3 are distinct. We thus only need to prove this expression for Δ. Abbreviate $g_2 = g_2(\Lambda)$ and $g_3 = g_3(\Lambda)$. Consider the Vandermonde matrix determined by e_1, e_2, e_3:

$$V := \begin{pmatrix} 1 & 1 & 1 \\ e_1 & e_2 & e_3 \\ e_1^2 & e_2^2 & e_3^2 \end{pmatrix}.$$

From linear algebra we know its determinant is given by $\det(V) = \prod_{1 \le i < j \le 3}(e_i - e_j)$. Hence

$$\Delta = 16 \det(V)^2 = 16 \det(VV^t) = 16 \det \begin{pmatrix} 3 & S_1 & S_2 \\ S_1 & S_2 & S_3 \\ S_2 & S_3 & S_4 \end{pmatrix},$$

where $S_k = e_1^k + e_2^k + e_3^k$ for $1 \le k \le 4$. From the equation

$$(x - e_1)(x - e_2)(x - e_3) = x^3 + Ax - B,$$

with $A = -g_2/4$ and $B = g_3/4$ we get

$$B = e_1 e_2 e_3, \quad A = e_1 e_2 + e_2 e_3 + e_3 e_1 \quad \text{and} \quad S_1 = e_1 + e_2 + e_3 = 0.$$

From the above third relation we get $S_2 + 2A = (e_1 + e_2 + e_3)^2 = 0$; hence $S_2 = -2A$. Moreover, we can check that for each $j = 1, 2, 3$, $e_j^3 = -Ae_j + B$. Hence $S_3 = -A(e_1 + e_2 + e_3) + 3B = 3B$ and $e_j^4 = -Ae_j^2 + Be_j$. The latter then implies that $S_4 = -AS_2 = 2A^2$. Thus get

$$\Delta = 16 \det \begin{pmatrix} 3 & 0 & -2A \\ 0 & -2A & 3B \\ -2A & 3B & 2A^2 \end{pmatrix} = 16(-4A^3 - 27B^2) = g_2^3 - 27g_4^2,$$

finishing the proof. $\square$

Definition 1.21. The *j-invariant* function is defined by

$$j(\Lambda) := \frac{1728 g_2(\Lambda)^3}{\Delta(\Lambda)}.$$

Since $\Delta(\Lambda) \neq 0$, this function is well-defined.

2. Modular forms: definition and examples

The functions $G_k(\Lambda), \Delta(\Lambda), j(\Lambda)$ can all be viewed as functions in

$$\mathcal{L} := \{\Lambda \subset \mathbb{C} : \Lambda \text{ is a lattice in } \mathbb{C}\},$$

the *space of lattices in* $\mathbb{C}$. Some of them are in fact examples of modular forms.

Definition 2.1. Let k be a non-negative integer. A *modular form of weight* k is a function

$$F : \mathcal{L} \to \mathbb{C}$$

satisfying the following properties:

(1) F is homogeneous of degree $-k$, i.e.

$$f(\lambda\Lambda) = \lambda^{-k} f(\Lambda) \quad \text{for any } \lambda \in \mathbb{C}^\times \text{ and } \Lambda \in \mathcal{L}, \tag{2.2}$$

(2) "F is holomorphic in $\mathcal{L}$",
(3) "F is holomorphic at ∞".

To make the above definition more clear, we need a parameterization of the domain for functions defined above, namely the space of latices $\mathcal{L}$.

Definition 2.3. A pair of complex numbers $\langle\omega_1, \omega_2\rangle$ is called *positive* if $\mathfrak{Im}(\frac{\omega_1}{\omega_2}) > 0$.

We can use positive pairs to parameterize $\mathcal{L}$: Clearly, each positive pair $\langle\omega_1, \omega_2\rangle$ is $\mathbb{R}$-linearly independent, thus gives rise to a lattice $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$. Moreover, each lattice can be realized by a positive pair: Given any lattice Λ, let $\{\omega_1, \omega_2\}$ be a basis so that $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$. Up to replacing ω_1 by $-\omega_1$, we can have $\mathfrak{Im}(\frac{\omega_1}{\omega_2}) > 0$, i.e. $\langle\omega_1, \omega_2\rangle$ is a positive pair. However, there is some redundancy in this parameterization as shown in the following lemma.

Lemma 2.1. *Two positive pairs $\langle\omega_1, \omega_2\rangle$ and $\langle\omega_1', \omega_2'\rangle$ give the same lattice if and only if there exists some $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ such that*

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}. \tag{2.4}$$

Here

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{Z}) : ad - bc = 1 \right\}$$

is the modular group, i.e. the group of two by two integral matrices with determinant 1.

Proof. This direction "$\Leftarrow$" is clear. We only need to prove the other direction. Assume $\omega_1\mathbb{Z} + \omega_2\mathbb{Z} = \omega_1'\mathbb{Z} + \omega_2'\mathbb{Z}$. First since $\omega_1', \omega_2' \in \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$, there exist $a, b, c, d \in \mathbb{Z}$ such that $\omega_1' = a\omega_1 + b\omega_2$ and $\omega_2' = c\omega_1 + d\omega_2$. Similarly, there exist $a', b', c', d' \in \mathbb{Z}$ such that $\omega_1 = a'\omega_1' + b'\omega_2'$ and $\omega_2 = c'\omega_1' + d'\omega_2'$. This implies that

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix}.$$

Thus $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) = I_2$, implying that $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is an invertible integral matrix, i.e. $\det(\gamma) = \pm 1$. It thus remains to show $\det(\gamma) > 0$. This is an easy exercise. $\square$

For any positive pair $\langle\omega_1,\omega_2\rangle$ we denote by $z=z(\omega_1,\omega_2)=\frac{\omega_1}{\omega_2}\in\mathbb{H}$, where

$$\mathbb{H}:=\{z=x+iy\in\mathbb{C}: y>0\}$$

is the usual *upper half plane* of the complex plane. Let F be a modular form of weight k. The homogeneity condition (2.2) implies that for any positive pair $\langle\omega_1,\omega_2\rangle$,

$$F(\omega_1\mathbb{Z}+\omega_2\mathbb{Z})=\omega_2^{-k}F(\tfrac{\omega_1}{\omega_2}\mathbb{Z}+\mathbb{Z}). \tag{2.5}$$

Thus F is uniquely determined by its values on the subset $\{z\mathbb{Z}+\mathbb{Z}: z\in\mathbb{H}\}\subset\mathcal{L}$ is can be naturally parameterized by $\mathbb{H}$. We thus define

$$f(z):=F(z\mathbb{Z}+\mathbb{Z}).$$

Now condition (2) in Definition 2.1 just means that f is holomorphic in the variable $z\in\mathbb{H}$. We also need to rephrase the homogeneity condition (2.2) in terms of f. For any $z\in\mathbb{H}$ and $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$. By definition we have

$$\begin{aligned}f(z)&=F(\mathbb{Z}+z\mathbb{Z})=F((az+b)\mathbb{Z}+(cz+d)\mathbb{Z})\\&=(cz+d)^{-k}F(\tfrac{az+b}{cz+d}\mathbb{Z}+\mathbb{Z})=(cz+d)^{-k}f\left(\tfrac{az+b}{cz+d}\right).\end{aligned}$$

Or equivalently,

$$f(\gamma z)=(cz+d)^kf(z),\quad\forall\,\gamma\in\mathrm{SL}_2(\mathbb{Z}),z\in\mathbb{H}.$$

Here $\gamma z=\frac{az+b}{cz+d}$ is the usual *linear fractional transformation.* (Lemma 2.2 below shows that this is a well-defined action on $\mathbb{H}$.) We now give an alternative definition of a modular form in terms of this function f.

Definition 2.6. Let k be a non-negative integer. A function $f:\mathbb{H}\to\mathbb{C}$ is a *modular form of weight* k if

(1) f satisfies the transformation rule

$$f(\gamma z)=(cz+d)^kf(z),\quad\forall\,z\in\mathbb{H},\ \gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z}), \tag{2.7}$$

(2) f is holomorphic on $\mathbb{H}$,
(3) f is holomorphic at ∞.

If f further vanishes at ∞ then f is called a *cusp form* of weight k.

We denote the set of modular forms (resp. cusp forms) of weight k by $\mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z}))$ (resp. $\mathcal{S}_k(\mathrm{SL}_2(\mathbb{Z}))$; when there is no ambiguity we simply write $\mathcal{M}_k$ and $\mathcal{S}_k$.

Remark 2.8. Let us make the following few remarks.

(1) From the homogeneity condition we see modular forms of different weights are linearly independent over $\mathbb{C}$.
(2)
$$\mathcal{M}_{k_1}\mathcal{M}_{k_2}\subset\mathcal{M}_{k_1+k_2}\quad\text{for any } k_1,k_2\geq 0$$
(3) Taking $\gamma=-I_2$, (2.10) becomes $f(z)=(-1)^kf(z)$. Hence $\mathcal{M}_k=\{0\}$ if k is odd.
(4) Let $T=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $S=\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$. Applying (2.10) for these two transformations we have

$$f(z+1)=f(z)\quad\text{and}\quad f(-1/z)=(-z)^kf(z),\quad\forall\,z\in\mathbb{H}. \tag{2.9}$$

We will see later that in order to verify (2.10), it suffices to verify it for these two transformations.

Here we need to explain the definition a bit. Let $q = q(z) = e^{2\pi i z} = e^{2\pi i x}e^{-2\pi y}$. This map sends $\mathbb{H}$ to the punctured disc $\mathbb{D}' := \{q \in \mathbb{C} \setminus \{0\} : |q| < 1\}$[2]. Define $g : \mathbb{D}' \to \mathbb{C}$, corresponding to f, by $g(q) = f(\log(q)/2\pi i)$. Note that apriori g may not be well-defined since the complex logarithmic function is defined only up to an integer multiple of $2\pi i$. However, the first transformation rule in (2.9) removes this ambiguity. Namely, around every $q \in \mathbb{D}'$ we can choose a branch for $\log q$. Then any other choice of branch is of the form $\log q + 2n\pi i$ and we have by the transformation rule

$$f((\log q + 2n\pi i)/2\pi i) = f(\log q/2\pi i + n) = f(\log q/2\pi i),$$

implying that g is well-defined and is holomorphic on $\mathbb{D}'$ and it has a Laurent expansion $g(q) = \sum_{n\in\mathbb{Z}} a_n q^n$ for $q \in \mathbb{D}'$. Now note that $|q| = e^{-2\pi y}$, thus $q \to 0$ as $y \to \infty$. Then the condition f is holomorphic at ∞ just means that g can extends holomorphically to the punctured point $q = 0$, i.e. the Laurent series sums only over $n \in \mathbb{N}$. This means that f has a Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a_n(f)e(nz). \tag{2.10}$$

Similarly, f vanishes at ∞ means that $a_0(f) = 0$ in the above Fourier expansion, that is,

$$f(z) = \sum_{n=1}^{\infty} a_n(f)e(nz). \tag{2.11}$$

Remark 2.12. Let $f : \mathbb{H} \to \mathbb{C}$ be a function satisfying conditions (1) and (2) in Definition 2.6 and denote by $f(\infty) := \lim_{y\to\infty} f(iy)$. Then we have the following criterion for whether f is a modular or cusp form:

$$f \text{ is a modular form (resp. cusp form)} \quad \Leftrightarrow \quad f(\infty) < \infty, \text{ (reps. } f(\infty) = 0).$$

For any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and non-negative integer k define the *weight k operator* $[\gamma]_k$ on functions $f : \mathbb{H} \to \mathbb{C}$ by

$$(f[\gamma])_k(z) := j_\gamma(z)^{-k} f(\gamma z), \quad \forall\ z \in \mathbb{H},$$

where $j_\gamma(z) = cz + d$ is called the *factor of automorphy*. Note that under this new terminology condition (2.7) becomes

$$f[\gamma]_k = f, \quad \forall\ \gamma \in \mathrm{SL}_2(\mathbb{Z}).$$

Below we list some basic properties of this operator.

Lemma 2.2. *For all $\gamma, \gamma' \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \mathbb{H}$,*

(1) $j_{\gamma\gamma'}(z) = j_\gamma(\gamma' z) j_{\gamma'}(z)$ *(chain rule),*
(2) $(\gamma\gamma')z = \gamma(\gamma' z)$,
(3) $f[\gamma\gamma']_k = (f[\gamma]_k)[\gamma']_k$,
(4) $\mathfrak{Im}(\gamma z) = \frac{\mathfrak{Im}(z)}{|j_\gamma(z)|^2}$.

[2]But it's not biholomorphic since q is $\mathbb{Z}$-periodic, i.e. $q(z+1) = q(z)$ for any $z \in \mathbb{H}$. In fact it sends each of the strip $\{z \in \mathbb{H} : \mathfrak{Re}(z) \in [a, a+1)\}$ biholomorphically to $\mathbb{D}'$.

Exercise 4. *Prove this lemma.*
Hint: Note that for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix} = j_\gamma(z)\begin{pmatrix} \gamma z \\ 1 \end{pmatrix}.$$

2.1. **Examples of modular forms.** In fact, we have already encountered some explicit modular forms. Recall for any integer $k \geq 3$,

$$G_k(\Lambda) = \sideset{}{'}\sum_{w\in\Lambda} \frac{1}{w^k}.$$

As a function on $\mathbb{H}$, G_k is given by

$$G_k(z) = G_k(z\mathbb{Z}+\mathbb{Z}) = \sideset{}{'}\sum_{(c,d)\in\mathbb{Z}^2} \frac{1}{(cz+d)^k}. \tag{2.13}$$

The following proposition asserts that G_k is a weight k modular form.

Proposition 2.3. *For any integer* $k \geq 3$, $G_k \in \mathcal{M}_k$ *but* $G_k \notin \mathcal{S}_k$.

Proof. It is clear from the original definition that $G_k(\Lambda)$ is homogeneous of degree $-k$ which is equivalent to the fact that $G_k(z)$ satisfies the transformation rule (2.7). Next, we show the defining series of $G_k(z)$ converges absolutely and uniformly in the compact set

$$K_B := \left\{z = x+iy : |x| \leq B,\ B^{-1} \leq y \leq B\right\}$$

for any $B > 1$, and hence defines a holomorphic function on $\mathbb{H}$. For this, first we show $|cz+d|^2 \gg_B c^2+d^2$ uniformly for any $z \in K_B$. By direct computation

$$|cz+d|^2 = |c(x+iy)+d|^2 = (cx+d)^2 + c^2y^2 \geq (cx+d)^2 + B^{-2}c^2.$$

If $|d| \geq 2|cx|$, then $(cx+d)^2 \geq (|d|-|cx|)^2 \geq \frac{1}{4}d^2$. Hence in this case

$$|cz+d|^2 \geq B^{-2}c^2 + \frac{1}{4}d^2 \gg_B c^2+d^2.$$

If $|d| < 2|cx|$, then $c^2+d^2 \leq c^2+4B^2c^2$, i.e. $c^2 \geq \frac{c^2+d^2}{1+4B^2}$. Hence in this case

$$|cz+d|^2 \geq B^{-2}c^2 \geq \frac{c^2+d^2}{B^2(1+4B^2)} \gg_B c^2+d^2.$$

In both cases we have for $z \in K_B$, $|cz+d| \gg_B (c^2+d^2)^{\frac{1}{2}}$. Hence for $z \in K_B$,

$$\sideset{}{'}\sum_{(c,d)\in\mathbb{Z}^2} \frac{1}{|cz+d|^k} \ll_B \sideset{}{'}\sum_{(c,d)\in\mathbb{Z}^2} \frac{1}{(c^2+d^2)^{\frac{k}{2}}} < \infty.$$

Here the convergence is guaranteed by Lemma 1.3. This proves that G_k is holomorphic on $\mathbb{H}$.

Finally, an easy computation shows that

$$G_k(\infty) = \lim_{y\to\infty} G_k(iy) = 2\zeta(k) < \infty, \tag{2.14}$$

implying that G is also holomorphic at ∞. $\square$

Remark 2.15. As mentioned before, $G_k = 0$ when $k \geq 3$ is odd.

Corollary 2.4. $\Delta \in \mathcal{S}_{12}$.

Proof. Recall
$$\Delta = g_2^3 - 27g_3^2 = 216000G_4^3 - 529200G_6^2.$$
Hence in view of Proposition 2.3, $\Delta \in \mathcal{M}_{12}$. Moreover, using identities $\zeta(4) = \frac{\pi^4}{90}$ and $\zeta(6) = \frac{\pi^6}{945}$ we have
$$\Delta(\infty) = \frac{64\pi^{12}}{27} - \frac{64\pi^{12}}{27} = 0. \tag{2.16}$$
This finishes the proof. □

Remark 2.17. The j-invariant function $j = 1728g_2^3/\Delta$ has a simple pole at ∞ and hence is not a modular form. It is an example of weight 0 modular function; see Definition 4.3 below.

We have the following precise Fourier expansion formulas for G_k.

Proposition 2.5. *For any even $k \geq 4$,*
$$G_k(z) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)e(nz),$$
where for any $s \geq 0$, $\sigma_s(n) = \sum_{d|n} d^s$ is the s-divisor function.

Remark 2.18. The divisor function σ_s is multiplicative, i.e. $\sigma_s(mn) = \sigma_s(m)\sigma_s(n)$ for any $\gcd(m,n) = 1$, and satisfies the growth condition that $\sigma_s(n) \ll_\epsilon n^{s+\epsilon}$.

To prove this Fourier expansion formula we need the following identity.

Lemma 2.6. *For any integer $k \geq 2$ and $z \in \mathbb{H}$,*
$$\sum_{m\in\mathbb{Z}}\frac{1}{(z+m)^k} = \frac{(-2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty}m^{k-1}e(mz). \tag{2.19}$$

Proof. It suffices to prove this identity for $k = 2$. We apply the following infinite product identity for the sine function which can be proved by comparing zeros on both sizes (see e.g. [SS03, p. 142, Equation (3)])
$$\sin(\pi z) = \pi z\prod_{m=1}^{\infty}(1 - \frac{z^2}{m^2}) = \pi z\prod_{m=1}^{\infty}(1 - \frac{z}{m})(1 + \frac{z}{m}).$$
Taking logarithmic derivatives in both sides we get
$$\begin{aligned}\text{LHS} &= \pi\frac{\cos(\pi z)}{\sin(\pi z)} = \pi i\frac{e(z/2) + e(-z/2)}{e(z/2) - e(-z/2)} = \pi i\frac{e(z)+1}{e(z)-1}\\ &= \pi i(1 + \frac{2}{e(z)-1}) = \pi i(1 - 2\sum_{m=0}^{\infty}e(mz)),\end{aligned}$$
and
$$\text{RHS} = \frac{1}{z} + \sum_{m=1}^{\infty}(\frac{1}{z+m} + \frac{1}{z-m}).$$
Further differentiating both sides we get
$$-2\pi i\sum_{m=0}^{\infty}(2\pi im)e(mz) = -\frac{1}{z^2} - \sum_{m=1}^{\infty}\left(\frac{1}{(z+m)^2} + \frac{1}{(z-m)^2}\right) = -\sum_{m\in\mathbb{Z}}\frac{1}{(z+m)^2}.$$
We can then finish the proof by negating both sides. □

We can now easily prove Proposition 2.5.

Proof of Proposition 2.5. Rearrange the sum we get

$$\begin{aligned} G_k(z) &= \sum_{d\neq 0}\frac{1}{d^k}+\sum_{c\neq 0}\sum_{d\in\mathbb{Z}}\frac{1}{(cz+d)^k}\\ &= 2\zeta(k)+2\sum_{c=1}^{\infty}\frac{(2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty}m^{k-1}e(cmz)\\ &= 2\zeta(k)+2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sum_{m\mid n}m^{k-1}e(nz)\\ &= 2\zeta(k)+2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)e(nz), \end{aligned}$$

as desired. □

Remark 2.20. Using this Fourier expansion formulas we can compute

$$\Delta(z)=(12\pi)^{12}\sum_{n=1}^{\infty}\tau(n)e(nz),$$

with $\tau(1)=1$, $\tau(2)=-24$, $\tau(3)=252$, $\tau(4)=-1472,\cdots$. The coefficient function τ is called the Ramanujan function. It is not a coincidence that these coefficients are all integers. In fact we will see later that Δ has the following infinite product expression

$$\Delta(z)=(2\pi)^{12}e(z)\prod_{n=1}^{\infty}(1-e(nz))^{24}.$$

from which it follows easily that $\tau(n)\in\mathbb{Z}$. Based on many numerical computations, Ramanujan (1916) made the following conjecture regarding this function.

Conjecture 2.21 (Ramanujan).

(1) *τ is multiplicative, i.e. $\tau(mn)=\tau(m)\tau(n)$ whenever* $\gcd(m,n)=1$,
(2) *$\tau(p^{r+1})=\tau(p)\tau(p^r)-p^{11}\tau(p^{r-1})$ for all primes p,*
(3) *$|\tau(p)|\leq 2p^{\frac{11}{2}}$ for all primes p.*

The first two statements were proved by Mordell [Mor17] one year later and the last statement was proved by Deligne [Del74] as a consequence of his proof of the Weil conjectures.

Remark 2.22. The *normalized Eisenstein series* is defined by

$$E_k(z):=\frac{1}{2\zeta(k)}G_k(z)=1+a_1e(z)+a_2e(2z)+\cdots.$$

We have seen from the first homework that E_k has the following series expression

$$E_k(z)=\frac{1}{2}\sum_{\gcd(c,d)=1}\frac{1}{(cz+d)^k}$$

and its Fourier expansion formula is given by

$$E_k(z)=1-\frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)e(nz), \tag{2.23}$$

where B_k is the k-th *Bernoulli number* defined by the formal power series expansion

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

3. The modular group and its fundamental domain

In this section we study more closely the $\mathrm{SL}_2(\mathbb{Z})$-linear fractional action on the upper half plane. Two important transformations are given by $S = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, where

$$Sz = -1/z \quad \text{and} \quad Tz = z + 1, \quad \forall z \in \mathbb{H}.$$

Theorem 3.1. *The modular group $\mathrm{SL}_2(\mathbb{Z})$ is generated by S and T.*

Proof. It suffices to show for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ after finitely many steps of left multiplying S and T we can reduce γ into the identity matrix. Note that

$$S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}$$

and

$$T^n \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + cn & b + dn \\ c & d \end{pmatrix}.$$

If $c = 0$, then $\gamma = \pm \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ for some $n \in \mathbb{Z}$. Applying T^{-n} we get $\pm I_2$. Applying $S^2 = -I$ if necessary to kill the negative sign, we can get the identity matrix. If $c \neq 0$, then apply T^n for some appropriate $n \in \mathbb{Z}$ we can get a new top left entry a' with $0 \leq a' < |c|$. Then apply S to get a new bottom left entry with absolute value strictly smaller than $|c|$. After applying this process finitely many times we can get a matrix with bottom left entry 0, reducing the argument to the first case. This finishes the proof. $\square$

Remark 3.1. In view of Lemma 2.2 and the above theorem, in order to check condition (2) in the definition of a modular form, it suffices to check (2.7) for $\gamma = S, T$. That is, (2.7) is equivalent to saying that f satisfies the two transformation rules in (2.9).

Definition 3.2. A set $\mathcal{F} \subset \mathbb{H}$ is called a *fundamental domain* for the modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$-action on $\mathbb{H}$ if

(1) $\mathcal{F}$ is a domain (i.e. a nonempty and connected open set) in $\mathbb{H}$,
(2) every orbit of Γ has a point in $\mathcal{F}$ or on the boundary $\partial\mathcal{F}$,
(3) distinct points in $\mathcal{F}$ are not in the same orbit of Γ.

Theorem 3.2. *The set*

$$\mathcal{F} := \left\{ z \in \mathbb{H} : |z| > 1,\ |\mathfrak{Re}(z)| < \tfrac{1}{2} \right\} \tag{3.3}$$

is a fundamental domain for the $\mathrm{SL}_2(\mathbb{Z})$-action on $\mathbb{H}$.

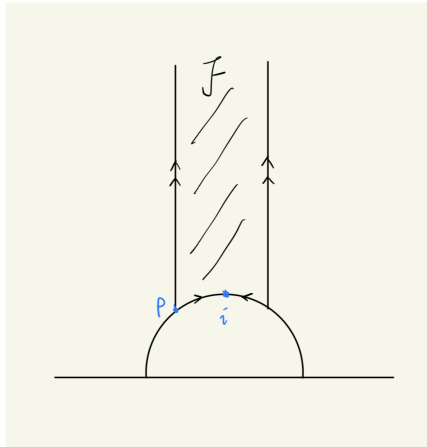

FIGURE 1. The fundamental domain $\mathcal{F}$ with boundary identified.

Remark 3.4. Roughly speaking, the quotient space $\Gamma\backslash\mathbb{H}$ (with $\Gamma = \mathrm{SL}_2(\mathbb{Z})$) is where a modular form lives[3] . It can be visualized as $\overline{\mathcal{F}}$ with the boundary of $\mathcal{F}$ identified by T and S respectively as shown in Figure 1. After gluing the equivalent sides of the boundary, $\overline{\mathcal{F}}$ becomes a punctured sphere. (The missing point is the point at ∞ and is called a *cusp* of $\Gamma\backslash\mathbb{H}$.) One can add this point at infinity to $\Gamma\backslash\mathbb{H}$ and equip suitable complex charts on $\Gamma\backslash\mathbb{H}\cup\{\infty\}$ to make it a compact Riemann surface, i.e. a connected 1-dimensional complex manifold.

Proof of Theorem 3.2. Clearly, $\mathcal{F}$ is a domain. We thus only need to verify conditions (2) and (3). For (2), take any $z = x + iy \in \mathbb{H}$, we need to show there exists $z' \in \Gamma z$ such that $z' \in \overline{\mathcal{F}} = \left\{z \in \mathbb{H} : |z| \leq 1,\ |\mathfrak{Re}(z)| \leq \frac{1}{2}\right\}$. First we claim that there exists $z'' \in \Gamma z$ attains the maximal height (the imaginary part) among all points in Γz. Moreover, each such a point satisfies the property that $|z''| \geq 1$. To prove this claim, note that for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$

$$\mathfrak{Im}(\gamma z) = \frac{y}{|cz+d|^2} = \frac{y}{(cx+d)^2 + c^2y^2} \begin{cases} \leq \frac{1}{c^2 y} \leq \frac{1}{y} & \text{if } c \neq 0, \\ = y & \text{if } c = 0. \end{cases}$$

In other words, heights of elements in Γz is uniformly bounded by $\max\{y, y^{-1}\}$. This implies the existence of such a point with maximal height. For the moreover part, suppose not, then we have some $z' \in \Gamma z$ with maximal height but $|z''| < 1$. Then clearly $Sz'' \in \Gamma z$, but

$$\mathfrak{Im}(Sz'') = \frac{\mathfrak{Im}(z'')}{|z''|^2} > \mathfrak{Im}(z''),$$

violating the maximality of the height of z''. Now take such a $z'' \in \Gamma z$, applying $T^n z'' = z'' + n$ for some appropriate $n \in \mathbb{Z}$ we can make $z' = T^n z''$ with $|\mathfrak{Re}(z')| \leq \frac{1}{2}$. But since this action does not change the height, z' is still of maximal height. Thus $|z'| \geq 1$. In other words, $z' \in \overline{\mathcal{F}}$. This verifies condition (2).

[3]Let f be a modular form. If f is of weight 0, then it is left Γ-invariant and can be viewed as a function on $\Gamma\backslash\mathbb{H}$. If f is of positive weight, then it is no longer a function on $\Gamma\backslash\mathbb{H}$ due to the factor $j_\gamma(z)$ in (2.7). However, it is still uniquely determined by its values on a fundamental domain of $\Gamma\backslash\mathbb{H}$.

For (3), let $z_1, z_2 \in \mathcal{F}$ with $z_1 = \gamma z_2$ and $\gamma \in \Gamma$. We would like to show $z_1 = z_2$. Without loss of generality we may assume $\mathfrak{Im}(z_1) \geq \mathfrak{Im}(z_2)$. Write $z_2 = x + iy$ and note that $y > \frac{\sqrt{3}}{2}$. Using similar computation as above we see that $\mathfrak{Im}(z_1) \geq \mathfrak{Im}(z_2)$ implies that

$$(cx + d)^2 + c^2 y^2 \leq 1. \tag{3.5}$$

In particular, we have $c^2y^2 \leq 1$, implying that $c^2 \leq y^{-2} < \frac{4}{3}$. Hence $c = 0, \pm 1$. If $c = 0$, then $\gamma = \pm T^n$ for some $n \in \mathbb{Z}$, implying that $z_1 = z_2 + n$. But $\max\{|\mathfrak{Re}(z_1)|, |\mathfrak{Re}(z_2)|\} < \frac{1}{2}$ forces $n = 0$. Hence $z_1 = z_2$ as desired.

If $c = \pm 1$, since $|x| < \frac{1}{2}$, we have $|cx + d| > \frac{1}{2}$ unless $d = 0$. If $d \neq 0$, then

$$(cx + d)^2 + c^2y^2 = (cx + d)^2 + y^2 > \frac{1}{4} + \frac{3}{4} = 1,$$

violating (3.5). If $d = 0$, then (3.5) is equivalent to $x^2 + y^2 < 1$, or equivalently, $|z_2| < 1$, violating $z_2 \in \mathcal{F}$. Hence we can not have $c = \pm 1$. This verifies condition (3) and hence finishes the proof. ☐

4. Dimension formula

The main goal of this section is to prove the following dimension formula for $\mathcal{M}_k$.

Theorem 4.1. *Let $k \geq 0$ be an even integer. The dimension of the space $\mathcal{M}_k$ is given by*

$$\dim_{\mathbb{C}} \mathcal{M}_k = \begin{cases} \lfloor k/12 \rfloor & k \equiv 2 \pmod{12}, \\ \lfloor k/12 \rfloor + 1 & k \not\equiv 2 \pmod{12}. \end{cases} \tag{4.1}$$

We first have the following preliminary lemma.

Lemma 4.2. *For $k \geq 4$ even, we have*

$$\dim_{\mathbb{C}} \mathcal{S}_k = \dim_{\mathbb{C}} \mathcal{M}_k - 1. \tag{4.2}$$

Proof. Let $k \geq 4$ be even. Consider the map $\mathcal{M}_k \to \mathbb{C}$ given by $f \mapsto f(\infty)$. Since $G_k \in \mathcal{M}_k$ and $G_k(\infty) \neq 0$, this map is surjective. One easily sees that this map is a group homomorphism with kernel $\mathcal{S}_k$, concluding this lemma. ☐

4.1. The zero-pole theorem.

The key ingredient to prove the dimension formula (4.1) is the following zero-pole theorem for modular forms. To state this theorem we need to introduce some more notation.

Definition 4.3. A *weight-k modular function* is a weight-k modular form except we only require it to be meromorphic on $\mathbb{H} \cup \{\infty\}$.

Example 4.4. *The j-invariant function $1728g_2^3/\Delta$ is a weight-0 modular function, which is holomorphic on $\mathbb{H}$ but has a simple pole at ∞ (since $g_2(\infty) \neq 0$ and Δ has a simple zero at ∞; see Remark 2.20).*

Let f be a weight k modular function. For any $z \in \mathbb{H}$, $\mathrm{ord}_z(f)$ is the order of f at z defined as before. We also need to define the order of f at ∞: Similar to modular forms, f also has a Fourier expansion at ∞

$$f(z) = \sum_{n=m}^{\infty} a_n(f) e(nz),$$

with $m \in \mathbb{Z}$ the smallest integer (not necessarily non-negative) such that $a_m(f) \neq 0$. Then m is defined as the *order of f at ∞*, denoted by $\operatorname{ord}_\infty(f)$.

We also let

$$\mathcal{F}' = \mathcal{F} \cup \{z \in \partial\mathcal{F} : \mathfrak{Re}(z) \leq 0\}$$

be a genuine fundamental domain for the $\mathrm{SL}_2(\mathbb{Z})$-action on $\mathbb{H}$, that is, every Γ-orbit has one and only one point in $\mathcal{F}'$. We now state the zero-pole theorem.

Theorem 4.3. *For a weight k modular function $f : \mathbb{H} \to \mathbb{C}$ not identically zero we have*

$$\operatorname{ord}_\infty(f) + \frac{1}{2}\operatorname{ord}_i(f) + \frac{1}{3}\operatorname{ord}_\rho(f) + \sum_{\substack{z \in \mathcal{F}' \\ \tau \neq \rho, i}} \operatorname{ord}_z(f) = \frac{k}{12}, \tag{4.5}$$

where $\rho = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$.

Lemma 4.4. *Let $A_\alpha(z_0, r)$ be the counter-clockwise arc boundary of the sector, centered at $z_0 \in \mathbb{C}$ with angle $\alpha \in (0, 2\pi)$ and radius r; see Figure 2. Then for any meromorphic function f, we have*

$$\lim_{r \to 0^+} \frac{1}{2\pi i} \int_{A_\alpha(z_0, r)} \frac{f'}{f}(z)\, dz = \frac{\alpha}{2\pi} \operatorname{ord}_{z_0}(f).$$

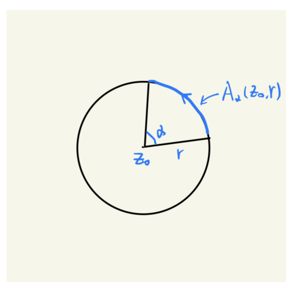

FIGURE 2. The arc $A_\alpha(z_0, r)$.

Proof. Write $f(z) = (z - z_0)^m g(z)$ with $m = \operatorname{ord}_{z_0}(f)$ and $g(z)$ holomorphic and nonzero around z_0. Then

$$\frac{f'}{f}(z) = \frac{m}{z - z_0} + \frac{g'}{g}(z).$$

Since g is holomorphic and nonzero around z_0, we have

$$\lim_{r \to 0^+} \frac{1}{2\pi i} \int_{A_\alpha(z_0, r)} \frac{g'}{g}(z)\, dz = 0.$$

This lemma then follows by a simple integration. $\square$

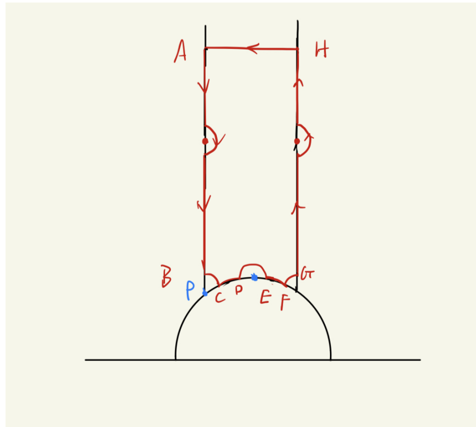

FIGURE 3. The integration path.

Proof of Theorem 4.3. Let L be the red path shown in Figure 3. Here we make small circular detours on the boundary of the standard fundamental domain $\mathcal{F}$ whenever f has a pole or zero there. The horizontal line segment $\overline{HA}$ is taken sufficiently high so that all the non-infinity poles or zeros of f in $\mathcal{F}$ are below $\overline{HA}$. Around the three points $w = \rho, i, \rho'$ with $\rho' = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, the path is the intersection between the small r-circle around w and $\mathcal{F}$. We can choose $r > 0$ sufficiently small so that f has neither poles nor zeros in the sectors enclosed by $\partial\mathcal{F}$ and this arc (except possibly at w). We integrate f'/f along this path to get

$$\frac{1}{2\pi i}\int_L \frac{f'}{f}(z)\ dz = \sum_{\substack{z\in\mathcal{F}'\\ z\neq\rho,i}} \mathrm{ord}_z(f).$$

On the horizontal path $\overline{HA}$, write $f(z) = \sum_{n=m}^{\infty} a_n(f)e(nz)$ with $m = \mathrm{ord}_\infty(f)$. Then $f'(z) = \sum_{n=m}^{\infty} a_n(f)(2\pi i n)e(nz)$ and we have

$$\frac{f'}{f}(z) = \sum_{\ell=0}^{\infty} b_\ell e(\ell z),$$

with $b_0 = 2\pi i m$. Hence

$$\frac{1}{2\pi i}\int_H^A \frac{f'}{f}(z)\ dz = -m = -\mathrm{ord}_\infty(f).$$

To compute the remaining integrals, for any $\gamma = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, let $F_\gamma(z) := f(\gamma z)$. By the chain rule we have

$$F_\gamma'(z) = f'(\gamma z)\frac{d\gamma z}{dz} = f'(\gamma z) j_\gamma(z)^{-2}.$$

On the other hand, $F_\gamma(z) = f(\gamma z) = j_\gamma(z)^k f(z)$, thus

$$F_\gamma'(z) = k j_\gamma(z)^{k-1} c f(z) + j_\gamma(z)^k f'(z).$$

This implies that

$$f'(\gamma z) = j_\gamma(z)^{k+1}(ckf(z) + j_\gamma(z)f'(z)).$$

Thus

$$\frac{f'}{f}(\gamma z)\ d\gamma z = \frac{j_\gamma(z)^{k+1}(ckf(z)+j_\gamma(z)f'(z))}{j_\gamma(z)^{k+2}f(z)}\ dz = \left(\frac{ck}{j_\gamma(z)}+\frac{f'}{f}(z)\right)\ dz. \tag{4.6}$$

Now we compute the remaining integrals. Note that $\overline{GH} = -T\overline{AB}$. Hence by (4.6) we get

$$\frac{1}{2\pi i}\int_G^H \frac{f'}{f}(z)\ dz = -\frac{1}{2\pi i}\int_A^B \frac{f'}{f}(Tz)\ dTz = -\frac{1}{2\pi i}\int_A^B \frac{f'}{f}(z)\ dz,$$

implying

$$\frac{1}{2\pi i}\left(\int_A^B + \int_G^H\right)\frac{f'}{f}(z)\ dz = 0.$$

We can apply Lemma 4.4 and (4.6) to get

$$\lim_{r\to 0^+}\frac{1}{2\pi i}\left(\int_B^C + \int_F^G\right)\frac{f'}{f}(z)\ dz = \lim_{r\to 0^+}\frac{1}{2\pi i}\left(\int_{F-1}^{G-1} + \int_B^C\right)\frac{f'}{f}(z)\ dz = -\frac{1}{3}\mathrm{ord}_\rho(f),$$

and

$$\lim_{r\to 0^+}\frac{1}{2\pi i}\int_D^E \frac{f'}{f}(z)\ dz = -\frac{1}{2}\mathrm{ord}_i(f).$$

It remains to evaluate the integral along the arcs $\overline{CD}$ and $\overline{EF}$. Note that $\overline{EF} = -S\overline{CD}$. Then by (4.6) we have

$$\frac{1}{2\pi i}\int_E^F \frac{f'}{f}(z)\ dz = -\frac{1}{2\pi i}\int_C^D \left(\frac{k}{z}+\frac{f'}{f}(z)\right)\ dz.$$

This implies that

$$\frac{1}{2\pi i}\left(\int_C^D + \int_E^F\right)\frac{f'}{f}(z)\ dz = \frac{1}{2\pi i}\int_D^C \frac{k}{z}\ dz \to \frac{\pi/6}{2\pi}k = \frac{k}{12}$$

as $r \to 0^+$. We then conclude the proof by collecting all the terms. □

4.2. **Proof of the dimension formula.** We can now give the

Proof of Theorem 4.1. We prove by induction. For a non-negative even integer k, let f be a weight k modular form which is not identically zero. In particular, when applying (4.5) to f, the terms in the left hand side of (4.5) are all non-negative.
Case I: $k = 0$**.** In this case f is a holomorphic function on $\Gamma\backslash\mathbb{H}$ which is also holomorphic at ∞. This forces that f to be entire and bounded, and thus is a constant, i.e. $\mathcal{M}_0 = \mathbb{C}$, consisting of constant functions.

Case II: $k = 2$**.** In this case $\mathcal{M}_2 = \{0\}$ since it is not possible for the left side of (4.5) to equal to $\frac{2}{12} = \frac{1}{6}$.

Case III: $k = 4, 6, 8, 10$**.** We claim for these k, $\mathcal{M}_k = \mathbb{C}G_k$. The containment "$\supset$" is clear. For the other containment by (4.2) it suffices to show $\mathcal{S}_k = \{0\}$ in view of Lemma 4.2. This is true since if there exists a nonzero $f \in \mathcal{S}_k$, applying (4.5) for f the left hand side is at least one while the right hand side is $\frac{k}{12}$, strictly smaller than 1, giving a contradiction.

Case IV: $k = 12$**.** In this case we claim $\mathcal{S}_{12} = \mathbb{C}\Delta$ and $\mathcal{M}_k = \mathcal{S}_k \oplus \mathbb{C}G_{12}$. The containment "$\supset$" is clear. For the other containment, again in view of Lemma 4.2 it suffices to show $\mathcal{S}_{12} = \mathbb{C}\Delta$. Let $f \in \mathcal{S}_{12}$ so that f vanishes at ∞. Since Δ has a simple zero at ∞ (see (2.16)) and it does not vanish on $\mathbb{H}$ (see Lemma 1.7) we have

$$\frac{f}{\Delta} \in \mathcal{M}_0 = \mathbb{C},$$

implying that there exists some $c \in \mathbb{C}$ such that $f = c\Delta$, finishing the proof of this case.

Case IV: $k > 12$**.** In general, the map

$$\mathcal{M}_{k-12} \to \mathcal{S}_k, \quad f \mapsto f\Delta$$

is a bijection since its inverse map $f \mapsto f/\Delta$ is well-defined. This implies that

$$\dim_{\mathbb{C}}\mathcal{M}_k = \dim_{\mathbb{C}}\mathcal{S}_k + 1 = \dim_{\mathbb{C}}\mathcal{M}_{k-12} + 1.$$

Applying this formula, together with our previous dimension formulas for $k \leq 12$ we can conclude the proof by induction. □

Remark 4.7. When $k = 4$, applying (4.3) to G_4 we see that the left hand side equals that right hand side, i.e. $\frac{1}{3}$, only if $\mathrm{ord}_\rho(f) = 1$ and $\mathrm{ord}_z(f) = 0$ for any other $z \in \mathcal{F}'$. Thus G_4 has a unique simple zero at ρ. Similarly, we can conclude G_6 has a unique simple zero at i.

Corollary 4.5. *The function G_4 and G_6 are algebraically independent, and*

$$\bigoplus_{k=0}^{\infty} \mathcal{M}_k = \mathbb{C}[G_4, G_6].$$

Proof. We first show that G_4, G_6 generate modular forms of all weights. This can be easily checked for $k < 12$. For $k \geq 12$ even, take $f \in \mathcal{M}_k$. We want to show f can be expressed as a polynomial in G_4 and G_6. First we can find $(i, j) \in (\mathbb{Z}_{\geq 0})^2$ such that $4i + 6j = k$. Then $G_4^i G_6^j$ is a weight k modular form which is nonzero at ∞. Hence we can find some $c \in \mathbb{C}$ such that $f - cG_4^i G_6^j \in \mathcal{S}_k$. Then the function $h(G_4, G_6) := \frac{f - cG_4^i G_6^j}{\Delta} \in \mathcal{M}_{k-12}$, which by induction, is a polynomial in G_4, G_6. Thus $f = cG_4^i G_6^j + \Delta h(G_4, G_6)$ is a polynomial in G_4, G_6.

Next, we show G_4 and G_6 are algebraically independent. Suppose not, then there is some nonzero complex polynomial $P(x, y)$ such that $P(G_4, G_6) = 0$. Since modular forms of different weights are $\mathbb{C}$-linearly independent (Remark 2.8), we may assume the monomials in $P(G_4, G_6)$ are of the same weight. We can also assume this weight is minimal. If a pure power of G_4 occurs in $P(G_4, G_6)$, i.e. there exists some positive integer m and some lower degree polynomial $P'(x, y)$ such that

$$0 = P(G_4, G_6) = G_4^m + G_6 P'(G_4, G_6).$$

Taking $z = i$ and noting that $G_6(i) = 0$ we get $G_4^m(i) = 0$ which is impossible since $G_4(i) \neq 0$; see Remark 4.7. Hence there is no pure power of G_4 appearing in $P(G_4, G_6)$. This shows that $P = G_6 P'(G_4, G_6)$ for some lower degree polynomial P'. But since G_6 only vanishes at one point, we have $P'(G_4, G_6) = 0$, contradicting the assumption that monomials of P are of minimal weight. □

5. Relations with elliptic curves

Recall that $(\wp, \wp')$ satisfies the relation (1.18). This gives a realization of the complex torus $\mathbb{C}/\Lambda$ as an *elliptic curve*.

Definition 5.1. A complex *elliptic curve* E is a smooth projective algebraic curve of genus 1. Up to isomorphism it is given by the Weierstrass form

$$E = \left\{(x, y) \in \mathbb{C}^2 : y^2 = 4x^3 - a_2 x - a_3\right\} \cup \{\infty\}$$

with $a_2^3 - 27a_3^2 \neq 0$[4].

In view of (1.18) and Lemma 1.7 every lattice $\Lambda \subset \mathbb{C}$ gives rise to an elliptic curve E_Λ with $a_2 = g_2(\Lambda)$ and $a_3 = g_3(\Lambda)$.

Lemma 5.1. *For any lattice $\Lambda \subset \mathbb{C}$, the map sending $u+\Lambda \in \mathbb{C}/\Lambda$ to $(\wp(u), \wp'(u)) \in E_\Lambda$ is bijective*[5].

Proof. We only need to show the inverse map exists. We define a map $\iota : E_\Lambda \to \mathbb{C}/\Lambda$ as following: Set $\iota(\infty) = \Lambda$. For a non infinity point $(x, y) \in E_\Lambda$, if $y \neq 0$ (so that $x \neq e_j$ for $j = 1, 2, 3$), then there exists u which is not a 2-torsion point such that $\wp(u) = x$. Then set $\iota(x, y) = u + \Lambda$ and $\iota(x, -y) = -u + \Lambda$. If $y = 0$ then $x = e_j$ for some $1 \le j \le 3$ and we set $\iota(x, y) = \frac{\omega_j}{2} + \Lambda$. ◻

This map transfers the group law from the complex torus to the elliptic curve. More precisely, if $P_1, P_2 \in E_\Lambda$ is the image of $u_1 + \Lambda$ and $u_2 + \Lambda$ under the above map. Then $P_1 + P_2 \in E_\Lambda$ is defined to be the image of $u_1 + u_2 + \Lambda$ under the above map. We have the following geometric description of this group law. We denote by O the ∞ point of E_Λ which is the image of $\Lambda \in \mathbb{C}/\Lambda$ under the above map, and thus is the identity element in E_Λ.

Proposition 5.2. *For any $P_1, P_2, P_3 \in E_\Lambda$, $P_1 + P_2 + P_3 = O$ if and only if P_1, P_2, P_3 are colinear, i.e. they are the intersection points of a line with E_Λ.*

Proof. For $i = 1, 2, 3$, assume $P_i = (\wp(u_i), \wp'(u_i))$ for some $u_i \in \mathbb{C}$. Let $L : ax + by + c = 0$ be the line passing through P_1 and P_2, that is, the function

$$f(u) := a\wp(u) + b\wp'(u) + c$$

vanishes at $u = u_1, u_2$. If $b \neq 0$, then f has a triple pole at $0 \pmod{\Lambda}$ (cf. Remark 1.17). Then by (1.8) f must have another zero. We thus have the following equivalent statements:

$$\begin{aligned} P_3 \in E_\Lambda \text{ is on the line } \overline{P_1P_2} \quad &\Longleftrightarrow \quad u_3 \text{ is a zero of } f \\ &\overset{(1.9)}{\Longleftrightarrow} \quad u_1 + u_2 + u_3 - 3 \cdot 0 \in \Lambda \\ &\Longleftrightarrow \quad P_1 + P_2 + P_3 = O. \end{aligned}$$

The case when $b = 0$ follows from similar analysis which implies that in this case $P_1 + P_2 = O = P_3$. ◻

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[4]More precisely, $E \subset \mathbb{PC}^3$ is cut out the projective homogeneous cubic polynomial $zy^2 = 4x^3 - a_2xz^2 - a_4z^3$ which consists of the affine curve $y^2 = 4x^3 - a_2x - a_3$ by setting $z = 1$ together with the point $[0, 1, 0]$ viewed as the ∞ point.

[5]Here $(\wp'(0), \wp(0))$ is understood as the ∞ point of E_Λ.

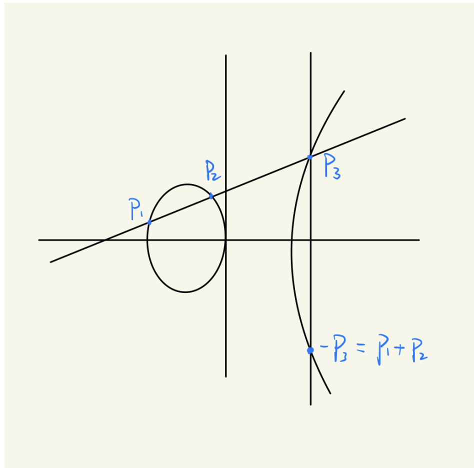

FIGURE 4. Group law on the elliptic curve $y^2 = 4x^3 - 4x$.

We have seen that every lattice Λ gives rise to an elliptic curve E_Λ. Indeed the converse is also true. For this we need the following simple application of the zero-pole theorem to the j-invariant function.

Lemma 5.3. *The j-invariant function $j : \mathbb{H} \to \mathbb{C}$ is a bijection.*

Proof. For any value $c \in \mathbb{C}$, consider the function $f(z) = j(z) - c$. Since j has a simple pole at ∞ and is holomorphic everywhere else, we can apply (4.5) to f to get

$$\frac{1}{2}\mathrm{ord}_i(f) + \frac{1}{3}\mathrm{ord}_\rho(f) + \sum_{\substack{z \in \mathcal{F}' \\ \tau \neq \rho, i}} \mathrm{ord}_z(f) = 1.$$

For this equality to hold the left side must have exactly one positive term, implying that j is a bijection. $\square$

The following theorem shows that indeed up to isomorphism every complex elliptic curve can be realized as E_Λ for some lattice Λ.

Theorem 5.4. *For any complex elliptic curve $y^2 = x^3 - a_2 x - a_3$ with $a_2^3 - 27a_3^2 \neq 0$. There exists some lattice Λ such that $g_2(\Lambda) = a_2$ and $g_3(\Lambda) = a_3$.*

Proof. Note that

$$j = \frac{1728 g_2^3}{\Delta} = \frac{1728}{1 - 27 g_3^2/g_2^3}.$$

By Lemma 5.3 there exists some $z \in \mathbb{H}$ such that $j(z) = \frac{1728}{1 - 27a_3^2/a_2^3}$. In terms of lattices, this means that the lattice $\Lambda = z\mathbb{Z} + \mathbb{Z}$ satisfies

$$\frac{g_2^3(\Lambda)}{g_3^2(\Lambda)} = \frac{a_3^2}{a_2^3}. \tag{5.2}$$

Note that $g_2(\lambda\Lambda) = \lambda^{-4} g_2(\Lambda)$ and $g_3(\lambda\Lambda) = \lambda^{-6} g_3(\Lambda)$ for any nonzero $\lambda \in \mathbb{C}$. We can find λ such that $g_2(\lambda\Lambda) = a_2$. This, together with (5.2) implies that $g_3^2(\lambda\Lambda) = a_3^2$. Thus up to changing λ to $-\lambda$, this lattice $\lambda\Lambda$ satisfies the desired property that $g_2(\lambda\Lambda) = a_2$ and $g_3(\lambda\Lambda) = a_3$. This concludes the proof. $\square$

6. Modular forms for congruence subgroups

6.1. A quick review of hyperbolic geometry. Let

$$\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$$

be the usual upper half plane. It is equipped with the hyperbolic metric

$$\frac{|dz|^2}{y^2} = \frac{dx^2 + dy^2}{y^2}.$$

This is to say, for each $z = x + iy \in \mathbb{H}$ the tangent space $T_z\mathbb{H} \cong \mathbb{C}$ of $\mathbb{H}$ at z is equipped with the inner product $\langle v, w\rangle_z := \frac{(v,w)}{y^2}$, where $(v, w) = v\overline{w}$ is the usual inner product on $\mathbb{C}$ with $\overline{w}$ the complex conjugation. Or equivalently, $T_z\mathbb{H}$ is equipped with the norm $\|\cdot\|_z := y^{-1}|\cdot|$, where $|\cdot|$ is the usual Euclidean norm on $\mathbb{C}$. The hyperbolic distance is defined such that for any $z, w \in \mathbb{H}$,

$$d_{\mathbb{H}}(z, w) = \inf_{\phi} \int_0^1 \|\phi'(t)\|_{\phi(t)} dt,$$

where $\phi(t) : [0, 1] \to \mathbb{H}$ runs through all smooth curves in $\mathbb{H}$ connecting z and w.

Lemma 6.1. *Let $z_1 = iT_1, z_2 = iT_2$ for some $T_1 > T_2 > 0$. Then $d_{\mathbb{H}}(z_1, z_2) = \log\left(\frac{T_1}{T_2}\right)$ with the geodesic given by the line segment connecting z_1 and z_2.*

Proof. Let $\phi(t) = x(t) + iy(t) : [0, 1] \to \mathbb{H}$ be any smooth curve connecting z_1, z_2. Then we have

$$\begin{aligned}\int_0^1 \|\phi'(t)\|_{\phi(t)} dt &= \int_0^1 y(t)^{-1}|\phi'(t)| dt \geq \int_0^1 \frac{|y'(t)|}{y(t)} dt \\ &\geq \int_0^1 \frac{y'(t)}{y(t)} dt = \log y(t)\Big|_0^1 = \log\left(\tfrac{T_1}{T_2}\right).\end{aligned}$$

Since ϕ is an arbitrary smooth curve connecting z_1 and z_2, we conclude that $d_{\mathbb{H}}(z_1, z_2) \geq \log\left(\frac{T_1}{T_2}\right)$. Moreover, we see from the computation that the equality holds when $x'(t) = 0$ and $y'(t) \geq 0$ for all $t \in [0, 1]$, implying that the geodesic from z_1 to z_2 is the line segment connecting z_1 and z_2. $\square$

6.2. The isometry group. The group $M_2(\mathbb{R})$ of 2×2 real matrices is a vector space with a norm given by

$$\|g\| = a^2 + b^2 + c^2 + d^2, \quad g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{R}).$$

This norm gives a metric topology to the subgroup

$$\mathrm{SL}_2(\mathbb{R}) = \{g \in M_2(\mathbb{R}) : \det(g) = 1\}$$

through the natural embedding. This subgroup $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ via linear fractional transformation:

$$gz = \frac{az + b}{cz + d}, \quad \forall\, g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), z \in \mathbb{H}.$$

Let $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$ be the boundary of $\mathbb{H}$. Then $\mathrm{SL}_2(\mathbb{R})$ also acts on $\partial\mathbb{H}$ with the same formula[6].

Lemma 6.2. *Consider the* $\mathrm{SL}_2(\mathbb{R})$*-action on* $\mathbb{H}$.

[6] Precisely, $g\infty = \frac{a}{c}$, and for any $x \in \mathbb{R}$, $gx = \frac{ax+b}{cx+d} \in \mathbb{R}$ unless $cx + d = 0$ for which gx is understood as ∞.

(1) *This action is transitive.*
(2) *This action is isometric, that is,*

$$d_{\mathbb{H}}(gz_1, gz_2) = d_{\mathbb{H}}(z_1, z_2), \quad \forall\ z_1, z_2 \in \mathbb{H}, g \in \mathrm{SL}_2(\mathbb{R}).$$

Proof. For (1), take any $z = x + iy \in \mathbb{H}$, we want to show there exists some $g \in \mathrm{SL}_2(\mathbb{R})$ such that $gi = z$. Note that the matrix $g = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \begin{pmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ suffices.

For (2), let $\phi : [0,1] \to \mathbb{H}$ be any smooth curve connecting z_1 and z_2. Then $g\phi$ is a smooth curve connecting gz_1 and gz_2. Thus

$$d_{\mathbb{H}}(gz_1, gz_2) \leq \int_0^1 \|dg\phi\|_{g\phi(t)}\ dt.$$

Note that for any $z \in \mathbb{H}$, $dgz = \frac{dz}{j_g(z)}$ and $\mathfrak{Im}(gz) = \frac{\mathfrak{Im}(z)}{|j_g(z)|^2}$, where $j_g(z) = cz + d$ is the automorphy factor as before. This implies that for any $0 \leq t \leq 1$,

$$\|dg\phi(t)\|_{g\phi(t)} = \mathfrak{Im}(g\phi(t))^{-1} \frac{|d\phi(t)|}{|j_g(\phi(t))|^2} = \mathfrak{Im}(\phi(t))^{-1}|d\phi(t)| = \|d\phi(t)\|_{\phi(t)}.$$

Thus

$$d_{\mathbb{H}}(gz_1, gz_2) \leq \int_0^1 \|d\phi\|_{\phi(t)}\ dt.$$

Since ϕ is arbitrary we deduce that $d_{\mathbb{H}}(gz_1, gz_2) \leq d_{\mathbb{H}}(z_1, z_2)$. On the other hand, since $g \in \mathrm{SL}_2(\mathbb{R})$ is arbitrary, we also have

$$d_{\mathbb{H}}(z_1, z_2) = d_{\mathbb{H}}(g^{-1}gz_1, g^{-1}gz_2) \leq d_{\mathbb{H}}(gz_1, gz_2),$$

implying that $d_{\mathbb{H}}(gz_1, gz_2) = d_{\mathbb{H}}(z_1, z_2)$. This finishes the proof. □

Remark 6.1. In fact, the group $\mathrm{PSL}_2(\mathbb{R}) := \mathrm{SL}_2(\mathbb{R})/\langle \pm I_2 \rangle$ is the orientation preserving isometry group of $\mathbb{H}$.

As a corollary we have the following description of geodesics on $\mathbb{H}$.

Corollary 6.3. *The geodesics on $\mathbb{H}$ are semi-circles connecting two points in $\partial\mathbb{H}$. Here we regard the vertical line connecting $x \in \mathbb{R}$ and ∞ as the semi-circle connecting these two points.*

Proof. Given $z_1, z_2 \in \mathbb{H}$, we want to show the geodesic from z_1 to z_2 is the arc from z_1 to z_2 on the semi-circle determined by z_1 and z_2. First we show that there exists $g \in \mathrm{SL}_2(\mathbb{R})$ satisfying $g^{-1}z_1 = i$ and $g^{-1}z_2 \in i\mathbb{R}_{>0}$. The first condition is equivalent to $gi = z_1$. For $x \in \mathbb{R}, y > 0, \theta \in [0, 2\pi)$ let $n_x = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$, $a_y = \begin{pmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{pmatrix}$ and $k_\theta = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$ respectively. We take $g = n_{x_1}a_{y_1}k_{\theta_1} \in \mathrm{SL}_2(\mathbb{R})$ with $x_1 = \mathfrak{Re}(z_1)$, $y_1 = \mathfrak{Im}(z_1)$ and $\theta_1 \in [0, 2\pi)$ to be determined. One easily checks that k_{θ_1} fixes i and thus $gi = n_{x_1}a_{y_1}i = x_1 + iy_1 = z_1$. Hence the first condition is satisfied. The second condition is equivalent to finding θ_1 such that $k_{-\theta_1}a_{y_1}^{-1}n_{x_1}^{-1}z_2 \in i\mathbb{R}_{>0}$ (note that $k_\theta^{-1} = k_{-\theta}$. Then by Exercise 5 below we can always find such θ_1 and hence this condition can be satisfied. Now let g be such that $g^{-1}z_1 = i$ and $g^{-1}z_2 \in i\mathbb{R}_{>0}$. Let $\mathcal{G}$ be the geodesic from $g^{-1}z_1$ to $g^{-1}z_2$. Then by Lemma 6.1 it is the vertical line segment from $g^{-1}z_1$ to $g^{-1}z_2$. Since g acts on $\mathbb{H}$ as isometries, it sends geodesics to geodesics. Thus $g\mathcal{G}$ is the geodesic from z_1 to z_2. By Exercise 6 below wee see that $g\mathcal{G}$ is the desired arc on the semi-circle determined by z_1 and z_2. □

Exercise 5. *For any $\theta \in [0, 2\pi)$ let k_θ be as above. Show that for any $z \in \mathbb{H}$ there exists some θ such that $k_\theta z$ has real part equaling 0.*

Exercise 6. *Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$. Show that g sends the vertical line $i\mathbb{R}$ to the semi-circle connecting $\frac{a}{c}$ and $\frac{b}{d}$.*

The group $\mathrm{SL}_2(\mathbb{R})$ has an Iwasawa decomposition that any $g \in \mathrm{SL}_2(\mathbb{R})$ can be written uniquely of the form

$$g = n_x a_y k_\theta, \quad \text{for some } x \in \mathbb{R}, y > 0, \theta \in [0, 2\pi),$$

where $n_x = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$, $a_y = \begin{pmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{pmatrix}$ and $k_\theta = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$. The matrix n_x acts as translations $n_x z = z + x$ and has exactly one fixed point ∞, a_y acts as dilations $a_y z = yz$ and has two fixed points $0, \infty$, while k_θ fixes i and acts as rotation around i.

The action of $\mathrm{SL}_2(\mathbb{R})$ can be classified by the fixed points of its action on $\mathbb{H} \cup \partial\mathbb{H}$.

Proposition 6.4. *Let $g \neq \pm I_2 \in \mathrm{SL}_2(\mathbb{R})$ and consider its action on $\mathbb{H} \cup \partial\mathbb{H}$.*

1. *g has exactly one fixed point on $\partial\mathbb{H}$ if $|\operatorname{tr}(g)| = 2$,*
2. *g has two fixed points both on $\partial\mathbb{H}$ if $|\operatorname{tr}(g)| > 2$,*
3. *g has exactly one fixed point in $\mathbb{H}$ if $|\operatorname{tr}(g)| < 2$.*

Definition 6.2. The transformations in these three cases are called *parabolic*, *hyperbolic* and *elliptic* respectively.

Proof of Proposition 6.4. Take $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. If $c = 0$ (so that $d = a^{-1}$), then ∞ is clearly has a fixed point. To see if it has any other fixed point, assume $z \in \mathbb{H} \cup \mathbb{R}$ such that $gz = z$. This is equivalent to $(1 - a^2)z = b$. Note that in this case $|\operatorname{tr}(g)| = |a + a^{-1}| \geq 2$. If $|\operatorname{tr}(g)| = 1$, i.e. $|a| = 1$, then this equation has no solution unless $b = 0$, but this is the case when $g = \pm I_2$, violating our assumption. If $|\operatorname{tr}(g)| > 2$ (i.e. $|a| \neq 1$), then $z = \frac{b}{1-a^2}$ is another fixed point and its on $\partial\mathbb{H}$. This proves the case when $c = 0$.

If $c \neq 0$ and suppose $z \in \overline{\mathbb{H}}$ is a fixed point of g, that is,

$$\gamma z = \frac{az + b}{cz + d} = z.$$

Note that in this case the denominator can not be zero since otherwise $z = -\frac{d}{c} \neq \infty = \gamma z$. Hence the above equation is equivalent to the quadratic equation $cz^2 + (d - a)z - b = 0$. Computing the discriminant we get

$$\Delta = (d - a)^2 + 4bc = a^2 - 2ad + d^2 + 4bc = (a + d)^2 - 4 = \operatorname{tr}(g)^2 - 4.$$

When $|\operatorname{tr}(g)| = 2$, this quadratic equation has only one real solution, lying on $\partial\mathbb{H}$. Hence $|\operatorname{tr}(g)| > 2$, it has two real solutions, thus both lying on $\partial\mathbb{H}$. When $\operatorname{tr}(g)| < 2$, it has two complex solutions, one lies in $\mathbb{H}$ and the other lies in the lower half plane. This finishes the proof. □

6.3. **Discrete subgroups.**

Definition 6.3. A subgroup $\Gamma < \mathrm{SL}_2(\mathbb{R})$ is *discrete* if the induced topology in Γ is discrete, i.e. if the sets $\{\gamma \in \Gamma : \|\gamma\| < r\}$ are finite for any $r > 0$.

Definition 6.4. A discrete subgroup $\Gamma < \mathrm{SL}_2(\mathbb{R})$ is called a *lattice* if its fundamental domain has finite area with respect to the hyperbolic measure

$$d\mu(z) = \frac{dxdy}{y^2}.$$

It is called *uniform* (resp. *non-uniform*) if the closure of its fundamental domain is compact (resp. non-compact).

Exercise 7. *Show that μ is $\mathrm{SL}_2(\mathbb{R})$-invariant, i.e. $d\mu(gz) = d\mu(z)$ for any $g \in \mathrm{SL}_2(\mathbb{R})$.*

Example 6.5. *The subgroup*

$$\Gamma = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$$

is discrete, but it's not a lattice since it has a fundamental domain

$$\mathcal{F} = \{z = x + iy \in \mathbb{H} : 0 \leq x < 1\}$$

with hyperbolic area

$$\mu(\mathcal{F}) = \int_0^1 \int_0^\infty \frac{dxdy}{y^2} = \infty.$$

Example 6.6. *The modular group $\mathrm{SL}_2(\mathbb{Z})$ is an example of a non-uniform lattice. It is clearly discrete. Recall the set $\mathcal{F}$ defined in (3.3) is a fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$. It is clearly non-compact, but its hyperbolic area is finite:*

$$\mu(\mathcal{F}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\sqrt{1-x^2}}^\infty \frac{dydx}{y^2} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} d\theta = \frac{\pi}{3}.$$

Let $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ be a finite-index subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Let $\{\gamma_i\} \subset \mathrm{SL}_2(\mathbb{Z})$ be such that $\mathrm{SL}_2(\mathbb{Z}) = \bigsqcup_i \Gamma\gamma_i$ if $-I_2 \in \Gamma$ and $\Gamma = \bigsqcup_i (\Gamma\gamma_i \sqcup \Gamma - \gamma_i)$ if $-I_2 \notin \Gamma$. We have the following description of the fundamental domain of Γ which implies that Γ is also a non-uniform lattice.

Proposition 6.5. *Let $\mathcal{F} \subset \mathbb{H}$ be a fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$, we claim without proof that the set*

$$\mathcal{F}_\Gamma := \bigcup_i \gamma_i \mathcal{F}$$

is a disjoint union and is a fundamental domain for Γ.

Remark 6.7. The set $\mathcal{F}_\Gamma$ clearly depends on the choice of coset representatives. For some choices $\mathcal{F}_\Gamma$ may not be a domain, and thus here we do not require condition (1) of Definition 3.2. This proposition easily implies that all finite-index subgroups of $\mathrm{SL}_2(\mathbb{Z})$ are also non-uniform lattices.

Proof of Proposition 6.5. We assume $-I_2 \in \Gamma$ and note that the case when $-I_2 \notin \Gamma$ follows from similar arguments. We first show that this union is disjoint. Suppose not, then there exists $i \neq j$ and $z_1, z_2 \in \mathcal{F}$ such that $\gamma_i z_1 = \gamma_j z_2$, or equivalently, $\gamma_2^{-1}\gamma_1 z_1 = z_2$. Since $\mathcal{F}$ is a fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$, $z_1, z_2 \in \mathcal{F}$ and $\gamma_2^{-1}\gamma_1 \in \mathrm{SL}_2(\mathbb{Z})$, we have $\gamma_2^{-1}\gamma_1 = \pm I_2 \in \Gamma$[7] which is a contradiction since γ_i and γ_j lie in different Γ cosets. Next, we show any two points in $\mathcal{F}_\Gamma$ are in distinct Γ orbits.

[7]We didn't state this fact, but it follows from our proof of Theorem 3.2.

Suppose $\gamma_i z_1, \gamma_j z_2 \in \mathcal{F}_\Gamma$ lie in the same Γ orbit, i.e. there exists some $\gamma \in \Gamma$ such that $\gamma\gamma_i z_1 = \gamma_j z_2$. Again this implies that $\gamma_j^{-1}\gamma\gamma_i = \pm I_2$ and $z_1 = z_2$. The first condition implies that γ_i and γ_j are in the same Γ coset and hence $i = j$, implying that $\gamma_i z_1 = \gamma_j z_2$.

Finally, we show that for any point $z \in \mathbb{H}$, there exists $\gamma \in \Gamma$ such that $\gamma z \in \overline{\mathcal{F}_\Gamma}$. Taking the inverse from the right coset decomposition we have $\mathrm{SL}_2(\mathbb{Z}) = \bigsqcup_i \gamma_i^{-1}\Gamma$. Now since $\mathcal{F}$ is a fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$, there exists some $\gamma' \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma' z \in \overline{\mathcal{F}}$. From the above left coset decomposition, we have $\gamma' = \gamma_i^{-1}\gamma$ for some i and $\gamma \in \Gamma$. That is, $\gamma z \in \gamma_i\overline{\mathcal{F}} \subset \overline{\mathcal{F}_\Gamma}$. This finishes the proof. $\square$

Example 6.8. *This shows that any finite-index subgroup of* $\mathrm{SL}_2(\mathbb{Z})$ *is also a non-uniform lattice. For instance, consider* $\Gamma = \langle T^2, S\rangle$ *generated by* T^2 *and* S. *Note that* $-I_2 = S^2 \in \Gamma$. *One easily sees that* $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma] = 3$ *and we can choose* $\{I_2, T, TS\}$ *as a set of right* Γ*-coset representatives. Let* $\mathcal{F}$ *be the fundamental domain as in* (3.3). *Then the resulting fundamental domain of* Γ *is as in the following first figure. The second fundamental domain is obtained by cutting the first one along the line* $\mathfrak{Re}(z) = 1$ *and moving the part to the right of the line to the left by the transformation* T^{-2}.

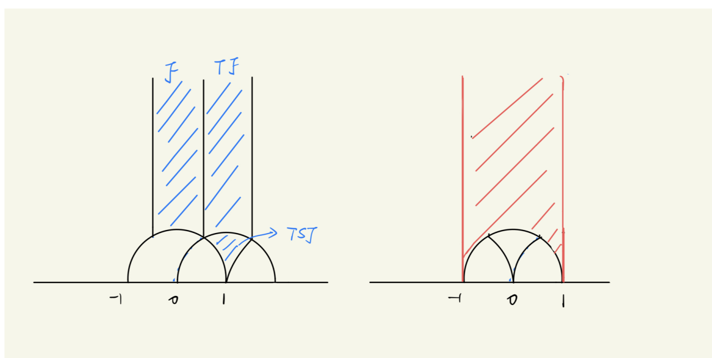

FIGURE 5. Fundamental domains for $\Gamma = \langle T^2, S\rangle$

Let Γ be a non-uniform lattice. For any $x \in \partial\mathbb{H}$, let us denote by

$$\Gamma_x := \{\gamma \in \Gamma : \gamma x = x\}$$

the stabilizer of x in Γ. We say Γ_x is trivial if Γ_x does not contain any parabolic motion, or equivalently, Γ_x is nontrivial if Γ_x contains some parabolic motion.

Lemma 6.6. *Let* Γ *be a finite-index subgroup of* $\mathrm{SL}_2(\mathbb{Z})$.

1. $\mathrm{SL}_2(\mathbb{Z})$ *acts on* $\mathbb{Q} \cup \{\infty\}$ *and this action is transitive.*
2. $\mathrm{SL}_2(\mathbb{Z})_x$ *is nontrivial if and only if* $x \in \mathbb{Q} \cup \{\infty\}$.
3. *For any* $x \in \mathbb{Q} \cup \{\infty\}$, Γ_x *is cyclic and generated by a parabolic motion*[8].
4. Γ*-action on* $\mathbb{Q} \cup \{\infty\}$ *has finitely many orbits.*

Proof. For (1), the first part is clear. For (2), for any $r \in \mathbb{Q} \cup \{\infty\}$ we need to show there exists $g \in \mathrm{SL}_2(\mathbb{Z})$ such that $g\infty = r$. If $r = \infty$, we simply take $g = I_2$. If $r = \frac{a}{c}$ is rational written in lowest terms, i.e. $\gcd(a, c) = 1$. Then there exist

[8]More precisely, when $-I_2 \in \Gamma$, it is generated by a parabolic motion together with $-I_2$.

$b, d \in \mathbb{Z}$ such that $ad - bc = 1$. Then $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $g\infty = \frac{a}{c} = r$ satisfying the desired property.

For (2) we only prove the easy direction "$\Leftarrow$". The other direction is left as an exercise. When $x = \infty$ this is clear since

$$\mathrm{SL}_2(\mathbb{Z})_\infty = \{\pm \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) : n \in \mathbb{Z}\}.$$

For general $x \in \mathbb{Q} \cup \{\infty\}$, we can find $\tau \in \mathrm{SL}_2(\mathbb{Z})$ such that $\tau\infty = x$. We claim that $\mathrm{SL}_2(\mathbb{Z})_x = \tau\, \mathrm{SL}_2(\mathbb{Z})_\infty \tau^{-1}$ from which it is clear that $\mathrm{SL}_2(\mathbb{Z})_x$ is nontrivial. To prove this claim, we prove a slightly more general statement that for any finite-index subgroup Γ and $\sigma \in \mathrm{SL}_2(\mathbb{R})$ such that $\sigma\infty = x$, we have

$$\sigma^{-1}\Gamma_x\sigma = (\sigma^{-1}\Gamma\sigma)_\infty. \tag{6.9}$$

Applying this statement for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and $\tau = \sigma$ we easily get the above claim. Take any element $\gamma = \sigma^{-1}\gamma'\sigma \in \sigma^{-1}\Gamma_x\sigma$ for some $\gamma' \in \Gamma_x$. Clearly $\gamma \in \sigma^{-1}\Gamma\sigma$. We also have $\gamma\infty = \sigma^{-1}\gamma'\sigma\infty = \sigma^{-1}\gamma' x = \sigma^{-1}x = \infty$, implying that $\sigma^{-1}\Gamma_x\sigma \subset (\sigma^{-1}\Gamma\sigma)_\infty$. The other containment follows by applying the same argument to the lattice $\Gamma' = \sigma^{-1}\Gamma\sigma$ and the translating matrix σ^{-1} sending x to ∞. This proves the statement.

For (3), by (2) for any $x \in \mathbb{Q} \cup \{\infty\}$ there exists some parabolic motion $\tau_x \in \mathrm{SL}_2(\mathbb{Z})$ such that $\mathrm{SL}_2(\mathbb{Z})_x = \langle \pm\tau_x \rangle$. In particular, this implies that Γ_x (a subgroup of $\mathrm{SL}_2(\mathbb{Z})_x$) can only consist of the trivial motion and parabolic motions. Suppose Γ_x is trivial, then $\tau_x^i \notin \Gamma$ for any nonzero integer i. In particular, this implies that $\Gamma\tau_x^i$, $i \in \mathbb{Z}$ are distinct Γ-cosets, contradicting the finite index assumption.

For (4), write $\mathrm{SL}_2(\mathbb{Z}) = \bigsqcup_i \Gamma\gamma_i$ in coset decompositions. Then $\mathbb{Q} \cup \{\infty\} = \mathrm{SL}_2(\mathbb{Z})\infty = \bigcup_i \Gamma\gamma_i\infty$. This finishes the proof. $\square$

Exercise 8. *Show that if* $\mathrm{SL}_2(\mathbb{Z})_x$ *is non-trivial then* $x \in \mathbb{Q} \cup \{\infty\}$.

Definition 6.10. A *cusp* of Γ is a Γ-orbit under the Γ-action on $\mathbb{Q} \cup \{\infty\}$. We also use elements in each Γ-orbit to represent this cusp. Two elements in the same Γ-orbit are called Γ-equivalent.

Example 6.11.

(1) *In view of (1) of Lemma 6.6,* $\mathrm{SL}_2(\mathbb{Z})$ *has only one cusp. We can say* ∞ *(or any other point in* $\mathbb{Q} \cup \{\infty\}$*) is a cusp of* $\mathrm{SL}_2(\mathbb{Z})$.

(2) *The lattice* $\Gamma = \langle T^2, S \rangle$ *is of index* 3 *with coset representatives given by* $\{I_2, T, TS\}$*, but it has only two cusps:* $\Gamma I_2\infty = \Gamma\infty = \Gamma T\infty$ *and* $\Gamma TS\infty = \Gamma 1$*. We can say* ∞ *and* 1 *are the two inequivalent cusps of* Γ.

Remark 6.12.

(1) In view of (4) of Lemma 6.6, a finite-index subgroup of Γ has finitely many cusps. However, in general the number of cusps is smaller than the index of Γ in $\mathrm{SL}_2(\mathbb{Z})$ as seen from the above example.

(2) Geometrically, cusps of a non-uniform lattice correspond to "cusps" of the quotient space $\Gamma\backslash\mathbb{H}$ (endowed with the quotient topology from the natural projection map $\mathbb{H} \to \Gamma\backslash\mathbb{H}$ and visualized by its fundamental domain with sides identified).

Let $\mathfrak{a} \in \mathbb{Q} \cup \{\infty\}$ be a cusp of Γ and let $\tau_\mathfrak{a} \in \mathrm{SL}_2(\mathbb{Z})$ be such that $\tau_\mathfrak{a}\infty = \mathfrak{a}$. Then

$$\tau_\mathfrak{a}^{-1}\Gamma_\mathfrak{a}\tau_\mathfrak{a} = (\tau_\mathfrak{a}^{-1}\Gamma\tau_\mathfrak{a})_\infty = \langle \text{``}\pm\text{''} \left(\begin{smallmatrix} 1 & m_\mathfrak{a} \\ 0 & 1 \end{smallmatrix}\right) \rangle,$$

for some positive integer $m_{\mathfrak{a}}$. This integer $m_{\mathfrak{a}}$ is called the *width* of the cusp $\mathfrak{a}$. Similarly, there exists $\sigma_{\mathfrak{a}} \in \mathrm{SL}_2(\mathbb{R})$ satisfying

$$\sigma_{\mathfrak{a}}\infty = \mathfrak{a}, \quad \tau_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\tau_{\mathfrak{a}} = (\tau_{\mathfrak{a}}^{-1}\Gamma\tau_{\mathfrak{a}})_\infty = \langle \text{``}\pm\text{''} \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\rangle.$$

For instance, we may take $\sigma_{\mathfrak{a}} = \tau_{\mathfrak{a}} \begin{pmatrix} \sqrt{m_{\mathfrak{a}}} & 0 \\ 0 & 1/\sqrt{m_{\mathfrak{a}}} \end{pmatrix}$. The matrix $\sigma_{\mathfrak{a}}$ is called a *scaling matrix* of the cusp $\mathfrak{a}$.

Remark 6.13.

(1) The width of a cusp measures the size of a cusp, e.g. for $\Gamma = \langle T^2, S\rangle$, one sees that $\Gamma_\infty = \langle \pm \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)\rangle$ and thus the cusp ∞ has width 2. For the cusp 1 one can take $\tau_1 = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. It is easy to see $\Gamma_1 = \langle \pm T^2 S\rangle$ and $\tau_1^{-1}\Gamma_1\tau_1 = (\tau_1^{-1}\Gamma\tau_1)_\infty = \langle \pm \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\rangle$. This implies that the cusp 1 has width 1.
(2) A scaling matrix is essentially used as a "change of variable" to transfer cusp $\mathfrak{a}$ (of Γ) to the cusp ∞ (of $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}$); see Remark 6.15 below.

Definition 6.14. Let $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ be a finite-index subgroup. A function $f : \mathbb{H} \to \mathbb{C}$ is a *modular form of weight k* with respect to Γ if

(1) f is holomorphic,
(2) f is weakly modular of weight k with respect to Γ, i.e. $f[\gamma]_k = f$, $\forall\, \gamma \in \Gamma$,
(3) f is holomorphic at each cusp of Γ.

If in addition f vanishes at each cusp, then f is called a *cusp form of weight k with respect to* Γ.

Remark 6.15.

(1) For any $\sigma \in \mathrm{SL}_2(\mathbb{R})$, one easily sees that f is weakly modular of weight k with respect to Γ if and only if $f[\sigma]_k$ is weakly modular of weight k with respect to $\sigma^{-1}\Gamma\sigma$. Let $\mathfrak{a} \in \mathbb{Q} \cup \{\infty\}$ be a cusp of Γ with a scaling matrix $\sigma_{\mathfrak{a}}$. Then f is holomorphic at $\mathfrak{a}$ means that $f[\sigma_{\mathfrak{a}}]_k$ is holomorphic at ∞, which as before means that it has the following Fourier expansion at ∞
$$f[\sigma_{\mathfrak{a}}]_k(z) = \sum_{n=0}^{\infty} \widehat{f_{\mathfrak{a}}}(n)e(nz).$$
Similarly, f vanishes at $\mathfrak{a}$ means we further have $\widehat{f_{\mathfrak{a}}}(0) = 0$.
(2) We denote by $\mathcal{M}_k(\Gamma)$ (resp. $\mathcal{S}_k(\Gamma)$) the set of weight k modular (resp. cusp) forms with respect to Γ. If $\Gamma_1 < \Gamma_2$, then we have the relations $\mathcal{M}_k(\Gamma_2) \subset \mathcal{M}_k(\Gamma_1)$ and $\mathcal{S}_k(\Gamma_2) \subset \mathcal{S}_k(\Gamma_1)$. (Smaller group means less restrictions from the weak modularity assumption (condition (2) of Definition 6.14).)
(3) If $-I_2 \in \Gamma$, then $\mathcal{M}_k(\Gamma)$ is empty for odd weights. If $-I_2 \notin \Gamma$, $\mathcal{M}_k(\Gamma)$ may not be empty.

6.4. **Congruence groups.** In this course we mainly work with modular forms with respect to certain family of subgroups of $\mathrm{SL}_2(\mathbb{Z})$ called congruence subgroups.

Definition 6.16. For any $N \in \mathbb{N}$, the *principle congruence group of level N* is defined by
$$\Gamma(N) := \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv I_2 \pmod{N}\}.$$

Remark 6.17. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, this notation $\gamma \equiv I_2 \pmod N$ is a shorthand for $a \equiv d \equiv 1 \pmod N$ and $b \equiv c \equiv 0 \pmod N$.

Definition 6.18. A discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ is called a *congruence subgroup* if $\Gamma(N) < \Gamma < \mathrm{SL}_2(\mathbb{Z})$ for some $N \in \mathbb{N}$.

The following two families of groups are common examples of congruence subgroups: For any $N \in \mathbb{N}$, define

$$\Gamma_1(N) := \left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \pmod N\right\},$$

and

$$\Gamma_0(N) := \left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \pmod N\right\}.$$

Proposition 6.7. *For any $N \in \mathbb{N}$, we have*

$$[\Gamma(1) : \Gamma(N)] = N^3 \prod_{p|N} (1 - p^{-2}), \tag{6.19}$$

$$[\Gamma(1) : \Gamma_1(N)] = N^2 \prod_{p|N} (1 - p^{-2}), \tag{6.20}$$

and

$$[\Gamma(1) : \Gamma_0(N)] = N \prod_{p|N} (1 + p^{-1}), \tag{6.21}$$

where the product $\prod_{p|N}$ runs over all prime divisors of N.

Lemma 6.8. *The natural reduction map from $\mathrm{SL}_2(\mathbb{Z})$ to $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ is surjective.*

Proof. Let π be this projection. The case when $N = 1$ is trivial since in this case $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ is trivial. Assume $N \geq 2$. Take any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{Z})$ such that its reduction modulo N lies in $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$, that is, $ad - bc \equiv 1 \pmod N$. We need to show that there exists $\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ whose reduction modulo N is the same as that of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We first note that the condition $ad - bc \equiv 1 \pmod N$ implies that $\gcd(c, d, N) = 1$. Changing c to $c + N$ we may assume $c \neq 0$. First we show that we can take $(c', d') \in \mathbb{Z}^2$ with $\gcd(c', d') = 1$ and $(c', d') \equiv (c, d) \pmod N$. We take $c' = c$ and $d' = d + kN$ with $k \in \mathbb{Z}$ to be determined. Let $P_1 = \prod_{\substack{p|c \\ p|d}} p$ and $P_2 = \prod_{\substack{p|c \\ p\nmid d}} p$. Clearly $\gcd(P_1, P_2) = 1$. By the Chinese Remainder theorem we can find k satisfying

$$k \equiv 1 \pmod{P_1} \quad \text{and} \quad k \equiv 0 \pmod{P_2}.$$

Let $p \mid c$ be a prime divisor of c. If $p \mid d$, then $\gcd(p, N) = 1$ (since $\gcd(c, d, N) = 1$) and $p \mid P_1$. Hence $\gcd(c, d + kN) = \gcd(c, kN) = 1$. If $p \nmid d$, then $p \mid P_2$ and thus $p \mid k$. Hence $(p, d + kN) = \gcd(p, d) = 1$. This shows that any prime divisor of $c' = c$ is coprime to $d' = d + kN$ and thus $\gcd(c', d') = 1$. Since $\gcd(c', d') = 1$, there exist $u, v \in \mathbb{Z}$ such that $ud' - vc' = 1$. Then we have

$$\begin{pmatrix} a & b \\ c' & d' \end{pmatrix} \begin{pmatrix} u & v \\ c' & d' \end{pmatrix}^{-1} = \begin{pmatrix} ad' - bc' & * \\ 0 & ud' - vc' \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod N.$$

Thus there exists some $\ell \in \mathbb{Z}$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} a & b \\ c' & d' \end{pmatrix} \equiv \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u & v \\ c' & d' \end{pmatrix} = \begin{pmatrix} u + \ell c' & v + \ell d' \\ c' & d' \end{pmatrix} \pmod N.$$

Then $a' = u + \ell c'$ and $b' = v + \ell d'$ satisfy the desired properties. $\square$

Proof of Proposition 6.7. Consider the reduction map $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. By Lemma 6.8 it is surjective and clearly with kernel $\Gamma(N)$. Hence

$$[\Gamma(1) : \Gamma(N)] = \# \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}).$$

Now (6.19) follows from the counting formula for $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ (see Exercise 9). For (6.21) consider the map $\Gamma_1(N) \to \mathbb{Z}/N\mathbb{Z}$ sending $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to $b \pmod N$. We easily check that it is a group homomorphism and is surjective with kernel $\Gamma(N)$. Hence $[\Gamma_1(N) : \Gamma(N)] = q$. This, together with (6.19) implies (6.20). Finally, for (6.21), note that the map $\Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^\times$ sending $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to $d \pmod N$ is a surjective group homomorphism with kernel given by $\Gamma_1(q)$. Hence $[\Gamma_0(1) : \Gamma_1(N)] = \phi(N)$, implying that

$$[\Gamma(1) : \Gamma_0(N)] = [\Gamma(1) : \Gamma_1(N)]\phi(N)^{-1} = \frac{N^2 \prod_{p|N}(1-p^{-2})}{N \prod_{p|N}(1-p^{-1})} = N \prod_{p|N}(1+p^{-1}),$$

as desired. $\square$

Exercise 9. *Show that* $\# \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) = N^3 \prod_{p|N}(1-p^{-2})$.

6.5. **Cusps of principle congruence subgroups.** In this subsection we give an explicit description of the cusps of the principle congruence subgroup $\Gamma(N)$. That is, we classify the orbits of the $\Gamma(N)$-action on $\mathbb{Q} \cup \{\infty\}$. Since we already know $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$ has one cusp, we assume $n \geq 2$ in the below.

The starting point is the following lemma giving necessary and sufficient conditions on when two elements in $\mathbb{Q} \cup \{\infty\}$ represent the same $\Gamma(N)$-orbit.

Lemma 6.9. *Let* $s = \frac{a}{c}$ *and* $s' = \frac{a'}{c'}$ *be elements of* $\mathbb{Q} \cup \{\infty\}$ *with* $\gcd(a,c) = \gcd(a',c') = 1$. *Then*

$$\Gamma(N)s = \Gamma(N)s' \quad \Longleftrightarrow \quad \left(\begin{smallmatrix} a' \\ c' \end{smallmatrix}\right) \equiv \pm \left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right) \pmod N.$$

Proof. For this direction "$\Rightarrow$", suppose there exists some $\gamma = \left(\begin{smallmatrix} p & q \\ r & t \end{smallmatrix}\right) \in \Gamma(N)$ such that $s' = \gamma s = \frac{pa+qc}{ra+tc}$. First note that $\gcd(pa+qc, ra+tc) = 1$ (see Exercise 10). This then implies that

$$\begin{pmatrix} a' \\ c' \end{pmatrix} = \pm \begin{pmatrix} pa+qc \\ ra+tc \end{pmatrix} \equiv \pm \begin{pmatrix} a \\ c \end{pmatrix} \pmod N.$$

For the other direction, assume $\left(\begin{smallmatrix} a' \\ c' \end{smallmatrix}\right) \equiv \pm \left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right) \pmod N$. we want to find $\gamma \in \Gamma(N)$ such that $\gamma s = s'$. Up to changing $\left(\begin{smallmatrix} a' \\ c' \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} -a' \\ -c' \end{smallmatrix}\right)$ we may assume $\left(\begin{smallmatrix} a' \\ c' \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right) \pmod N$. We first assume $\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ so that $s = \frac{1}{0} = \infty$ and $\left(\begin{smallmatrix} a' \\ c' \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \pmod N$. Since $\gcd(a',c') = 1$ and $a' \equiv 1 \pmod N$, there exists $\beta, \delta \in \mathbb{Z}$ such that $a'\delta - c'\beta = \frac{1-a'}{N}$, or equivalently, $a'(N\delta+1) - c'\beta N = 1$. Then the matrix $\gamma = \left(\begin{smallmatrix} a' & \beta N \\ c' & 1+\delta N \end{smallmatrix}\right)$ lies in $\Gamma(N)$ and satisfies that $\gamma\infty = \frac{a'}{c'} = s'$. In general, there exists $b, d \in \mathbb{Z}$ such that $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Note that $\alpha \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right)$. Hence the vector $\left(\begin{smallmatrix} a'' \\ c'' \end{smallmatrix}\right) := \alpha^{-1} \left(\begin{smallmatrix} a' \\ c' \end{smallmatrix}\right)$ satisfies $\left(\begin{smallmatrix} a'' \\ c'' \end{smallmatrix}\right) \equiv \alpha^{-1} \left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \pmod N$. Then by the previous argument, there exists $\gamma' \in \Gamma(N)$ satisfying $\gamma'\infty = \frac{a''}{c''} = \alpha^{-1}\frac{a'}{c'}$. Let $\gamma = \alpha\gamma'\alpha^{-1}$ Then we have

$$\gamma s = \alpha\gamma'\alpha^{-1}\frac{a}{c} = \alpha\gamma'\infty = \alpha\alpha^{-1}\frac{a'}{c'} = s'.$$

Moreover, since $\Gamma(N) < \mathrm{SL}_2(\mathbb{Z})$ is normal, we also have $\gamma \in \Gamma(N)$. This finishes the proof. $\square$

Exercise 10. *Let* $\binom{a}{c}, \binom{a'}{c'} \in \mathbb{Z}^2$ *be two nonzero integer vectors. Suppose* $\binom{a}{c} = \gamma\binom{a}{c}$ *for some* $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. *Show that* $\gcd(a,c) = \gcd(a',c')$.

Lemma 6.10. *Let* $a, c \in \mathbb{Z}^2$ *and let* $\overline{a}, \overline{c} \in \mathbb{Z}/N\mathbb{Z}$ *be their reductions in* $\mathbb{Z}/N\mathbb{Z}$. *The following are equivalent.*

1. *There exists a lift* $\binom{a'}{c'} \in \mathbb{Z}^2$ *of* $\binom{\overline{a}}{\overline{c}}$ *with* $\gcd(a',c') = 1$,
2. $\gcd(a,c,N) = 1$,
3. $\binom{\overline{a}}{\overline{c}}$ *has order* N *in* $(\mathbb{Z}/N\mathbb{Z})^2$.

Proof. If condition (1) holds, then there exists k, l, s, t such that $k(a+sN) + l(c+tN) = 1$, i.e. $ka + lc + (ks+lt)N = 1$. This implies that $\gcd(a,c,N) = 1$, giving condition (2).

If condition (2) holds, then there exist $b, d, k \in \mathbb{Z}$ such that $ad - bc + kN = 1$, implying that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv I_2 \pmod N$. Then by Lemma 6.8 there exists a lift $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. In particular, $\binom{a'}{c'} \in \mathbb{Z}^2$ satisfies $\gcd(a',c') = 1$ and $\binom{a'}{c'} \equiv \binom{a}{c} \pmod N$, giving condition (1).

Finally, we show (2) $\Leftrightarrow$ (3). Note that $k\binom{\overline{a}}{\overline{c}} \equiv \begin{pmatrix} \overline{0} \\ \overline{0} \end{pmatrix} \pmod N$ is equivalent to $N \mid k\gcd(a,c)$. Thus condition (3) is equivalent to the statement that $N \mid k\gcd(a,c)$ if and only if $N \mid k$, which is equivalent to $\gcd(a,c,N) = 1$, i.e. condition (2). $\square$

Proposition 6.11. *Let* h_N *be the number of cusps of* $\Gamma(N)$. *We have*

$$h_N = \begin{cases} \frac{1}{2}N^2\prod_{p\mid N}(1-p^{-2}) & N > 2, \\ 3 & N = 2. \end{cases}$$

Proof. In view of Lemma 6.9 and Lemma 6.10 and the fact that $N \geq 1$, $1 \equiv -1 \pmod N$ if and only if $N = 2$ we know that

$$h_N = \delta_N \# \left\{(a,c) \in (\mathbb{Z}/N\mathbb{Z})^2 : \gcd(a,c,N) = 1\right\},$$

where $\delta_N = 1$ if $N = 2$ and $\delta_N = \frac{1}{2}$ if $N > 2$. Let $\varphi(N)$ be the above counting function. We have

$$\varphi(N) = \sum_{d\mid N} \sum_{\substack{a \in \mathbb{Z}/N\mathbb{Z} \\ \gcd(a,N)=d}} \sum_{\substack{c \in \mathbb{Z}/N\mathbb{Z} \\ \gcd(c,d)=1}} 1 = \sum_{d\mid N} \phi(N/d)(N/d)\phi(d)$$

with ϕ the Euler's totient function. With a standard computation one gets that

$$\varphi(N) = N^2\prod_{p\mid N}(1-p^{-2})$$

implying the desired formula for h_N. $\square$

Exercise 11. *Fill in the details of the above computation.*

Example 6.22.

1. *For* $\Gamma(3)$ *as* $\binom{a}{c}$ *runs through* $\{0,1,2\}^2$ *and after ruling out the ones with* $\gcd(a,c,3) > 1$ *we get* 4 *pairs of integral vectors* $\binom{0}{1} \sim \binom{0}{2}$, $\binom{1}{0} \sim \binom{2}{0}$, $\binom{1}{1} \sim \binom{2}{2}$ *and* $\binom{1}{2} \sim \binom{2}{1}$ *which gives* 4 *inequivalent cusps* $0, \infty, 1, \frac{1}{2}$.
2. *For* $\Gamma(4)$ *one can similarly get that* $\Gamma(4)$ *has* 6 *cusps with a complete list of cusp representatives given by* $0, \infty, 1, \frac{1}{2}, \frac{1}{3}, 2$.

7. Eisenstein series and Poincaré series

7.1. **Poincaré series for the modular group.** Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and let $k \geq 4$ is be an even integer. Recall that the normalized weight k Eisenstein series is defined by

$$E_k(z) = \frac{1}{2\zeta(k)} G_k(z) = \frac{1}{2} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ \gcd(c,d)=1}} \frac{1}{(cz+d)^k}.$$

There is a more intrinsic way of defining this series which can be generalized to produce more modular forms. Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and let

$$\Gamma_\infty = \{\gamma \in \Gamma : \gamma\infty = \infty\} = \langle \pm \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle$$

be the stabilizer of the cusp ∞ in Γ as before. We have the following simple lemma.

Lemma 7.1. *Let $\Gamma'_\infty = \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle$ be the index two subgroup of Γ_∞ and let*

$$\mathbb{Z}^2_{\mathrm{pr}} := \left\{(c,d) \in \mathbb{Z}^2 : \gcd(c,d) = 1\right\}$$

be the set of primitive integer points in $\mathbb{R}^2$. The map from $\Gamma'_\infty\backslash\Gamma$ to $\mathbb{Z}^2_{\mathrm{pr}}$ sending $\Gamma'_\infty\gamma$ to $(0,1)\gamma$ is well-defined and a bijection.

Remark 7.1. Similarly, $\Gamma_\infty\backslash\Gamma$ is in bijection with $\mathbb{Z}^2_{\mathrm{pr}}/\pm$ which can be further identified with the set $\{(0,1)\} \cup \{(c,d) \in \mathbb{Z}^2_{\mathrm{pr}} : c > 0\}$.

Proof. First note that $(0,1)\gamma$ is exactly the bottom row of γ and left multiplying elements of Γ'_∞ does not change the bottom row of γ. This implies that this map is well-defined. It is also surjective since for any $(c,d) \in \mathbb{Z}^2_{\mathrm{pr}}$ we can find $(a,b) \in \mathbb{Z}^2$ such that $ad - bc = 1$, i.e. $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and satisfies $(0,1)\gamma = (c,d)$. It is also injective: Suppose $(0,1)\gamma_1 = (0,1)\gamma_2$ for some $\gamma_1, \gamma_2 \in \Gamma$. Then we have $(0,1)\gamma_1\gamma_2^{-1} = (0,1)$, implying that $\gamma_1\gamma_2^{-1} \in \Gamma'_\infty$, i.e. $\Gamma'_\infty\gamma_1 = \Gamma'_\infty\gamma_2$. $\square$

Recall that for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $j_\gamma(z) = cz + d$ is the factor of automorphy. Thus the normalized Eisenstein series can be rewritten as

$$E_k(z) = \frac{1}{2} \sum_{\gamma \in \tilde{\Gamma}_\infty\backslash\Gamma} j_\gamma(z)^{-k} = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j_\gamma(z)^{-k}.$$

We can construct more general modular forms via this averaging technique.

Definition 7.2. Let $p : \mathbb{H} \to \mathbb{C}$ be holomorphic and periodic with period 1.

(1) The corresponding *weight k Poincaré series* is defined by

$$P(z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j_\gamma(z)^{-k} p(\gamma z)$$

whenever this series is absolutely convergent.

(2) When $p(z) = e(mz)$ for some integer $m \geq 0$, the corresponding Poincaré series, denoted by $P_{m,k}$ is called the *m-th Poincaré series of weight k.*

Remark 7.3. The 0-th Poincaré series $P_{0,k}$ is simply the normalized Eisensetien series E_k.

Lemma 7.2. *The Poincaré series is well-defined.*

Proof. We need to show the definition is independent of the choice of coset representatives. First note that p is periodic with period 1 means that $p(\gamma z) = p(z)$ for any $\gamma \in \Gamma_\infty$. Let $\{\gamma_i\}, \{\gamma_i'\}$ be two complete sets of coset representatives of $\Gamma_\infty \backslash \Gamma$. Up to reordering we may assume $\Gamma_\infty \gamma_i = \Gamma_\infty \gamma_i'$ for each i. Then there exists $n_i \in \Gamma_\infty$ such that $\gamma_i' = n_i \gamma_i$. Then we have $j_{\gamma_i'}(z) = j_{n_i\gamma_i}(z) = j_{n_i}(\gamma_i z) j_{\gamma_i}(z) = j_{\gamma_i}(z)$ and $p(\gamma_i' z) = p(n_i \gamma_i z) = p(\gamma_i z)$. Hence

$$\sum_i j_{\gamma_i'}(z)^{-k} p(\gamma_i' z) = \sum_i j_{\gamma_i}(z)^{-k} p(\gamma_i z)$$

is independent of the choice of coset representatives. This finishes the proof. $\square$

Lemma 7.3. *$P_{m,k} \in \mathcal{S}_k$ for any $m \geq 1$.*

Proof. Since $e(mz)$ is uniformly bounded by 1 and is holomorphic and $k \geq 4$, the defining series converges absolutely and uniformly on compact sets, it defines a holomorphic function on $\mathbb{H}$. Moreover, one can show

$$\lim_{y \to \infty} P_{m,k}(iy) = \lim_{y \to \infty} e(mz) = 0,$$

vanishing at ∞. Here the last equality holds since $m \geq 1$ and $z \in \mathbb{H}$. It thus remains to show $P_{m,k}$ is weakly modular with respect to Γ, i.e. $P_{m,k}[\alpha]_k = P_{m,k}$ for any $\alpha \in \Gamma$. Take any $\alpha \in \Gamma$ we have

$$\begin{aligned} P_{m,k}[\alpha]_k(z) &= j_\alpha(z)^{-k} P_{m,k}(\alpha z) = j_\alpha(z)^{-k} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} j_\gamma(\alpha z)^{-k} e(m\gamma\alpha z) \\ &= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} j_{\gamma\alpha}(z)^{-k} e(m\gamma\alpha z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} j_\gamma(z)^{-k} e(m\gamma z) = P_{m,k}(z). \end{aligned}$$

Here for the third equation we used the chain rule of $j_\gamma(z)$ (see Lemma 2.2) and for the second last equality we used the fact that if $\{\gamma_i\}$ is a set of representatives for $\Gamma_\infty \backslash \Gamma$, so is $\{\gamma_i \alpha\}$ for any $\alpha \in \Gamma$. This concludes the proof. $\square$

Remark 7.4. When $k = 12$, for any $m \geq 1$, $P_{m,12} = c_m \Delta$ for some $c_m \in \mathbb{C}$.

7.2. **Poincaré series for congruence subgroups.** The same construction also works for a general congruence subgroup. Let $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup and let $\mathcal{C}_\Gamma \subset \mathbb{Q} \cup \{\infty\}$ be a complete set of inequivalent cusps of Γ, e.g. $\mathcal{C}_{\mathrm{SL}_2(\mathbb{Z})} = \{\infty\}$ and $\mathcal{C}_{\Gamma(3)} = \{0, \infty, 1, \frac{1}{2}\}$.

Definition 7.5. Let $\mathfrak{a} \in \mathcal{C}_\Gamma$ be a cusp of Γ and let $p : \mathbb{H} \to \mathbb{C}$ be holomorphic and periodic with period 1.

(1) The corresponding *weight k Poincaré series of* Γ *at the cusp* $\mathfrak{a}$ is defined by

$$P_{\mathfrak{a}}(z) = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} j_{\sigma_{\mathfrak{a}}^{-1}\gamma}(z)^{-k} p(\sigma_{\mathfrak{a}}^{-1} \gamma z)$$

whenever this series is absolutely convergent. Here $\sigma_{\mathfrak{a}}$ is a scaling matrix at the cusp $\mathfrak{a}$.

(2) When $p(z) = e(mz)$ for some $m \geq 0$, the corresponding Poincaré series, denoted by $P_{\mathfrak{a},m}(z)$, is called the *m-th Poincaré series of* Γ *at the cusp* $\mathfrak{a}$. The 0-th Poincaré series is called the *Eisenstein series of* Γ *at the cusp* $\mathfrak{a}$ and we denote it by $E_{\mathfrak{a}}(z)$.

Remark 7.6. Each cusp defines a Poincaré series.

Lemma 7.4. *For each $\mathfrak{a} \in \mathcal{C}_\Gamma$, $P_{\mathfrak{a}}(z)$ is well-defined.*

Proof. The proof is similar to that of Lemma 7.2. Let $\{\gamma_i\}$ and $\{\gamma_i'\}$ be two sets of coset representatives for $\Gamma_{\mathfrak{a}}\backslash\Gamma$. Up to reordering we may assume $\gamma_i' = \tau_i\gamma_i$ for some $\tau_i \in \Gamma_{\mathfrak{a}}$. Recall that $\sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \langle \text{“}\pm\text{”} \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\rangle$. Thus τ_i is of the form $\tau_i = \sigma_{\mathfrak{a}} n_i \sigma_{\mathfrak{a}}^{-1}$ for some $n_i \in \langle \text{“}\pm\text{”} \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\rangle$. We then have

$$\begin{aligned}\sum_i j_{\sigma_{\mathfrak{a}}^{-1}\gamma_i'}(z)^{-k} p(\sigma_{\mathfrak{a}}^{-1}\gamma_i' z) &= \sum_i j_{n_i\sigma_{\mathfrak{a}}^{-1}\gamma_i}(z)^{-k} p(n_i\sigma_{\mathfrak{a}}^{-1}\gamma_i z) \\ &= \sum_i j_{\sigma_{\mathfrak{a}}^{-1}\gamma_i}(z)^{-k} p(\sigma_{\mathfrak{a}}^{-1}\gamma_i z),\end{aligned}$$

implying this definition is independent of the choice of coset representatives. □

Lemma 7.5. *For $k \geq 3$ and $m \geq 0$, $P_{\mathfrak{a},m}(z)$ is holomorphic and weakly modular of weight k with respect to Γ.*

Proof. Since $k \geq 3$, by Proposition 2.3 the defining series converges absolutely and uniformly on compact sets of $\mathbb{H}$, thus defines a holomorphic function. Next we show $P_{\mathfrak{a},m}$ is weakly modular of weight k with respect to Γ. Take any $\alpha \in \Gamma$, we have

$$\begin{aligned}P_{\mathfrak{a},m}[\alpha]_k(z) &= j_\alpha(z)^{-k} \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} j_{\sigma_{\mathfrak{a}}^{-1}\gamma}(\alpha z)^{-k} e(m\sigma_{\mathfrak{a}}^{-1}\gamma\alpha z) \\ &= \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} j_{\sigma_{\mathfrak{a}}^{-1}\gamma\alpha}(z)^{-k} e(m\sigma_{\mathfrak{a}}^{-1}\gamma\alpha z) \\ &= \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} j_{\sigma_{\mathfrak{a}}^{-1}\gamma}(z)^{-k} e(m\sigma_{\mathfrak{a}}^{-1}\gamma z) = P_{\mathfrak{a},m}(z).\end{aligned}$$

This finishes the proof. □

In order to show that $P_{\mathfrak{a},m}$ is a modular form with respect to Γ, we need to show that it is holomorphic at all cusps of Γ. For this we prove explicitly the Fourier expansion of $P_{\mathfrak{a},m}$. We first carry out the computation for the modular group.

7.3. **Fourier expansion of Poincaré series for the modular group.** Let $k \geq 4$ be even and $m \geq 1$. Since $P_{m,k} \in \mathcal{S}_k$, it has a Fourier expansion

$$P_{m,k}(z) = \sum_{n=1}^{\infty} \widehat{p}_k(m,n)e(nz)$$

with

$$\widehat{p}_k(m,n) = \int_0^1 p_{m,k}(z)e(-nz)\, dx.$$

Here as usual $z = x + iy$. Before presenting the theorem, we first introduce some notation.

Definition 7.7. For any $m, n \in \mathbb{Z}$ and $c \geq 1$, the classical *Kloosterman sum* is defined by

$$S(m,n;c) = \sum_{ad\equiv 1 \pmod c} e\left(\frac{ma+nd}{c}\right).$$

Exercise 12. *Let $c \geq 1$ be a positive integer and $m, n \in \mathbb{Z}$. Show that*

(1) $S(m,n;c) = S(n,m;c)$.

(2) $S(am,n;c)=S(m,an;c)$ *if* $\gcd(a,c)=1$.
(3) *(multiplicativity in c)*

$$S(m,n;c)=S(\overline{q}m,\overline{q}n;r)S(\overline{r}m,\overline{r}n;q)$$

if $c=qr$ *with* $(q,r)=1$ *and* $\overline{q}q\equiv 1 \pmod r$ *and* $\overline{r}r\equiv 1 \pmod q$.

Definition 7.8. Let $\nu\in\mathbb{Z}$, the J-Bessel function of type ν is defined by the following formal power series expression

$$e^{x(z-\frac{1}{z})/2}=\sum_{\nu=-\infty}^{\infty}J_\nu(x)z^\nu.$$

Exercise 13. *Use the power series expression for the exponential function to show that*

$$J_\nu(x)=\sum_{m=\max\{0,-\nu\}}\frac{(-1)^m}{m!(m+\nu)!}\left(\frac{x}{2}\right)^{\nu+2m}.$$

We have the following explicit formula for $\widehat{p}_k(m,n)$, generalizing Proposition 2.5.

Theorem 7.6. *For any $m\geq 1$ and $n\geq 1$,*

$$\widehat{p}_k(m,n)=\delta_{mn}+2\pi i^{-k}\left(\frac{n}{m}\right)^{\frac{k-1}{2}}\sum_{c=1}^{\infty}c^{-1}S(m,n;c)J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

Remark 7.9. We can alternatively write

$$\widehat{p}_k(m,n)=\left(\frac{n}{m}\right)^{\frac{k-1}{2}}\left(\delta_{mn}+2\pi i^{-k}\sum_{c=1}^{\infty}c^{-1}S(m,n;c)J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)\right). \tag{7.10}$$

By (1) of Exercise 12 we see that $\widehat{p}(m,n)$ has a symmetry in (m,n) in the sense that

$$\left(\frac{m}{n}\right)^{\frac{k-1}{2}}\widehat{p}_k(m,n)=\left(\frac{n}{m}\right)^{\frac{k-1}{2}}\widehat{p}_k(n,m).$$

Here $\widehat{p}_k(m,n)$ in the left hand side is the n-th Fourier coefficient of the m-th Poincaré series while $\widehat{p}_k(n,m)$ in the right hand side is the m-th Fourier coefficient of the n-th Poincaré series. This is why we write this Fourier coefficient as $\widehat{p}_k(m,n)$ rather than the more conventional $\widehat{P}_{m,k}(n)$.

Proof of Theorem 7.6. For any $\gamma\in\Gamma$, we denote by $\gamma_{(c,d)}$ to indicate that (c,d) is the bottom row of γ. Then by definition and the bijection in Remark 7.1 we have

$$\begin{aligned}
\widehat{p}_k(m,n)&=\int_0^1\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}j_\gamma(z)^{-k}e(m\gamma z)e(-nz)\,dx\\
&=\int_0^1 e((m-n)z)\,dx+\sum_{\substack{(c,d)\in\mathbb{Z}^2_{\mathrm{pr}}\\ c>0}}\int_0^1\frac{e(m\gamma_{(c,d)}z)}{(cz+d)^k}e(-nz)\,dx\\
&=\delta_{mn}+\sum_{c=1}^{\infty}\sum_{\substack{d\in\mathbb{Z}\\ \gcd(c,d)=1}}\int_0^1\frac{e(m\gamma_{(c,d)}z)}{(cz+d)^k}e(-nz)\,dx\\
&=\delta_{mn}+\sum_{c=1}^{\infty}\sum_{d\in(\mathbb{Z}/c\mathbb{Z})^\times}\sum_{d'\equiv d \pmod c}\int_0^1\frac{e(m\gamma_{(c,d')}z)}{(cz+d')^k}e(-nz)\,dx,
\end{aligned}$$

where in the last equality we split the sum over d into congruence classes modulo c. This theorem then follows from the following calculation of the innermost sum in the above equation; see also Remark 7.11 below. $\square$

Proposition 7.7. *For any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ with $c > 0$ and $0 \le d < c$ and for any $m \ge 0$, $n \in \mathbb{Z}$ define*

$$\mathcal{I}_\gamma(m,n) := \sum_{\ell \in \mathbb{Z}} \int_0^1 \frac{e(m\gamma n_\ell z)}{j_{\gamma n_\ell}(z)^k} e(-nz)\, dx,$$

where and $n_\ell = \left(\begin{smallmatrix} 1 & \ell \\ 0 & 1 \end{smallmatrix}\right)$. Then we have $\mathcal{I}_\gamma(m,n) = 0$ if $n \le 0$ and for $n > 0$

$$\mathcal{I}_\gamma(m,n) = \begin{cases} e\left(\frac{nd}{c}\right)\left(\frac{2\pi}{ic}\right)^k \frac{n^{k-1}}{(k-1)!} & m = 0, \\ \frac{2\pi}{i^k c}\left(\frac{n}{m}\right)^{\frac{k-1}{2}} e\left(\frac{ma+nd}{c}\right) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) & m \ge 1. \end{cases}$$

Remark 7.11. When $\gamma = \gamma_{(c,d)}$ is some element in $\mathrm{SL}_2(\mathbb{Z})$ with the bottom row given by (c,d), one easily sees that as ℓ runs through $\mathbb{Z}$, the bottom right entry of $\gamma_{(c,d)} n_\ell$, being $d' = d + c\ell$, runs through all integers in the same congruence class as $d \pmod c$. Hence in this case $I_{\gamma_{(c,d)}}(m,n)$ agrees with the above innermost sum in the proof of Theorem 7.6, namely,

$$\sum_{d' \equiv d \pmod c} \int_0^1 \frac{e(m\gamma_{(c,d')} z)}{(cz+d')^k} e(-nz)\, dx.$$

Proof of Proposition 7.7. Note that

$$\gamma z = \frac{az+b}{cz+d} = \frac{a(z+\frac{d}{c}) + b - \frac{ad}{c}}{c(z+\frac{d}{c})} = \frac{a}{c} + \frac{bc-ad}{c^2(z+\frac{d}{c})} = \frac{a}{c} - \frac{1}{c^2(c+\frac{d}{c})}.$$

Hence

$$\gamma n_\ell z = \gamma(z+\ell) = \frac{a}{c} - \frac{1}{c^2(z+\ell+\frac{d}{c})}.$$

Hence we have

$$\begin{aligned} \mathcal{I}_\gamma(m,n) &= \sum_{\ell \in \mathbb{Z}} \int_0^1 \frac{e\left(m\left(\frac{a}{c} - \frac{1}{c^2(z+\ell+\frac{d}{c})}\right)\right)}{c^k((z+\ell+\frac{d}{c})^k} e(-nz)\, dx \\ &\overset{z+\ell \mapsto z}{=} \int_{\mathbb{R}} \frac{e\left(m\left(\frac{a}{c} - \frac{1}{c^2(z+\frac{d}{c})}\right)\right)}{c^k((z+\frac{d}{c})^k} e(-nz)\, dx \\ &\overset{z+\frac{d}{c} \mapsto z}{=} c^{-k} e\left(\frac{ma+nd}{c}\right) \int_{\mathbb{R}} \frac{e\left(-\frac{m}{c^2 z} - nz\right)}{z^k}\, dx \\ &= c^{-k} e\left(\frac{ma+nd}{c}\right) \int_{-\infty+iy}^{\infty+iy} \frac{e\left(-\frac{m}{c^2 z} - nz\right)}{z^k}\, dz. \end{aligned}$$

By Cauchy's theorem the above integral is independent of the choice of $y > 0$. Note that since $m \ge 0$ and $z \in \mathbb{H}$ we have $\left|e\left(-\frac{m}{c^2 z} - nz\right)\right| = e^{2\pi ny}$. In particular, if $n \le 0$ we have

$$\begin{aligned}\left|\int_{-\infty+iy}^{\infty+iy} \frac{e\left(-\frac{m}{c^2 z}-nz\right)}{z^k}\,dz\right| &= \left|\lim_{Y\to\infty}\int_{-\infty+iY}^{\infty+iY} \frac{e\left(-\frac{m}{c^2 z}-nz\right)}{z^k}\,dz\right| \\ &\le \lim_{Y\to\infty}\int_{-\infty+iY}^{\infty+iY}\frac{dx}{(x^2+Y^2)^{k/2}} \\ &= 2\lim_{Y\to\infty} Y^{1-k}\int_0^\infty \frac{dx}{(x^2+1)^{k/2}} = 0\end{aligned}$$

whenever $k>1$. For $n>0$ and $m=0$, we can show again by Cauchy's theorem

$$\begin{aligned}\mathcal{I}_\gamma(0,n) &= c^{-k} e\left(\frac{nd}{c}\right)\int_{-\infty+iy}^{\infty+iy}\frac{e(-nz)}{z^k}\,dz \\ &= e\left(\frac{nd}{c}\right)\left(\frac{2\pi}{ic}\right)^k \frac{n^{k-1}}{(k-1)!}.\end{aligned}$$

For $n>0$ and $m>0$, making a change of $z\mapsto \sqrt{\frac{m}{n}}\frac{z}{c}$ we get

$$\mathcal{I}_\gamma(m,n) = c^{-1}\left(\frac{n}{m}\right)^{\frac{k-1}{2}} e\left(\frac{ma+nd}{c}\right)\int_{-\infty+i\tilde{y}}^{\infty+i\tilde{y}}\frac{e\left(-\frac{\sqrt{mn}}{c}(z+z^{-1})\right)}{z^k}\,dz,$$

where $\tilde{y} = \frac{\sqrt{m}}{\sqrt{n}c}y$. Again by Cauchy's theorem, this above integral is independent of $\tilde{y}>0$ and equals $2\pi i^{-k}J_{k-1}(4\pi\sqrt{mn}/c)$ (Exercise 14). Plugging this into the above equation we get

$$\mathcal{I}_{c,d}(m,n) = \frac{2\pi}{i^k c}\left(\frac{n}{m}\right)^{\frac{k-1}{2}} e\left(\frac{ma+nd}{c}\right) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right),$$

finishing the proof. $\square$

Exercise 14. *Show that for any $\lambda>0$, $y>0$ and $k\in\mathbb{N}$,*

$$\frac{1}{2\pi i}\int_{-\infty+iy}^{\infty+iy}\frac{e^{-\frac{\lambda}{2}i(z+z^{-1})}}{z^k}\,dz = -i^{1-k}J_{k-1}(\lambda).$$

7.4. **Fourier expansion of Poincaré series for congruence subgroups.** Let $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. For simplicity of presentation we assume $-I_2\in\Gamma$ so that

$$\sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \langle \pm\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\rangle =: B,\quad \forall\ \mathfrak{a}\in\mathcal{C}_\Gamma.$$

In this subsection we compute the Fourier expansion of the Poincaré series of Γ. More precisely, let $\mathfrak{a},\mathfrak{b}\in\mathcal{C}_\Gamma$ be two (not necessarily distinct) cusps of Γ. We compute the Fourier expansion of the Poincaré series $P_{\mathfrak{a},m}$ at the cusp $\mathfrak{b}$, that is, the Fourier expansion of $P_{\mathfrak{a},m}[\sigma_{\mathfrak{b}}]_k$ at ∞. We denote by

$$\widehat{p}_{\mathfrak{a},\mathfrak{b}}(m,n) = \int_0^1 P_{\mathfrak{a},m}[\sigma_{\mathfrak{b}}]_k(z)e(-nz)\,dx$$

the n-Fourier coefficient of $P_{\mathfrak{a},m}[\sigma_{\mathfrak{b}}]_k$ so that

$$P_{\mathfrak{a},m}[\sigma_{\mathfrak{b}}]_k(z) = \sum_{n\in\mathbb{Z}}\widehat{p}_{\mathfrak{a},\mathfrak{b}}(m,n)e(nz).$$

In order to compute these Fourier coefficients, we first prove the following preliminary expression for $P_{\mathfrak{a},m}[\sigma_{\mathfrak{b}}]_k$.

Lemma 7.8. *For any $z \in \mathbb{H}$ we have*

$$P_{\mathfrak{a},m}[\sigma_{\mathfrak{b}}]_k(z) = \sum_{\gamma \in B\backslash \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}} j_\gamma(z)^{-k} e(m\gamma z).$$

Proof. By definition we have

$$\begin{aligned}
P_{\mathfrak{a},m}[\sigma_{\mathfrak{b}}]_k(z) &= j_{\sigma_{\mathfrak{b}}}(z)^{-k} \sum_{\gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma} j_{\sigma_{\mathfrak{a}}^{-1}\gamma}(\sigma_{\mathfrak{b}} z)^{-k} e(m\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}} z) \\
&= \sum_{\gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma} j_{\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}}}(z)^{-k} e(m\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}} z) \\
&= \sum_{\gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma} j_{\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}}}(z)^{-k} e(m\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}} z).
\end{aligned}$$

Now note that if $\{\gamma_i\}$ is a set of coset representatives for $\Gamma_{\mathfrak{a}}\backslash\Gamma$, i.e. $\Gamma = \bigsqcup_i \Gamma_{\mathfrak{a}}\gamma_i$, then

$$\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}} = \bigsqcup_i \sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}}^{-1}\sigma_{\mathfrak{a}}\gamma_i\sigma_{\mathfrak{b}} = \bigsqcup_i B\sigma_{\mathfrak{a}}\gamma_i\sigma_{\mathfrak{b}}.$$

This implies that $\{\sigma_{\mathfrak{a}}^{-1}\gamma_i\sigma_{\mathfrak{b}}\}$ is a set of coset representatives for $B\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$. Hence

$$P_{\mathfrak{a},m}[\sigma_{\mathfrak{b}}]_k(z) = \sum_{\gamma \in B\backslash \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}} j_\gamma(z)^{-k} e(m\gamma z),$$

as desired. □

To further proceed the computation, we need the following double coset decomposition of $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$.

Proposition 7.9. *The set $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$ is bi-B-invariant and*

$$\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}} = \delta_{\mathfrak{a}\mathfrak{b}} B \bigsqcup \bigsqcup_{(c,d)\in C(\mathfrak{a},\mathfrak{b})} B \begin{pmatrix} * & * \\ c & d \end{pmatrix} B, \tag{7.12}$$

where $\delta_{\mathfrak{a}\mathfrak{b}}$ is the Kronecker symbol and

$$C(\mathfrak{a},\mathfrak{b}) = \left\{(c,d) \in \mathbb{R}^2 : c > 0,\ 0 \le d < c,\ \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}\right\}.$$

Proof. We first show the right hand side of (7.12) is well-defined, that is the double cosets there are independent of the choice of representatives. For some $(c,d) \in C(\mathfrak{a},\mathfrak{b})$, suppose $\omega = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\omega' = \left(\begin{smallmatrix} a' & b' \\ c & d \end{smallmatrix}\right)$ both lie in $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$, we want to show they represent the same double B-coset, i.e. $B\omega B = B\omega' B$. For this write $\omega = \sigma_{\mathfrak{a}}^{-1}\tilde{\omega}\sigma_{\mathfrak{b}}$ and $\omega' = \sigma_{\mathfrak{a}}^{-1}\tilde{\omega}'\sigma_{\mathfrak{b}}$ with $\tilde{\omega}, \tilde{\omega}' \in \Gamma$. Note that

$$\gamma := \sigma_{\mathfrak{a}}\omega'\omega^{-1}\sigma_{\mathfrak{a}}^{-1} = \sigma_{\mathfrak{a}}\sigma_{\mathfrak{a}}^{-1}\tilde{\omega}'\sigma_{\mathfrak{b}}\sigma_{\mathfrak{b}}^{-1}\tilde{\omega}^{-1}\sigma_{\mathfrak{a}}\sigma_{\mathfrak{a}}^{-1} = \tilde{\omega}'\tilde{\omega}^{-1} \in \Gamma.$$

Moreover, since ω and ω' have the same bottom row, $\omega'\omega^{-1} = \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ fixes ∞. This implies that

$$\gamma\mathfrak{a} = \sigma_{\mathfrak{a}}\omega'\omega^{-1}\sigma_{\mathfrak{a}}^{-1}\mathfrak{a} = \sigma_{\mathfrak{a}}\infty = \mathfrak{a}.$$

In other words $\gamma \in \Gamma_{\mathfrak{a}}$. Hence $\omega'\omega^{-1} = \sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{a}} \in \sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = B$, giving $B\omega' B = B\omega B$ as desired. Next we show the right hand side is a disjoint union. This is clear from the following matrix computation:

$$\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a + cm & * \\ c & d + cn \end{pmatrix}. \tag{7.13}$$

Indeed, if $(c,d),(c',d')\in C(\mathfrak{a},\mathfrak{b})$ satisfy $B\left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right)B = B\left(\begin{smallmatrix} * & * \\ c' & d'\end{smallmatrix}\right)B$, then (7.13) implies that $c=c'$ and $d\equiv d' \pmod c$[9]. But since $0\le d,d'<c$, we must have $d=d'$.

Now we proceed to prove the statements in this proposition. First we show $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$ is bi-B-invariant. Note that $B=\sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}}=\sigma_{\mathfrak{b}}^{-1}\Gamma_{\mathfrak{b}}\sigma_{\mathfrak{b}}$. Thus

$$B\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}B=\sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\Gamma\Gamma_{\mathfrak{b}}\sigma_{\mathfrak{b}}=\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}.$$

Next, we prove the equality (7.12). From the bi-B-invariance of $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$ and the definition of $C(\mathfrak{a},\mathfrak{b})$ it is clear that the right hand side of (7.12) is contained in the left hand side. It thus remains to prove the other containment. Let $\gamma=\left(\begin{smallmatrix} a' & b' \\ c' & d'\end{smallmatrix}\right)\in\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$. We can write $\gamma=\sigma_{\mathfrak{a}}^{-1}\gamma'\sigma_{\mathfrak{b}}$ for some $\gamma'\in\Gamma$. We first show that if $c'=0$ then $\mathfrak{a}=\mathfrak{b}$ and $\gamma\in B$. Suppose $c'=0$, then $\gamma\infty=\infty$. This implies that

$$\gamma'\sigma_{\mathfrak{b}}\infty=\sigma_{\mathfrak{a}}\infty \iff \gamma'\mathfrak{b}=\mathfrak{a}.$$

Hence $\mathfrak{a}$ and $\mathfrak{b}$ are Γ-equivalent, implying that $\mathfrak{a}=\mathfrak{b}$ and thus $\gamma\in\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}=B$. Next, we assume $c'\neq 0$, we want to show that γ lie in one of the double coset $B\left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right)B$ for some $(c,d)\in C(\mathfrak{a},\mathfrak{b})$. Since $-I_2\in B$, up to change γ to $-\gamma$ we may assume $c'>0$. By right multiplication by $\left(\begin{smallmatrix} 1 & n \\ 0 & 1\end{smallmatrix}\right)$ for some $n\in\mathbb{Z}$ we can have $\gamma\left(\begin{smallmatrix} 1 & n \\ 0 & 1\end{smallmatrix}\right)=\left(\begin{smallmatrix} a' & b'+c'n \\ c' & d'+c'n\end{smallmatrix}\right)\in\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$ with $0\le d'+c'n<c'$. Thus by definition $(c,d):=(c',d'+c'n)\in C(\mathfrak{a},\mathfrak{b})$ and $B\left(\begin{smallmatrix} a' & b'+c'n \\ c' & d'+c'n\end{smallmatrix}\right)B$ is one of the double cosets in the right hand side of (7.12). It is then clear that γ lies in this double coset. $\square$

Remark 7.14. If $-I_2\notin\Gamma$. Then by almost identical arguments one can prove the same double coset decomposition as in (7.12) but with B and $C(\mathfrak{a},\mathfrak{b})$ modified to be $\langle\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right)\rangle$ and $\left\{(c,d)\in\mathbb{R}^2 : c\neq 0,\ 0\le d<|c|,\ \left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right)\in\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}\right\}$ respectively. The Fourier expansion formula obtained later should also be modified accordingly.

Before stating the Fourier expansion we need to introduce some more notation. Let $C^1(\mathfrak{a},\mathfrak{b})=\mathrm{pr}_1(C(\mathfrak{a},\mathfrak{b}))$, where $\mathrm{pr}_1:\mathbb{R}^2\to\mathbb{R}$ is the projection to the first coordinate. For any $c\in C^1(\mathfrak{a},\mathfrak{b})$ let

$$C(\mathfrak{a},\mathfrak{b};c)=\{d\in[0,c):(c,d)\in C(\mathfrak{a},\mathfrak{b})\}.$$

Theorem 7.10. *We have*

$$E_{\mathfrak{a}}[\sigma_{\mathfrak{b}}]_k(z)=P_{\mathfrak{a},0}[\sigma_{\mathfrak{b}}]_k(z)=\delta_{\mathfrak{a}\mathfrak{b}}+\sum_{n=1}^{\infty}\widehat{p}_{\mathfrak{a}\mathfrak{b}}(0,n)e(nz)$$

with

$$\widehat{p}_{\mathfrak{a}\mathfrak{b}}(0,n)=\left(\frac{2\pi}{i}\right)^k\frac{n^{k-1}}{(k-1)!}\sum_{c\in C^1(\mathfrak{a},\mathfrak{b})}c^{-k}S_{\mathfrak{a}\mathfrak{b}}(0,n;c).$$

For $m\ge 1$ we have

$$P_{\mathfrak{a},m}[\sigma_{\mathfrak{b}}]_k(z)=\sum_{n=1}^{\infty}\widehat{p}_{\mathfrak{a}\mathfrak{b}}(m,n)e(nz)$$

with

$$\widehat{p}_{\mathfrak{a}\mathfrak{b}}(m,n)=\left(\frac{n}{m}\right)^{\frac{k-1}{2}}\left(\delta_{\mathfrak{a}\mathfrak{b}}\delta_{mn}+2\pi i^{-k}\sum_{c\in C^1(\mathfrak{a},\mathfrak{b})}c^{-1}S_{\mathfrak{a}\mathfrak{b}}(m,n;c)J_{k-1}\left(\tfrac{4\pi\sqrt{mn}}{c}\right)\right).$$

[9]Here d,d',c may not be integers. This notation just means $d'-d$ is an integral multiple of c.

Here for any $m,n\in\mathbb{Z}$ and $c\in C(\mathfrak{a},\mathfrak{b})$,

$$S_{\mathfrak{a}\mathfrak{b}}(m,n;c)=\sum_{d\in C(\mathfrak{a},\mathfrak{b};c)}e\left(\frac{ma+nd}{c}\right).$$

Remark 7.15. Here $a\in\mathbb{R}$ in the above definition is such that there exists some $\omega=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ such that $\omega\in\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$. We have seen from the proof of Proposition 7.9 that if $\omega'=\left(\begin{smallmatrix}a'&b'\\c&d\end{smallmatrix}\right)$ is another element in $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$ with bottom row (c,d). Then $\omega'=\left(\begin{smallmatrix}1&m\\0&1\end{smallmatrix}\right)\omega'$ for some $m\in\mathbb{Z}$. Thus $a\equiv a' \pmod{c}$ and the above definition is well-defined.

Example 7.16. *Consider $\Gamma=\Gamma_0(N)$ and $\mathfrak{a}=\mathfrak{b}=\infty$. Then we can choose $\sigma_{\mathfrak{a}}=I_2$ and $C(\infty,\infty)=\left\{(c,d)\in\mathbb{Z}^2_{\mathrm{pr}}: c>0,\ 0\le d<c,\ N\mid c\right\}$ and for $c=N\ell$ with $\ell\geq 1$, $C(\mathfrak{a},\mathfrak{b};c)=\{0\le d<c:\gcd(c,d)=1\}\cong(\mathbb{Z}/c\mathbb{Z})^{\times}$. Hence $S_{\infty\infty}(m,n;c)=S(m,n;c)$ and*

$$\widehat{p}_{\infty\infty}(0,n)=\left(\frac{2\pi}{i}\right)^k\frac{n^{k-1}}{(k-1)!}\sum_{\ell=1}^{\infty}(N\ell)^{-k}S(0,n;N\ell).$$

One can obtain similar formula for $\widehat{p}_{\infty\infty}(m,n)$ for $m\geq 1$.

Combining this Fourier expansion and Lemma 7.5 we have the following corollary.

Corollary 7.11. *For any $k\geq 3$ and any $\mathfrak{a}\in\mathcal{C}_{\Gamma}$, $E_{\mathfrak{a}}\in\mathcal{M}_k(\Gamma)\setminus\mathcal{S}_k(\Gamma)$, while $P_{\mathfrak{a},m}\in\mathcal{S}_k(\Gamma)$ for $m\geq 1$.*

Proof of Theorem 7.10. By the double coset decomposition Proposition 7.9 we have

$$B\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}=\delta_{\mathfrak{a}\mathfrak{b}}B\backslash B\bigsqcup\bigsqcup_{(c,d)\in C(\mathfrak{a},\mathfrak{b})}B\backslash B\gamma_{(c,d)}B,$$

where $\gamma_{(c,d)}$ is some element in $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$ with bottom row given by (c,d). We can choose a set of coset representatives for $B\backslash B\gamma_{(c,d)}B$ to be $\{\gamma_{(c,d)}n_\ell\}_{\ell\in\mathbb{Z}}$ with $n_\ell=\left(\begin{smallmatrix}1&\ell\\0&1\end{smallmatrix}\right)$ as in Proposition 7.7. Then by Lemma 7.8 we have for any $n\in\mathbb{Z}$

$$\begin{aligned}\widehat{p}_{\mathfrak{a}\mathfrak{b}}(m,n)&=\int_0^1\sum_{\gamma\in B\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}}j_\gamma(z)^{-k}e(m\gamma z)e(-nz)\,dx\\&=\delta_{\mathfrak{a}\mathfrak{b}}\delta_{mn}+\sum_{(c,d)\in C(\mathfrak{a},\mathfrak{b})}\sum_{\ell\in\mathbb{Z}}\int_0^1\frac{e(m\gamma_{(c,d)}n_\ell z)}{j_{\gamma_{(c,d)}n_\ell}(z)^k}e(-nz)\,dx\\&=\delta_{\mathfrak{a}\mathfrak{b}}\delta_{mn}+\sum_{c\in C^1(\mathfrak{a},\mathfrak{b})}\sum_{d\in C(\mathfrak{a},\mathfrak{b};c)}\mathcal{I}_{\gamma_{(c,d)}}(m,n).\end{aligned}$$

The desired formulas for $\widehat{p}_{\mathfrak{a}\mathfrak{b}}(m,n)$ cna then be obtained by applying the formulas of $\mathcal{I}_{\gamma_{(c,d)}}(m,n)$ obtained in Proposition 7.7. □

8. Petersson inner product on the space of cusp forms

8.1. **Ptersson inner product on $\mathcal{S}_k$.** Let $\Gamma=\mathrm{SL}_2(\mathbb{Z})$ and $k\geq 4$ be even. In this subsection we define an inner product on $\mathcal{S}_k=\mathcal{S}_k(\Gamma)$ to make it a Hilbert space. Recall that the hyperbolic measure $d\mu(z)=\frac{dxdy}{y^2}$ is $\mathrm{SL}_2(\mathbb{R})$-invariant.

Lemma 8.1. *For any $f,g\in\mathcal{M}_k$, the function $F(z):=\mathfrak{Im}(z)^kf(z)\overline{g(z)}$ is Γ-invariant.*

Proof. Given $\gamma \in \Gamma$,

$$\begin{aligned} F(\gamma z) &= \mathfrak{Im}(\gamma z)^k f(\gamma z)\overline{g(\gamma z)} \\ &= |j_\gamma(z)|^{-2k}\mathfrak{Im}(z)^k j_\gamma(z)^k f(z)\overline{j_\gamma(z)}^k\overline{g(z)} \\ &= \mathfrak{Im}(z)^k\overline{g(z)} = F(z), \end{aligned}$$

as desired. □

Definition 8.1. The *Petersson inner product* on $\mathcal{S}_k$ is defined by

$$\langle f, g\rangle = \int_{\mathcal{F}} y^k f(z)\overline{g(z)}\ d\mu(z), \quad \forall\ f, g \in \mathcal{S}_k,$$

where $\mathcal{F} \subset \mathbb{H}$ is a fundamental domain of $\Gamma\backslash\mathbb{H}$.

Remark 8.2.

(1) The Γ-invariance of $\mathfrak{Im}(z)^k f(z)\overline{g(z)}$ and μ imply that the above definition is independent of the choice of fundamental domains. Hence we may replace $\mathcal{F}$ by the notation $\Gamma\backslash\mathbb{H}$.

(2) The assumption $f, g \in \mathcal{S}_k$ is to ensure integrability. Indeed,

$$f(z) = e(z)\sum_{n=1}^{\infty}\widehat{f}(n)e((n-1)z)$$

satisfies $|f(z)| = e^{-2\pi y}O_f(1)$. Similarly, $|g(z)| = e^{-2\pi y}O_g(1)$. Thus if we pick the standard fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ as given in (3.3), then for any $T > 2$,

$$\langle f, g\rangle \ll_{f,g,T} \int_T^\infty\int_0^1 y^{k-2}e^{-4\pi y}dxdy + 1 < \infty.$$

In fact the same analysis shows that we only need to require one of f, g to be in $\mathcal{S}_k$ to ensure integrability.

(3) $(\mathcal{S}_k, \langle,\rangle)$ becomes a (finite dimensional) Hilbert space. This is important for later discussions on Hecke theory (in order to apply certain linear algebra results).

The following computation shows that integrate against $P_{m,k}$ picks up the m-th Fourier coefficient of f.

Proposition 8.2. *Let $f \in \mathcal{S}_k$ with a Fourier expansion $f(z) = \sum_{n=1}^\infty \widehat{f}(n)e(nz)$. Then we have for any $m \in \mathbb{N}$,*

$$\langle f, P_{m,k}\rangle = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}}\widehat{f}(m).$$

Proof. By definition

$$\langle f, P_{m,k}\rangle = \int_{\mathcal{F}} y^k f(z)\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\overline{j_\gamma(z)}^{-k}\overline{e(m\gamma z)}\ d\mu(z),$$

where $\mathcal{F}$ is a fundamental domain for $\Gamma\backslash\mathbb{H}$. Now making a change of variable $\gamma z \mapsto z$ and using the Γ-invariance of μ we get

$$\begin{aligned}\langle f, P_{m,k}\rangle &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\int_{\gamma\mathcal{F}} \mathfrak{Im}(\gamma^{-1}z)^k f(\gamma^{-1}z)\overline{j_\gamma(\gamma^{-1}z)}^{-k}\overline{e(mz)}\ d\mu(z)\\ &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\int_{\gamma\mathcal{F}} |j_{\gamma^{-1}}(z)|^{-2k} y^k j_{\gamma^{-1}}(z)^k f(z)\overline{j_{\gamma^{-1}}(z)}^k\overline{e(mz)}\ d\mu(z)\\ &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\int_{\gamma\mathcal{F}} y^k f(z)\overline{e(mz)}\ d\mu(z),\end{aligned}$$

where for the second equality we used that $1 = j_{I_2}(z) = j_{\gamma\gamma^{-1}}(z) = j_\gamma(\gamma^{-1}z)j_{\gamma^{-1}}(z)$. Now recall that the union $\bigcup_{\gamma\in\Gamma_\infty\backslash\Gamma}\gamma\mathcal{F}$ is disjoint and forms a fundamental domain for Γ_∞ which can be chosen to be the region $\mathcal{F}_\infty := \{z\in\mathbb{H} : 0\le x<1\}$. Hence (and recall that $d\mu(z) = \frac{dxdy}{y^2}$)

$$\langle f, P_{m,k}\rangle = \int_0^\infty\int_0^1 f(z)\overline{e(mz)}y^{k-2}\ dxdy. \tag{8.3}$$

Note that

$$\int_0^1 f(z)\overline{e(mz)}\ dx = \sum_{n=1}^\infty \widehat{f}(n)e^{-2\pi(n+m)y}\int_0^1 e((n-m)x)\ dx = \widehat{f}(m)e^{-4\pi my}.$$

Thus

$$\langle f, P_{m,k}\rangle = \widehat{f}(m)\int_0^\infty e^{-4\pi my}y^{k-2}\ dy \overset{4\pi my\mapsto y}{=} \frac{\Gamma(k-1)}{(4\pi m)^{k-1}}\widehat{f}(m)$$

as desired. □

Remark 8.4. The process of changing the integrating region from $\mathcal{F}$ (a fundamental domain of Γ) to $\mathcal{F}_\infty$ (a fundamental domain of the subgroup Γ_∞) is called the *unfolding argument* or *unfolding trick*. This is a very useful argument when computing certain integrals involving modular forms constructed via the averaging technique. We note that while the integral $\langle f, P_{m,k}\rangle$ is absolutely convergent for $f\in\mathcal{M}_k$ and $m\ge 1$, **we do need** the extra assumption that $f\in\mathcal{S}_k$ to ensure absolute convergence in the integral in (8.3) which validates the unfolding argument.

We now discuss some consequences of this inner product formula.

Corollary 8.3. *The set* $\{P_{m,k}\}_{m\ge1}$ *spans* $\mathcal{S}_k$.

Proof. If $f\in\mathcal{S}_k$ is orthogonal to the subspace spanned by the above set. Then by Proposition 8.2 we have $\widehat{f}(m) = 0$ for all $m\ge 1$, implying that $f = 0$. □

Remark 8.5. Indeed let $d_k = \dim_{\mathbb{C}}\mathcal{S}_k$. Then one can show that $\{P_{m,k}\}_{1\le m\le d_k}$ spans $\mathcal{S}_k$.

Next, we discuss some consequences of Proposition 8.2 on the Ramanujan τ-function. Recall that

$$\Delta(z) = (2\pi)^{12}\sum_{n=1}^\infty \tau(n)e(nz)$$

and for each $m \geq 1$ there exists $c_m \in \mathbb{C}$ such that $P_{m,12} = c_m \Delta$. Using this relation and Proposition 8.2 we can compute the inner product $\langle \Delta, P_{m,k} \rangle$ in two different ways to get

$$\overline{c}_m \|\Delta\|^2 = \langle \Delta, P_{m,12} \rangle = \frac{\Gamma(11)}{(4\pi m)^{11}} (2\pi)^{12} \tau(m).$$

Note that $\tau(m) \in \mathbb{R}$ since $\Delta = g_2^3 - 27g_3^2$ and both g_2 and g_3 has real Fourier coefficients. Thus

$$c_m = \overline{c}_m = 2\pi\Gamma(11)(2m)^{-11}\tau(m)\|\Delta\|^{-2}. \tag{8.6}$$

Further computing $\langle P_{m,12}, P_{n,12} \rangle$ we can get

$$c_m c_n \|\Delta\|^2 = \langle P_{m,12}, P_{n,12} \rangle = \frac{\Gamma(11)}{(4\pi n)^{11}} \widehat{p}_{12}(m,n).$$

Applying the formulas (7.10) and (8.6) for $\widehat{p}_{12}(m,n)$ and c_m respectively we get

$$\tau(m)\tau(n) = \nu (mn)^{\frac{11}{2}} \left(\delta_{mn} + 2\pi \sum_{c=1}^{\infty} c^{-1} S(m,n;c) J_{11}\left(\tfrac{4\pi\sqrt{mn}}{c}\right) \right), \tag{8.7}$$

where $\nu = \frac{\|\Delta\|^2}{4\pi^{13}\Gamma(11)}$ is some fixed absolute constant. Taking $m = 1$ in (8.7) and recall that $\tau(1) = 1$ we get the following explicit formula for $\tau(n)$:

$$\tau(n) = 2\pi\nu n^{\frac{11}{2}} \sum_{c=1}^{\infty} c^{-1} S(1,n;c) J_{11}\left(\tfrac{4\pi\sqrt{n}}{c}\right), \quad \forall\, n \geq 2. \tag{8.8}$$

Proposition 8.4. *The Ramanujan τ-function satisfies the following recursive relation*

$$\tau(m)\tau(n) = \sum_{d|(m,n)} d^{11}\tau(mnd^{-2}) \tag{8.9}$$

and the and growth condition

$$|\tau(n)| \ll_\epsilon n^{\frac{23}{4}+\epsilon}.$$

Remark 8.10. Recall that the Ramanujan's conjecture asserts that $|\tau(n)| \ll_\epsilon n^{\frac{11}{2}+\epsilon}$. Here we get slightly worse exponent (noting that $\frac{23}{4} = \frac{11}{2} + \frac{1}{4}$).

Proof of Proposition 8.4. In order to prove the recursive relation we apply the Selberg's identity[10] on the classical Kloosterman sum which states that

$$S(m,n;c) = \sum_{d|(m,n,c)} dS(mnd^{-2}, 1; cd^{-1}).$$

We also have the following simple identity involving the δ-symbol:

$$\delta_{mn} = \sum_{d|(m,n)} \delta_{1,mnd^{-2}}.$$

[10]This identity was stated by Selberg in 1938 without proof. The first rigorous proof was given by Kuznetsov [Kuz81] using trace formula. See also [Mat90] for an elementary proof of this identity.

Applying the above two formulas to (8.7) and changing summation order we get

$$\begin{aligned}\tau(m)\tau(n) &= \nu(mn)^{\frac{11}{2}}\left(\sum_{d|(m,n)}\delta_{1,mnd^{-2}} + 2\pi\sum_{c=1}^{\infty}c^{-1}\sum_{d|(m,n,c)}dS(mnd^{-2},1;cd^{-1})J_{11}\left(\tfrac{4\pi\sqrt{mn}}{c}\right)\right)\\ &= \sum_{d|(m,n)}d^{11}\nu(mnd^{-2})^{\frac{11}{2}}\left(\delta_{1,mnd^{-2}} + 2\pi\sum_{c=1}^{\infty}c^{-1}S(mnd^{-2},1;cd^{-1})J_{11}\left(\tfrac{4\pi\sqrt{mnd^{-2}}}{c}\right)\right)\\ &= \sum_{d|(m,n)}d^{11}\tau(mnd^{-2}).\end{aligned}$$

For the growth condition, we apply Weil's bound on Kloosterman sum [Wei48] that

$$|S(m,n;c)| \ll_\epsilon (m,n,c)^{\frac{1}{2}}c^{\frac{1}{2}+\epsilon} \tag{8.11}$$

and bounds on J-Bessel function that

$$|J_\nu(x)| \ll \min\{x^\nu, x^{-\frac{1}{2}}\}, \quad \forall\ x>0 \tag{8.12}$$

to get

$$\begin{aligned}|\tau(n)| &\ll_\epsilon n^{\frac{11}{2}}\sum_{c=1}^{\infty}c^{-\frac{1}{2}+\epsilon}\min\left\{\left(\tfrac{\sqrt{n}}{c}\right)^{11}, \left(\tfrac{\sqrt{n}}{c}\right)^{-\frac{1}{2}}\right\}\\ &\le n^{\frac{11}{2}}\left(\sum_{c=1}^{\lfloor\sqrt{n}\rfloor}c^{-\frac{1}{2}+\epsilon}\left(\tfrac{\sqrt{n}}{c}\right)^{-\frac{1}{2}} + \sum_{\lfloor\sqrt{n}\rfloor+1}^{\infty}c^{-\frac{1}{2}+\epsilon}\left(\tfrac{\sqrt{n}}{c}\right)^{11}\right) \ll n^{\frac{23}{4}+\epsilon}.\end{aligned}$$ □

Next, we discuss another application of Proposition 8.2 which is useful to generalize the above Fourier coefficient bounds to general cusp forms.

Proposition 8.5 (Petersson trace formula)**.** *Let $\{f_j\} \subset \mathcal{S}_k$ be an orthonormal basis with respect to the Petersson inner product. Then for any positive integers m, n we have*

$$\sum_j \widehat{f}_j(n)\overline{\widehat{f}_j}(m) = \frac{(4\pi\sqrt{mn})^{k-1}}{\Gamma(k-1)}\left(\delta_{mn} + 2\pi i^{-k}\sum_{c=1}^{\infty}c^{-1}S(m,n;c)J_{k-1}\left(\tfrac{4\pi\sqrt{mn}}{c}\right)\right).$$

Proof. Expand $P_{m,k}$ with respect to $\{f_j\}$ and apply Proposition 8.2 to get

$$\begin{aligned}P_{m,k} &= \sum_j\langle P_{m,k}, f_j\rangle f_j = \sum_j\overline{\langle f_j, P_{m,k}\rangle}f_j\\ &= \sum_j\frac{\Gamma(k-1)}{(4\pi m)^{k-1}}\overline{\widehat{f}_j(m)}f_j.\end{aligned}$$

Hence

$$\langle P_{m,k}, P_{n,k}\rangle = \sum_j\frac{\Gamma(k-1)^2}{(4\pi)^{2k-2}(mn)^{k-1}}\overline{\widehat{f}_j(m)}\widehat{f}_j(n).$$

On the other hand, again applying Proposition 8.2 we get

$$\langle P_{m,k}, P_{n,k}\rangle = \frac{\Gamma(k-1)}{(4\pi n)^{k-1}}\widehat{p}_k(m,n).$$

Equating both equations and applying the formula for $\widehat{p}_k(m,n)$ (see (7.10)) we get the desired identity. □

As a corollary of this trace formula, we get the following bound on Fourier coefficients of a general weight k cusp form which generalizes the above bound on $\tau(n)$.

Corollary 8.6. *For any $f \in \mathcal{S}_k$ with a Fourier expansion $f(z) = \sum_{n=1}^{\infty} \widehat{f}(n)e(nz)$, we have*

$$|\widehat{f}(n)| \ll_{f,\epsilon} n^{\frac{k-1}{2}+\frac{1}{4}+\epsilon}.$$

Proof. Choose an orthonormal basis $\{f_j\}$ which includes $f/\|f\|$. Applying Proposition 8.5 for $\{f_j\}$ and $m = n$ we get

$$\frac{|\widehat{f}(n)|^2}{\|f\|^2} \leq \sum_j |\widehat{f_j}(n)|^2 \ll_f n^{k-1} + n^{k-1} \sum_{c=1}^{\infty} c^{-1}|S(n,n;c)||J_{k-1}\left(\tfrac{4\pi n}{c}\right)|.$$

This bound then follows from the following bound

$$\sum_{c=1}^{\infty} c^{-1}|S(n,n;c)||J_{k-1}\left(\tfrac{4\pi n}{c}\right)| \ll_\epsilon n^{\frac{1}{2}+\epsilon}. \tag{8.13}$$

□

Exercise 15. *Use Weil's bound on Kloosterman sum (8.11) and the bound (8.12) on J-Bessel function to prove (8.13).*

8.2. **Petersson inner product for congruence subgroups.** Let $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup and $k \geq 3$. In this subsection we define the Petersson inner product on $\mathcal{S}_k(\Gamma)$. The definition is similar to the modular group case.

Definition 8.14. The *Petersson inner product* on $\mathcal{S}_k(\Gamma)$ is defined by

$$\langle f, g\rangle_\Gamma := \int_{\Gamma\backslash\mathbb{H}} y^k f(z)\overline{g(z)}\, d\mu(z), \quad \forall\ f, g \in \mathcal{S}_k(\Gamma).$$

Remark 8.15. Similar to the modular group case, for $f, g \in \mathcal{S}_k(\Gamma)$, the above integrand is left Γ-invariant and thus the integral is independent of the choice of fundamental domains of Γ. Moreover, this integral is absolutely convergent as long as one of f, g is a cusp form.

Proposition 8.7. *Let $f \in \mathcal{S}_k(\Gamma)$. Then for any $\mathfrak{a} \in \mathcal{C}_\Gamma$ and any $m \geq 1$,*

$$\langle f, P_{\mathfrak{a},m}\rangle_\Gamma = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}} \widehat{f}_{\mathfrak{a}}(m).$$

Proof (Sketch). Let $\mathcal{F}_\Gamma \subset \mathbb{H}$ be a fundamental domain for $\Gamma\backslash\mathbb{H}$. Doing similar computations as in the proof of Proposition 8.2 we have

$$\begin{aligned}
\langle f, P_{\mathfrak{a},m}\rangle_\Gamma &= \int_{\mathcal{F}_\Gamma} y^k f(z) \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \overline{j_{\sigma_{\mathfrak{a}}^{-1}\gamma}(z)}^{-k}\, \overline{e(m\sigma_{\mathfrak{a}}^{-1}\gamma z)}\, d\mu(z)\\
&\overset{\gamma z\mapsto z}{=} \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \int_{\gamma\mathcal{F}_\Gamma} y^k f(z)\overline{j_{\sigma_{\mathfrak{a}}}(\sigma_{\mathfrak{a}}^{-1}z)}^k\, \overline{e(m\sigma_{\mathfrak{a}}^{-1}z)}\, d\mu(z)\\
&\overset{\sigma_{\mathfrak{a}}^{-1}z\mapsto z}{=} \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \int_{\sigma_{\mathfrak{a}}^{-1}\gamma\mathcal{F}_\Gamma} \mathfrak{Im}(\sigma_{\mathfrak{a}}z)^k f(\sigma_{\mathfrak{a}}z)\overline{j_{\sigma_{\mathfrak{a}}}(z)}^k\, \overline{e(mz)}\, d\mu(z)\\
&= \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \int_{\sigma_{\mathfrak{a}}^{-1}\gamma\mathcal{F}_\Gamma} f[\sigma_{\mathfrak{a}}]_k(z)\overline{e(mz)}\, d\mu(z).
\end{aligned}$$

Now note that $\bigcup_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma}\sigma_{\mathfrak{a}}^{-1}\gamma\mathcal{F}_\Gamma$ is disjoint and forms a fundamental domain for $\sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}}\backslash\Gamma$ which as before can be chosen to be $\{z\in\mathbb{H}:0\le x<1\}$. Hence

$$\langle f,P_{\mathfrak{a},m}\rangle_\Gamma=\int_0^\infty\int_0^1 f[\sigma_{\mathfrak{a}}]_k(z)\overline{e(mz)}\,dxy^{k-2}\,dy=\frac{\Gamma(k-1)}{(4\pi m)^{k-1}}\widehat{f}_{\mathfrak{a}}(m),$$

as desired. $\square$

As a consequence of this inner product formula, we also have the following Petersson trace formula on $\mathcal{S}_k(\Gamma)$. We omit the proof which is similar to that of Corollary 8.5.

Corollary 8.8. *Let $\mathcal{B}\subset\mathcal{S}_k$ be an orthonormal basis on $\mathcal{S}_k(\Gamma)$ with respect to the Petersson inner product. For any $\mathfrak{a},\mathfrak{b}\in\mathcal{C}_\Gamma$ and for any $m,n\ge1$ we have*

$$\sum_{f\in\mathcal{B}}\overline{\widehat{f}_{\mathfrak{a}}}(m)\widehat{f}_{\mathfrak{b}}(n)=\frac{(4\pi\sqrt{mn})^{k-1}}{\Gamma(k-1)}\left(\delta_{mn}\delta_{\mathfrak{a}\mathfrak{b}}+2\pi i^{-k}\sum_{c\in C^1(\mathfrak{a},\mathfrak{b})}c^{-1}S_{\mathfrak{a}\mathfrak{b}}(m,n;c)J_{k-1}\left(\tfrac{4\pi\sqrt{mn}}{c}\right)\right).$$

Remark 8.16. When $\Gamma=\Gamma_0(N)$, the corresponding Kloosterman sum $S_{\mathfrak{a}\mathfrak{b}}(m,n;c)$ can be related to the classical Kloosterman sum. For example, when $\mathfrak{a}=\mathfrak{b}=\infty$, the above formula becomes

$$\sum_{f\in\mathcal{B}}\overline{\widehat{f}_\infty}(m)\widehat{f}_\infty(n)=\frac{(4\pi\sqrt{mn})^{k-1}}{\Gamma(k-1)}\left(\delta_{mn}+2\pi i^{-k}\sum_{c=1}^\infty(cN)^{-1}S(m,n;cN)J_{k-1}\left(\tfrac{4\pi\sqrt{mn}}{cN}\right)\right).$$

From this trace formula we can similarly get

$$|\widehat{f}_\infty(n)|\ll_{\epsilon,f,N}n^{\frac{k-1}{2}+\frac14+\epsilon}.$$

More generally, one can get

$$|\widehat{f}_{\mathfrak{a}}(n)|\ll_{\epsilon,f,N}n^{\frac{k-1}{2}+\frac14+\epsilon},\qquad\forall\ \mathfrak{a}\in\mathcal{C}_{\Gamma_0(N)}.\tag{8.17}$$

8.3. **General bounds on Fourier coefficients of cusp forms.** In this subsection we give very soft arguments bounding Fourier coefficients of cusp forms for a general congruence subgroup. The estimate we get is not as good as (8.17), but it holds in a much greater generality.

Let $\Gamma<\mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. The argument is based on the following simple observation.

Lemma 8.9. *For any $f\in\mathcal{S}_k(\Gamma)$, the function $F(z)=y^{\frac{k}{2}}|f(z)|$ is left Γ-invariant and bounded on $\mathbb{H}$.*

Proof. The assertion that F is left-Γ-invariant is easy: Take any $\gamma\in\Gamma$ we have

$$F(\gamma z)=\mathfrak{Im}(\gamma z)^{\frac{k}{2}}|f(\gamma z)|=\mathfrak{Im}(z)^{\frac{k}{2}}|j_\gamma(z)|^{-k}|j_\gamma(z)^kf(z)|=y^{\frac{k}{2}}|f(z)|=F(z).$$

Hence F is determined by its values on a fixed fundamental domain of $\Gamma\backslash\mathbb{H}$. Let $\mathcal{F}_\Gamma\subset\mathbb{H}$ be a fundamental domain of $\Gamma\backslash\mathbb{H}$ with cusps $\mathcal{C}_\Gamma\subset\mathbb{Q}\cup\{\infty\}$. We need to show F is bounded on $\mathcal{F}_\Gamma$, which suffices to show F is bounded around every cusp in $\mathcal{C}_\Gamma$. For any $\mathfrak{a}\in\mathcal{C}_\Gamma$, since $\sigma_{\mathfrak{a}}\infty=\mathfrak{a}$, to show F is bounded around $\mathfrak{a}$, it suffices to show the function $g(z)=F(\sigma_{\mathfrak{a}}z)$ is bounded around ∞, that is, it is bounded as $\mathfrak{Im}(z)\to\infty$ (with $\mathfrak{Re}(z)$ uniformly bounded). We have

$$g(z)=F(\sigma_{\mathfrak{a}}z)=\mathfrak{Im}(\sigma_{\mathfrak{a}}z)^{\frac{k}{2}}|f(\sigma_{\mathfrak{a}}z)|=y^{\frac{k}{2}}|f[\sigma_{\mathfrak{a}}]_k(z)|.$$

Since $f \in \mathcal{S}_k(\Gamma)$, we have

$$|f[\sigma_{\mathfrak{a}}]_k(z)| = |e(z) \sum_{n=1}^{\infty} \widehat{f}_{\mathfrak{a}}(n) e((n-1)z)| \ll_f e^{-2\pi y}.$$

Hence

$$|g(z)| \ll_f y^{\frac{k}{2}} e^{-2\pi y} \to 0 \quad \text{as } y \to \infty.$$

This proves the lemma. □

As a corollary, we have the following bounds on Fourier coefficients which is due to Hecke.

Proposition 8.10 (Hecke). *Let $f \in \mathcal{S}_k(\Gamma)$. For any $N \geq 1$ and $\mathfrak{a} \in \mathcal{C}_\Gamma$, we have*

$$\sum_{n=1}^{N} |\widehat{f}_{\mathfrak{a}}(n)|^2 \ll_f N^k.$$

Proof. Consider the function $g(z) = f[\sigma_{\mathfrak{a}}]_k \in \mathcal{S}_k(\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}})$. By the same arguments as above, we have the function $y^{\frac{k}{2}}|g(z)|$ is bounded. In other words,

$$|g(z)| \ll_f y^{-\frac{k}{2}}.$$

Hence

$$\int_0^1 |g(z)|^2 \, dx \ll y^{-k}.$$

On the other hand, by the Fourier expansion

$$g(z) = \sum_{n=1}^{\infty} \widehat{f}_{\mathfrak{a}}(n) e(nz)$$

we have

$$\int_0^1 |g(z)|^2 \, dx = \sum_{n=1}^{\infty} |\widehat{f}_{\mathfrak{a}}(n)|^2 e^{-4\pi n y}.$$

Thus for any $N \geq 1$,

$$e^{-4\pi N y} \sum_{n=1}^{N} |\widehat{f}_{\mathfrak{a}}(n)|^2 \leq \sum_{n=1}^{N} |\widehat{f}_{\mathfrak{a}}(n)|^2 e^{-4\pi n y} \leq \int_0^1 |g(z)|^2 \, dx \ll_f y^{-k}.$$

Taking $y = N^{-1}$ gives the desired inequality. □

We have the following two immediate corollaries.

Corollary 8.11. *Keep the notation and assumptions as in Proposition 8.10. We have*

$$|\widehat{f}_{\mathfrak{a}}(n)| \ll_f n^{\frac{k}{2}}, \quad \forall\, n \in \mathbb{N}, \tag{8.18}$$

and

$$\sum_{n \leq N} |\widehat{f}_{\mathfrak{a}}(n)| \ll_f N^{\frac{k+1}{2}}, \quad \forall\, N \in \mathbb{N}. \tag{8.19}$$

The next result shows that if we remove the absolute value sign in the left hand side of (8.19), we can get an extra square root cancelation.

Proposition 8.12. *We have for any $N \geq 2$,*

$$|\sum_{n=1}^{N} \widehat{f}_{\mathfrak{a}}(n)| \ll_f N^{\frac{k}{2}} \log N.$$

Proof. Let $g(z) = f[\sigma_{\mathfrak{a}}]_k(z)$. Using the relation

$$\widehat{f}_{\mathfrak{a}}(n) = \int_0^1 g(z) e(-nz)\ dx$$

we have

$$\sum_{n=1}^{N} \widehat{f}_{\mathfrak{a}}(n) = \int_0^1 g(z) \sum_{n=1}^{N} e(-nz)\ dx = \int_0^1 g(z) \frac{e(-Nz)-1}{1-e(z)}\ dx.$$

Note that

$$\left| \frac{e(-Nz)-1}{1-e(z)} \right| \ll e^{2\pi N y} |1-e(z)|^{-1}.$$

This, together with the bound $|g(z)| \ll_f y^{-\frac{k}{2}}$ implies that

$$|\sum_{n=1}^{N} \widehat{f}_{\mathfrak{a}}(n)| \ll y^{-\frac{k}{2}} e^{2\pi N y} \int_0^1 |1-e(z)|^{-1}\ dx \ll y^{-\frac{k}{2}} e^{2\pi N y} \log(2+y^{-1}),$$

where for the last estimate we use Exercise 16 below. The desired bound then follows by taking $y = N^{-1}$. □

Exercise 16. *Show that for any $z = x + iy \in \mathbb{H}$,*

$$\int_0^1 |1-e(z)|^{-1}\ dx \ll \log(2+y^{-1}).$$

9. Double coset operator

Let Γ_1, Γ_2 be two congruence subgroups. Let

$$\mathrm{GL}_2^+(\mathbb{Q}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{M}_2(\mathbb{Q}) : ad - bc > 0 \right\}$$

be the group of 2 by 2 matrices with rational entries and positive determinants. Each $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$ defines a double coset

$$\Gamma_1 \alpha \Gamma_2 := \{\gamma_1 \alpha \gamma_2 : \gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2\}.$$

Clearly, $\Gamma_1 \alpha \Gamma_2$ is left Γ_1- and right Γ_2-invariant. In particular, there is a right Γ_1-coset decomposition

$$\Gamma_1 \alpha \Gamma_2 = \bigsqcup_j \Gamma_1 \beta_j.$$

The starting point of our discussion is that this coset decomposition is finite which will be proved by the following two lemmas.

Lemma 9.1. *Let $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup and let $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. Then $\alpha^{-1}\Gamma\alpha \cap \mathrm{SL}_2(\mathbb{Z})$ is still a congruence subgroup.*

Proof. By definition it suffices to show $\alpha^{-1}\Gamma\alpha$ contains some principle congruence subgroup. Since Γ is congruence and $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, there exists some $\tilde{N} \in \mathbb{N}$ such that $\Gamma(\tilde{N}) \subset \Gamma$ and $\tilde{N}\alpha, \tilde{N}\alpha^{-1} \in M_2(\mathbb{Z})$. Let $N = \tilde{N}^3$. We claim that $\Gamma(N) \subset \alpha^{-1}\Gamma\alpha$, or equivalently, $\alpha\Gamma(N)\alpha^{-1} \subset \Gamma$. To see this, note that

$$\alpha\Gamma(N)\alpha^{-1} \subset \alpha(I_2 + NM_2(\mathbb{Z}))\alpha^{-1} \subset I_2 + \tilde{N}\alpha\tilde{N}M_2(\mathbb{Z})\tilde{N}\alpha^{-1} \subset I_2 + \tilde{N}M_2(\mathbb{Z}).$$

Moreover, since $\alpha\Gamma(N)\alpha^{-1} \subset \mathrm{SL}_2(\mathbb{R})$, we have

$$\alpha\Gamma(N)\alpha^{-1} \subset (I_2 + \tilde{N}M_2(\mathbb{Z})) \cap \mathrm{SL}_2(\mathbb{R}) = \Gamma(\tilde{N}) \subset \Gamma,$$

finishing the proof. □

Remark 9.1. Let Γ_1, Γ_2 be any two congruence subgroups and $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, we note that $\alpha^{-1}\Gamma_1\alpha \cap \Gamma_2$ is also a congruence subgroup. This is true since by the above lemma there is some $N_1 \in \mathbb{N}$ such that $\Gamma(N_1) \subset \alpha^{-1}\Gamma_1\alpha$. Moreover, by definition $\Gamma(N_2) \subset \Gamma_2$ for some $N_2 \in \mathbb{N}$. Then $\Gamma(N_1N_2) \subset \alpha^{-1}\Gamma_1\alpha \cap \Gamma_2$, proving this claim.

Lemma 9.2. *Let $\Gamma_1, \Gamma_2 < \mathrm{SL}_2(\mathbb{Z})$ be two congruence subgroups and let $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. Set $\Gamma_3 = \alpha^{-1}\Gamma_1\alpha \cap \Gamma_2$. Then there is a bijection between the two quotient sets $\Gamma_3\backslash\Gamma_2$ and $\Gamma_1\backslash\Gamma_1\alpha\Gamma_2$. In particular,*

$$\#(\Gamma_1\backslash\Gamma_1\alpha\Gamma_2) = \#\Gamma_3\backslash\Gamma_1 < \infty.$$

Proof. Consider the map sending $\Gamma_2 \to \Gamma_1\backslash\Gamma_1\alpha\Gamma_2$ given by $\gamma \mapsto \Gamma_1\alpha\gamma_2$. It is clearly surjective. We just need to figure out when two elements in Γ_2 give the same Γ_1-coset. Let $\gamma_1, \gamma_2' \in \Gamma_2$. Suppose $\Gamma_1\alpha\gamma_2 = \Gamma_1\alpha\gamma_2'$. This means that there exists some $\gamma_1 \in \Gamma_1$ such that $\gamma_1\alpha\gamma_2 = \alpha\gamma_2'$, or equivalently, $\alpha^{-1}\gamma_1\alpha = \gamma_2'\gamma_2^{-1}$. This implies that $\gamma_2'\gamma_2^{-1} \in \alpha^{-1}\Gamma_1\alpha \cap \Gamma_2 = \Gamma_3$, i.e. $\Gamma_3\gamma_2 = \Gamma_3\gamma_2'$. We thus have shown two elements in Γ_2 give the same Γ_1-coset if and only if they represent the same Γ_3-coset. This finishes the proof. □

Definition 9.2. The $[\Gamma_1\alpha\Gamma_2]_k$*-operator* is an operator on $\mathcal{M}_k(\Gamma_1)$ defined by

$$f[\Gamma_1\alpha\Gamma_2]_k := \sum_{\beta_j \in \Gamma_1\backslash\Gamma_1\alpha\Gamma_2} f[\beta_j]_k,$$

where for any $\beta \in \mathrm{GL}_2^+(\mathbb{R})$ and $f : \mathbb{H} \to \mathbb{C}$,

$$f[\beta]_k(z) = (\det(\beta))^{\frac{k}{2}} j_\beta(z)^{-k} f(\beta z).$$

Remark 9.3. The weight k-operator $[\beta]_k$ coincides with the previous weight-k operator when $\beta \in \mathrm{SL}_2(\mathbb{R})$. Similar as before, one can check that $f[\beta_1\beta_2]_k = f[\beta_1]_k[\beta_2]_k$ for any $\beta_1, \beta_2 \in \mathrm{GL}_2^+(\mathbb{R})$. The factor $(\det(\beta))^{\frac{k}{2}}$ is such that $[\beta]_k$ is invariant under scaling, i.e.

$$f[\lambda\beta]_k = f[\beta]_k, \quad \forall\ \lambda > 0.$$

Proposition 9.3. *The double coset operator is well-defined and sends $\mathcal{M}_k(\Gamma_1)$ and $\mathcal{S}_k(\Gamma_1)$ to $\mathcal{M}_k(\Gamma_2)$ and $\mathcal{S}_k(\Gamma_2)$ respectively.*

Proof. We first show that it is well-defined. Let $\{\beta_j'\}$ be another set of representatives for the quotient $\Gamma_1\backslash\Gamma_1\alpha\Gamma_2$. Up to reordering, we may assume $\Gamma_1\beta_j = \Gamma_1\beta_j'$. Thus there exists some $\gamma_j \in \Gamma_1$ such that $\beta_j = \gamma_j\beta_j'$. Then we have

$$\sum_j f[\beta_j]_k = \sum_j f[\gamma_j\beta_j']_k = \sum_j f[\beta_j']_k,$$

proving the definition of $[\Gamma_1\alpha\Gamma_2]_k$ is independent of the choice of coset representatives. Next, we show $[\Gamma_1\alpha\Gamma_2]_k$ maps $\mathcal{M}_k(\Gamma_1)$ to $\mathcal{M}_k(\Gamma_2)$. Take $f \in \mathcal{M}_k(\Gamma_1)$. In particular, f is holomorphic on $\mathbb{H}$. It is then clear that for any $\beta \in \mathrm{GL}_2^+(\mathbb{Q})$, $f[\beta]_k(z)$ is also holomorphic on $\mathbb{H}$. Hence each summand in the definition of $f[\Gamma_1\alpha\Gamma_2]_k$ is holomorphic, implying that $f[\Gamma_1\alpha\Gamma_2]_k$ is holomorphic. Next we show $f[\Gamma_1\alpha\Gamma_2]_k$ is weakly modular of weight k with respect to Γ_2. For this, take any $\gamma_2 \in \Gamma_2$, we have

$$f[\Gamma_1\alpha\Gamma_2]_k[\gamma_2]_k(z) = \sum_{\beta_j \in \Gamma_1\backslash\Gamma_1\alpha\Gamma_2} f[\beta_j\gamma_2]_k = f[\Gamma_1\alpha\Gamma_2]_k(z)$$

as desired, where for the last equality we used the fact that if $\{\beta_j\}$ is a set of coset representatives for $\Gamma_1\backslash\Gamma_1\alpha\Gamma_2$, then so is $\{\beta_j\gamma_2\}$ for any $\gamma_2 \in \Gamma_2$. This is true since

$$\Gamma_1\alpha\Gamma_2 = \Gamma_1\alpha\Gamma_2\gamma_2 = \bigsqcup_j \Gamma_1\beta_j\gamma_2.$$

Finally, we need to show $f[\Gamma_1\alpha\Gamma_2]_k$ is holomorphic at every cusp of Γ_2. For this, we note that since Γ_2 is a congruence subgroup, it suffices to show $f[\Gamma_1\alpha\Gamma_2]_k$ is holomorphic at every $x \in \mathbb{Q}\cup\{\infty\}$, or equivalently, $f[\Gamma_1\alpha\Gamma_2]_k[\tau]_k$ is holomorphic at ∞ for every $\tau \in \mathrm{SL}_2(\mathbb{Z})$. Since $f \in \mathcal{M}_k(\Gamma_1)$ and Γ_1 is also a congruence subgroup, we have $f[\tau]_k$ is holomorphic at every $x \in \mathbb{Q}\cup\{\infty\}$. By Exercise 17, $f[\beta]_k$ is holomorphic at every $x \in \mathbb{Q}\cup\{\infty\}$, implying every summand in the definition of $f[\Gamma_1\alpha\Gamma_2]_k$ is holomorphic at every $x \in \mathbb{Q}\cup\{\infty\}$. In particular, $f[\Gamma_1\alpha\Gamma_2]_k$ is holomorphic at every $x \in \mathbb{Q}\cup\{\infty\}$. Hence $f[\Gamma_1\alpha\Gamma_2]_k \in \mathcal{M}_k(\Gamma_2)$. The statement that $[\Gamma_1\alpha\Gamma_2]_k : \mathcal{S}_k(\Gamma_1) \to \mathcal{S}_k(\Gamma_2)$ follows similarly from Exercise 17. $\square$

Exercise 17. *Let $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup and $f \in \mathcal{M}_k(\Gamma)$.*

(1) *Show that $f[\beta]_k$ is holomorphic at ∞ for any $\beta \in \mathrm{GL}_2^+(\mathbb{Q})$.*
(2) *Show that if $f \in \mathcal{S}_k(\Gamma)$, then $f[\beta]_k$ vanishes at ∞ for any $\beta \in \mathrm{GL}_2^+(\mathbb{Q})$.*

10. Hecke operators for the modular group

In this section we define Hecke operators for the modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. For any $n \in \mathbb{N}$, let

$$G_n := \{g \in M_2(\mathbb{Z}) : \det(g) = n\}$$

be the set of 2 by 2 integral matrices with determinant n. Clearly, G_n is bi-Γ-invariant.

Definition 10.1. For any $n \in \mathbb{N}$, the *n-th Hecke operator* T_n is an operator on $\mathcal{M}_k$ defined by

$$T_n f := n^{\frac{k}{2}-1} \sum_{\beta \in \Gamma\backslash G_n} f[\beta]_k, \quad \forall\, f \in \mathcal{M}_k.$$

Remark 10.2. The Hecke operator T_n also implicitly depends on the weight parameter k. Since this parameter is fixed throughout our discussion, we omit it in our notation.

The following proposition decomposes G_n as a double Γ-coset which implies that T_n is indeed a sum of double coset operators.

Proposition 10.1. *For any $n \in \mathbb{N}$,*

$$G_n = \bigsqcup_{d^2 \mid n} \Gamma \begin{pmatrix} d & \\ & \frac{n}{d} \end{pmatrix} \Gamma, \tag{10.3}$$

where the union is over all positive d such that $d^2 \mid n$.

Corollary 10.2. *For any $n \in \mathbb{N}$ we have*

$$T_n f = n^{\frac{k}{2}-1} \sum_{d^2 \mid n} f[\Gamma \begin{pmatrix} d & \\ & \frac{n}{d} \end{pmatrix} \Gamma]_k.$$

In particular, T_n maps $\mathcal{M}_k$ and $\mathcal{S}_k$ to $\mathcal{M}_k$ and $\mathcal{S}_k$ respectively.

Proof of Proposition 10.1. First note that since G_n is bi-Γ-invariant, the right hand side of (10.3) is clearly contained in G_n. Next, we show the right hand side is a disjoint union. Indeed, for any $g \in G_n$, we denote by $\gcd(g)$ the GCD of all four entries of g. Then for any $d^2 \mid n$ and $g \in \Gamma \begin{pmatrix} d & \\ & \frac{n}{d} \end{pmatrix} \Gamma$, $\gcd(g) = \gcd(d, \frac{n}{d}) = d$, implying that each coset in the right hand side of (10.3) can be distinguished by the GCD of its elements. Hence it is a disjoint union. Finally, we show G_n is contained in the right hand side of (10.3). Take $g \in G_n$ and let $d = \gcd(g)$. Then $g' = d^{-1} g \in M_2(\mathbb{Z})$ and thus $\det(g') = d^{-2}\det(g) = d^{-2} n \in \mathbb{Z}$. Thus $d^2 \mid n$ and the double coset $\Gamma \begin{pmatrix} d & \\ & \frac{n}{d} \end{pmatrix} \Gamma$ appears in the right hand side of (10.3). We would to show $g \in \Gamma \begin{pmatrix} d & \\ & \frac{n}{d} \end{pmatrix} \Gamma$, or equivalently, $g' \in \Gamma \begin{pmatrix} 1 & \\ & \frac{n}{d^2} \end{pmatrix} \Gamma$. This follows from the following lemma. ∎

Lemma 10.3. *For any $n \in \mathbb{N}$, we have*

$$G_n^{\mathrm{pr}} := \{g \in M_2(\mathbb{Z}) : \det(g) = n,\ \gcd(g) = 1\} = \Gamma \begin{pmatrix} 1 & \\ & n \end{pmatrix} \Gamma.$$

Proof. The double coset is clearly contained in G_n^{pr}. Take $g \in G_n^{\mathrm{pr}}$, we want to show $g \in \Gamma \left(\begin{smallmatrix} 1 & \\ & n \end{smallmatrix}\right) \Gamma$, or equivalently, there exists some $\gamma_1, \gamma_2 \in \Gamma$ such that $\gamma_1 g \gamma_2 = \left(\begin{smallmatrix} 1 & \\ & n \end{smallmatrix}\right)$. This amounts to perform row and column operations to g to reduce it to $\left(\begin{smallmatrix} 1 & \\ & n \end{smallmatrix}\right)$. By applying the Euclidean algorithm to the first column[11] and switching the first and second rows if necessary we can reduce g to the upper triangular matrix $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$. Next, by adding multiples of the first column and row to the second column and row we can further reduce it to $\left(\begin{smallmatrix} a & b+\ell\gcd(a,d) \\ 0 & d \end{smallmatrix}\right)$. We claim (see Exercise 18 below) that there exists $\ell \in \mathbb{Z}$ such that $\gcd(a, b + \ell \gcd(a,d)) = 1$. Taking such a ℓ we reduce g to $\left(\begin{smallmatrix} a & b' \\ 0 & d \end{smallmatrix}\right)$ with $\gcd(a, b') = 1$. Applying the Euclidean algorithm to the first row and switching the columns if necessary we can reduce it to $\left(\begin{smallmatrix} 1 & 0 \\ c' & d' \end{smallmatrix}\right)$ for some $c', d' \in \mathbb{Z}$. Finally, subtracting the c'-multiple of the first row from the second row we get $\left(\begin{smallmatrix} 1 & 0 \\ 0 & d' \end{smallmatrix}\right)$. Since these operations does not change the determinant, we must have $d' = \det(g) = n$, finishing the claim, and hence also this lemma. ∎

Exercise 18. *Let (a, b) be a pair of co-prime integers. Show that for any positive integer c, there exists $\ell \in \mathbb{Z}$ such that $\gcd(a + \ell b, c) = 1$. Use this to prove the above claim.*

---

[11] This amount to multiplying g from the left by $\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ n & 1 \end{smallmatrix}\right)$ consecutively. (Noting that $\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a+cn & b+dn \\ c & d \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ n & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & b \\ c+an & d+bn \end{smallmatrix}\right)$.)

The following lemma gives an explicit coset representatives for the quotient $\Gamma\backslash G_n$ from which we get a more explicit formula of the Hecke operators.

Proposition 10.4. *The set*

$$\Delta_n = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) : ad = n,\ a, d > 0,\ 0 \le b < d \right\}$$

forms a complete set of right coset representatives of $\Gamma\backslash G_n$*, that is,*

$$G_n = \bigsqcup_{ad=n} \bigsqcup_{0\le b<d} \Gamma \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}. \tag{10.4}$$

In particular, we have

$$T_n f(z) = \frac{1}{n} \sum_{ad=n} a^k \sum_{0\le b<d} f\left(\tfrac{az+b}{d}\right), \quad \forall\, f \in \mathcal{M}_k. \tag{10.5}$$

Remark 10.6. Since $f \in \mathcal{M}_k$ is periodic of period 1, the sum over $0 \le b < d$ can be rewritten as the sum over congruence classes modulo d. We formally write T_n as following:

$$T_n = \frac{1}{n} \sum_{ad=n} a^k \sum_{b \pmod d} \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}. \tag{10.7}$$

Proof of Proposition 10.4. The containment "$\supset$" is clear. We thus only need to prove the other containment and that the right side of (10.5) is a disjoint union. For the first statement, take any $g \in G_n$, as above by applying Euclidean algorithm to the first column of g we may assume $g = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$. Next, by left multiplying $\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ for suitable n we can make that $0 \le b < d$. This proves the containment "$\subset$". Next, we show the union is disjoint. Take $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right), \left(\begin{smallmatrix} a' & b' \\ 0 & d' \end{smallmatrix}\right) \in \Delta_n$, suppose $\Gamma \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) = \Gamma \left(\begin{smallmatrix} a' & b' \\ 0 & d' \end{smallmatrix}\right)$, we would like to show $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a' & b' \\ 0 & d' \end{smallmatrix}\right)$. By assumption there exists $\tau = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma$ such that $\tau \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a' & b' \\ 0 & d' \end{smallmatrix}\right)$. Note that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} * & * \\ \gamma a & * \end{pmatrix}.$$

Hence we have $\gamma = 0$, implying that $\tau = \pm \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$. The case $\tau = -\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ is impossible since $a, a' > 0$. Hence $\tau = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ and the relation $\tau \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a' & b' \\ 0 & d' \end{smallmatrix}\right)$ implies that $a = a'$, $d = d'$ and $b \equiv b' \pmod d$. But the condition $0 \le b, b' < d$ forces $b = b'$. This finishes the proof.

For the in particular part, use the above coset representatives we have

$$\begin{aligned} T_n f(z) &= n^{\frac{k}{2}-1} \sum_{ad=n} \sum_{0\le b<d} n^{\frac{k}{2}} d^{-k} f\left(\tfrac{az+b}{d}\right) \\ &= \frac{1}{n} \sum_{ad=n} a^k \sum_{0\le b<d} f\left(\tfrac{az+b}{d}\right) \end{aligned}$$

as desired. ∎

10.1. **Properties of Hecke operators.** In this subsection we prove various properties of Hecke operators. First we study relations between different Hecke operators.

Proposition 10.5. *We have*

(1) $T_m T_n = T_{mn}$ *whenever* $(m, n) = 1$.

(2) $T_{p^{r+1}} = T_p T_{p^r} - p^{k-1} T_{p^{r-1}}$ *for any prime* p *and* $r \geq 1$.

Remark 10.8. The two relations can be combined into one that

$$T_m T_n = \sum_{d|(m,n)} d^{k-1} T_{mnd^{-2}}.$$

In particular, this implies that Hecke operators commute with each other, i.e.

$$T_m T_n = T_n T_m, \quad \forall\ m, n \geq 1.$$

Proof of Proposition 10.5. We use the formal sum expression (10.7) for T_n. For (1) we have

$$\begin{aligned} mnT_m T_n &= \sum_{\substack{a_1 d_1 = m \\ a_2 d_2 = n}} (a_1 a_2)^k \sum_{\substack{b_1 \pmod{d_1} \\ b_2 \pmod{d_2}}} \begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ 0 & d_2 \end{bmatrix} \\ &= \sum_{\substack{a_1 d_1 = m \\ a_2 d_2 = n}} (a_1 a_2)^k \sum_{\substack{b_1 \pmod{d_1} \\ b_2 \pmod{d_2}}} \begin{bmatrix} a_1 a_2 & a_1 b_2 + b_1 d_2 \\ 0 & d_1 d_2 \end{bmatrix}. \end{aligned}$$

We claim that the map

$$\pi : \mathbb{Z}/d_1\mathbb{Z} \times \mathbb{Z}/d_2\mathbb{Z} \to \mathbb{Z}/d_1 d_2 \mathbb{Z}, \quad (b_1, b_2) \mapsto a_1 b_2 + b_1 d_2 \pmod{d_1 d_2}$$

is bijective. Assuming this claim and making changing of variables $a = a_1 a_2, d = d_1 d_2$ and $b = a_1 b_2 + b_1 d_2$ we have

$$mnT_m T_n = \sum_{ad = mn} a^k \sum_{b \pmod{d}} \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = mnT_{mn}$$

as desired. We now prove the claim. It suffices to show π is injective. Assume $\pi(b_1, b_2) = \pi(b_1', b_2')$ for some $b_1, b_1' \in \mathbb{Z}/d_1\mathbb{Z}$ and $b_2, b_2' \in \mathbb{Z}/d_1\mathbb{Z}$. We would like to show $b_1 \equiv b_1' \pmod{d_1}$ and $b_2 \equiv b_2' \pmod{d_2}$. By assumption, we have $a_1 b_2 + b_1 d_2 \equiv a_1 b_1' + b_1' d_2 \pmod{d_1 d_2}$. Reducing to congruence classes modulo d_2 we have $a_1 b_2 \equiv a_1 b_2' \pmod{d_2}$. But since $a_1 \mid m, d_2 \mid n$ and $(m, n) = 1$, we have $(a_1, d_2) = 1$. Thus the above congruence equation implies that $b_2 \equiv b_2' \pmod{d_2}$. Plugging this relation into the original congruence equation, we get $b_1 d_2 \equiv b_1' d_2 \pmod{d_1 d_2}$ which is equivalent to $b_1 \equiv b_1' \pmod{d_1}$. This proves the claim.

For (2) we note that

$$T_p = \frac{1}{p} \sum_{b \pmod{p}} \begin{bmatrix} 1 & b \\ 0 & p \end{bmatrix} + p^{k-1} \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}.$$

More generally, we have

$$T_{p^r} = \sum_{i=0}^{r} p^{ki-r} \sum_{b \pmod{p^{r-i}}} \begin{bmatrix} p^i & b \\ 0 & p^{r-i} \end{bmatrix}, \quad \forall\ r \geq 1.$$

Thus

$$T_p T_{p^r} = \sum_{i=0}^{r} p^{ki-r-1} \sum_{\substack{b_1 (p) \\ b_2 (p^{r-i})}} \begin{bmatrix} p^i & b_2 + b_1 p^{r-i} \\ 0 & p^{r+1-i} \end{bmatrix} + \sum_{i=0}^{r} p^{k(i+1)-r-1} \sum_{b_2 (p^{r-i})} \begin{bmatrix} p^{i+1} & pb_2 \\ 0 & p^{r-i} \end{bmatrix}.$$

By similar arguments as above, we can replace the sum $\sum_{\substack{b_1(p)\\ b_2(p^{r-i})}}$ by the sum $\sum_{b(p^{r+1-i})}$ after making a change of variable $b = b_2 + b_1 p^{r-i}$. We also split the above second sum into $\sum_{i=0}^{r-1}$ and $i = r$ to get

$$
\begin{aligned}
T_p T_{p^r} &= \sum_{i=0}^{r} p^{ki-r-1} \sum_{b(p^{r+1-i})} \begin{bmatrix} p^i & b \\ 0 & p^{r+1-i} \end{bmatrix} + p^{k(r+1)-r-1} \begin{bmatrix} p^{r+1} & 0 \\ 0 & 1 \end{bmatrix} \\
&\quad + \sum_{i=0}^{r-1} p^{k(i+1)-r-1} \sum_{b_2(p^{r-i})} \begin{bmatrix} p^{i+1} & pb_2 \\ 0 & p^{r-i} \end{bmatrix} \\
&= T_{p^{r+1}} + \sum_{i=0}^{r-1} p^{k(i+1)-r-1} \sum_{b_2(p^{r-i})} \begin{bmatrix} p^i & b_2 \\ 0 & p^{r-1-i} \end{bmatrix} \\
&= T_{p^{r+1}} + \sum_{i=0}^{r-1} p^{k(i+1)-r} \sum_{b_2(p^{r-1-i})} \begin{bmatrix} p^i & b_2 \\ 0 & p^{r-1-i} \end{bmatrix} \\
&= T_{p^{r+1}} + p^{k-1} T_{p^{r-1}}
\end{aligned}
$$

as desired. Here for the second equality we used that $\begin{pmatrix} p^{i+1} & pb_2 \\ 0 & p^{r-i} \end{pmatrix} z = \begin{pmatrix} p^i & b_2 \\ 0 & p^{r-1-i} \end{pmatrix} z$ for any $z \in \mathbb{H}$. $\square$

Next, we study effects of Hecke operators on Fourier coefficients.

Proposition 10.6. *Let $f \in \mathcal{M}_k$ with a Fourier expansion $f(z) = \sum_{m=0}^{\infty} \widehat{f}(m) e(mz)$. Then*

$$T_n f = \sum_{m=0}^{\infty} \widehat{T_n f}(m) e(mz)$$

with

$$\widehat{T_n f}(m) = \sum_{d|(m,n)} d^{k-1} \widehat{f}(mnd^{-2}).$$

Proof. By definition

$$
\begin{aligned}
T_n f(z) &= \frac{1}{n} \sum_{m=0}^{\infty} \widehat{f}(m) \sum_{ad=n} a^k \sum_{b \pmod d} e\left(m \tfrac{az+b}{d}\right) \\
&= \frac{1}{n} \sum_{m=0}^{\infty} \widehat{f}(m) \sum_{ad=n} a^k e\left(\tfrac{maz}{d}\right) \sum_{b \pmod d} e\left(\tfrac{mb}{d}\right) \\
&= \frac{1}{n} \sum_{m=0}^{\infty} \widehat{f}(m) \sum_{ad=n} a^k e\left(\tfrac{maz}{d}\right) dI(d \mid m) \\
&\overset{m=d\ell}{=} \sum_{ad=n} a^{k-1} \sum_{\ell=0}^{\infty} \widehat{f}(d\ell) e(a\ell z) \\
&\overset{a\ell=m}{=} \sum_{m=0}^{\infty} \left(\sum_{a|(m,n)} a^{k-1} \widehat{f}(mna^{-2}) \right) e(mz),
\end{aligned}
$$

finishing the proof. Here for the third equality we used the identity that

$$\sum_{b \pmod d} e\left(\tfrac{mb}{d}\right) = dI(d \mid m)$$

with $(I(d \mid m)$ the indicator function of the condition $d \mid m$, i.e. $I(d \mid m)$ equals 1 if $d \mid m$ and equals 0 otherwise. $\square$

Proposition 10.6 has the following immediate consequences whose proof we omit.

Corollary 10.7. *For any $n \geq 1$ and $f \in \mathcal{M}_k$ we have*

(1) $\widehat{T_n f}(m) = \widehat{T_m f}(n)$ *for any $m \geq 1$,*
(2) $\widehat{T_n f}(0) = \sigma_{k-1}(n)\widehat{f}(0)$,
(3) $\widehat{T_n f}(1) = \widehat{f}(n)$.

Let us now apply Hecke operators to the modular discriminant function Δ to see what we can get from them. First since $\mathcal{S}_{12} = \mathbb{C}\Delta$ is one dimensional, for any $n \geq 1$ there exists $\lambda(n) \in \mathbb{C}$ such that $T_n\Delta = \lambda(n)\Delta$. This implies that

$$\widehat{T_n\Delta}(1) = \lambda(n)\widehat{\Delta}(1) = (2\pi)^{12}\lambda(n).$$

On the other hand, by (3) of Corollary 10.7 we have

$$\widehat{T_n\Delta}(1) = \widehat{\Delta}(n) = (2\pi)^{12}\tau(n).$$

Equating both equations we get $\lambda(n) = \tau(n)$. This, together with Proposition 10.6 implies that

$$(2\pi)^{12}\tau(n)\tau(m) = \lambda(n)\widehat{\Delta}(m) = \widehat{T_n\Delta}(m) = (2\pi)^{12} \sum_{d \mid (m,n)} d^{k-1}\tau(mnd^{-2}).$$

Note that this recovers the recursive relation (8.9) for the Ramanujan's tau function.

10.2. **Self-adjointness of Hecke operators.** We wish to generalize the above discussion on Ramanujan's τ-function to Fourier coefficients of a general weight k cusp form. The main ingredient of the above discussion is the existence of joint eigenfunctions for all Hecke operators. While this property is immediate for $\mathcal{S}_{12}$ (since it is of one dimensional), it is no longer the case for larger k. The first guess for this joint eigenfunction basis is the basis given by Poincaré series. However, the following proposition shows that they are in general not the desired joint eigenfunctions. Below we abbreviate the Poincaré series $P_{m,k}$ by P_m.

Proposition 10.8. *For $k \geq 4$ even, $m \geq 0$ and $n \geq 1$ we have*

$$T_n P_m = \sum_{d \mid (m,n)} \left(\frac{n}{d}\right)^{k-1} P_{mnd^{-2}}. \tag{10.9}$$

In particular, when $m = 0$ the Eisenstein series E_k is a joint eigenfunction for all Hecek operators with

$$T_n E_k = \sigma_{k-1}(n)E_k, \quad \forall\ n \geq 1.$$

Proof. The in particular part follows easily from (10.9); we thus only prove (10.9). By definition

$$\begin{aligned}T_nP_m(z) &= n^{\frac{k}{2}-1}\sum_{\beta\in\Gamma\backslash G_n} P_m[\beta]_k(z)\\ &= n^{k-1}\sum_{\beta\in\Gamma\backslash G_n} j_\beta(z)^{-k}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j_\gamma(\beta z)^{-k}e(m\gamma\beta z)\\ &= n^{k-1}\sum_{\beta\in\Gamma\backslash G_n}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j_{\gamma\beta}(z)^{-k}e(m\gamma\beta z).\end{aligned}$$

Now we take Δ_n as before to be the fixed set of coset representatives of $\Gamma\backslash G_n$. Let $\mathcal{H}$ be a set of coset representatives of $\Gamma_\infty\backslash\Gamma$. Then clearly $\mathcal{H}\Delta_n$ is a set of coset representatives of $\Gamma_\infty\backslash G_n$. But by Exercise 19 the set $\Delta_n\mathcal{H}$ is also a set of coset representatives of $\Gamma_\infty\backslash G_n$. Thus we have

$$T_nP_m(z) = n^{k-1}\sum_{\beta\in\Delta_n}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j_{\beta\gamma}(z)^{-k}e(m\beta\gamma z).$$

Note that for $\beta = \left(\begin{smallmatrix} a & b\\ 0 & d\end{smallmatrix}\right)\in\Delta_n$,

$$j_{\beta\gamma}(z) = j_\beta(\gamma z)j_\gamma(z) = dj_\gamma(z).$$

Thus

$$\begin{aligned}T_nP_m(z) &= n^{k-1}\sum_{ad=n}\sum_{b \pmod d}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} d^{-k}j_\gamma(z)^{-k}e\left(m\tfrac{a\gamma z+b}{d}\right)\\ &= n^{k-1}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\sum_{ad=n} d^{-k}j_\gamma(z)^{-k}e\left(\tfrac{ma\gamma z}{d}\right)\sum_{b \pmod d} e\left(\tfrac{mb}{d}\right)\\ &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\sum_{\substack{ad=n\\ d\mid m}} (n/d)^{k-1}j_\gamma(z)^{-k}e\left(\tfrac{ma\gamma z}{d}\right)\\ &= \sum_{d\mid(m,n)}(n/d)^{k-1}P_{mnd^{-2}}.\end{aligned}$$

□

Exercise 19. *Let Δ_n and $\mathcal{H}$ be as above. Show that the set $\Delta_n\mathcal{H}$ is a set of coset representatives of $\Gamma_\infty\backslash G$*

In order to produce a joint eigenfunction basis for Hecke operators we apply the spectral theorem of linear algebra.

Theorem 10.9. *A commuting family of normal operators*[12] *on a finite dimensional Hilbert space can be simultaneously diagonalized, that is, there exists an orthogonal basis of simultaneous eigenvectors for this family of operators.*

To apply this theorem we show that Hecke operators are self-adjoint. Before stating the main result we record a symmetric identity which follows immediately from Proposition 10.8

$$m^{k-1}T_nP_m = n^{k-1}T_mP_n, \quad \forall\ m,n\geq 1. \tag{10.10}$$

[12]A continuous linear operator $T: H\to H$ on a complex Hilbert space H is called *normal* if it commutes with its adjoint operator T^* which is defined such that $\langle Tv,w\rangle = \langle v,T^*w\rangle$ for any $v,w\in H$.

We also need the following symmetric identity regarding the Petersson inner product.

Proposition 10.10. *For any $f \in \mathcal{M}_k$ and for any $m, n \geq 1$,*

$$m^{k-1}\langle T_n f, P_m\rangle = n^{k-1}\langle T_m f, P_n\rangle. \tag{10.11}$$

Proof. By Proposition 8.2 and (1) of Corollary 10.7 we have

$$m^{k-1}\langle T_n f, P_m\rangle = \frac{\Gamma(k-1)}{(4\pi)^{k-1}}\widehat{T_n f}(m) = \frac{\Gamma(k-1)}{(4\pi)^{k-1}}\widehat{T_m f}(n) = n^{k-1}\langle T_m f, P_n\rangle. \qquad \square$$

Theorem 10.11. *For any $n \geq 1$, the Hecke operator T_n is self-adjoint, that is,*

$$\langle T_n f, g\rangle = \langle f, T_n g\rangle, \quad \forall\ f, g \in \mathcal{S}_k.$$

Proof. Since the Poincaré series $\{P_m\}_{m\geq 1}$ spans $\mathcal{S}_k$ (Corollary 8.3), it suffices to prove the above identity for $f = P_m$ and $g = P_l$ for some $m, l \geq 1$. We then have

$$\langle T_n P_m, P_l\rangle = \left(\tfrac{n}{m}\right)^{k-1}\langle T_m P_n, P_l\rangle = \left(\tfrac{n}{l}\right)^{k-1}\langle T_l P_n, P_m\rangle = \langle T_n P_l, P_m\rangle,$$

where we applied (10.10) for the first and third identity and (10.11) for the second identity. Finally we note that

$$\langle T_n P_l, P_m\rangle = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}}\widehat{T_n P_l}(m)$$

is real since $\widehat{T_n P_l}(m)$ is a $\mathbb{R}$-linear combination of Fourier coefficients of Poincaré series and the latter is known to be real by its explicit formula derived in Theorem 7.6. Thus

$$\langle T_n P_m, P_l\rangle = \langle T_n P_l, P_m\rangle = \langle P_m, T_n P_l\rangle$$

as desired. $\square$

Combining Theorem 10.9, Theorem 10.11 and the commutativity of Hecke operators (see Remark 10.8) we can now produce a joint eigenfunction basis of $\mathcal{S}_k$ for all Hecke operators.

Corollary 10.12. *The space $\mathcal{S}_k$ has an orthogonal basis (with respect to the Petersson inner product) of joint eigenfunctions for all Hecke operators $\{T_n\}_{n\geq 1}$.*

Definition 10.12. A joint eigenfunction $f \in \mathcal{S}_k$ for all Hecke operators is called *normalized* if $\widehat{f}(1) = 1$.

Proposition 10.13. *Let $f \in \mathcal{S}_k$ be a normalized joint eigenfunction for all Hecke operators, that is, $T_n f = \lambda(n) f$ for some $\lambda(n) \in \mathbb{C}$ and $\widehat{f}(1) = 1$. Then we have*

1. *$\widehat{f}(n) = \lambda(n)$ for all $n \geq 1$.*
2. *$\widehat{f}(m)\widehat{f}(n) = \sum_{d\mid(m,n)} d^{k-1}\widehat{f}(mnd^{-2})$.*
3. *(Multiplicity one theorem) If f and g are two normalized joint eigenfunctions with same eigenvalues, then $f = g$.*

Proof. For (1) by (3) of Corollary 10.7 and the relation $T_n f = \lambda(n) f$ we have

$$\lambda(n) = \lambda\widehat{f}(1) = \widehat{T_n f}(1) = \widehat{f}(n).$$

The relation in (2) follows similarly since by (1), Proposition 10.6 and the relation $T_n f = \lambda(n) f$. For (3) by (1) we have $\lambda(n) = \widehat{f}(n) = \widehat{g}(n)$ for all $n \geq 1$. Hence $f = g$. $\square$

Remark 10.13. If $f \in \mathcal{S}_k$ is a joint eigenfunction (not necessarily normalized) with $T_n f = \lambda(n) f$, then using the same argument as above we get that $\widehat{f}(n) = \lambda(n)\widehat{f}(1)$, implying that if $\widehat{f}(1) = 0$ we must have $f = 0$. Hence a nonzero joint eigenfunction can always be normalized.

11. Hecke operators for congruence groups

In this section we define the Hecke operators for the congruence subgroup $\Gamma_1(N)$. We will do so by first decomposing the space $\mathcal{M}_k(\Gamma_1(N))$ into subspaces consisting of modular forms with respect to $\Gamma_0(N)$ and twisted with a Dirichlet character (see Proposition 11.3 below) and then define the Hecke operators in each of these subspaces.

We first give the definition of modular forms twisted with a character.

Definition 11.1. Let Γ be a congruence subgroup and $\vartheta : \Gamma \to \mathbb{C}^\times$ a unitary character[13], the space of weight k modular forms twisted with ϑ and with respect to Γ is defined by

$$\mathcal{M}_k(\Gamma, \vartheta) := \left\{ f : \mathbb{H} \to \mathbb{C} : \left| \begin{array}{l} f \text{ is holomorphic on } \mathbb{H} \text{ and at cusps and satisfies} \\ f[\gamma]_k = \vartheta(\gamma) f \text{ for any } \gamma \in \Gamma \end{array} \right. \right\}.$$

Similarly, we denote by $\mathcal{S}_k(\Gamma, \vartheta)$ the subspace of $\mathcal{M}_k(\Gamma, \vartheta)$ by further requiring f to be vanishing at all cusps. We will always assume that image of ϑ is finite, that is $\ker(\vartheta)$ is also a congruence subgroup.

Since Dirichlet characters will appear naturally in our decomposition, in the next subsection we give a quick review on these characters.

11.1. Dirichlet characters.

Definition 11.2. Let N be a positive integer. A *Dirichlet character of modulus N* is a function $\chi : \mathbb{Z} \to \mathbb{C}$ satisfying

(1) $\chi(n + N) = \chi(n)$ for any $n \in \mathbb{Z}$,
(2) $\chi(n) = 0$ if and only if $(n, N) > 1$,
(3) $\chi(mn) = \chi(m)\chi(n)$ for all $m, n \in \mathbb{Z}$.

Remark 11.3. Indeed a Dirichlet character comes from a character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ by first viewing it as a function on $\{n \in \mathbb{Z} : (n, N) = 1\}$ and then extending it trivially to $\mathbb{Z}$ to satisfy property (2) above.

We denote by $\widehat{(\mathbb{Z}/N\mathbb{Z})^\times}$ the set of all Dirichlet characters of modulus N. Note that $\widehat{(\mathbb{Z}/N\mathbb{Z})^\times}$ has a group structure with the group law given by multiplication and identity given by the trivial character 1_N defined such that $1_N(n) = 1$ if $(n, N) = 1$ and $1_N(n) = 0$ otherwise. It is called the *dual group* of $\widehat{(\mathbb{Z}/N\mathbb{Z})^\times}$.

Proposition 11.1. *The dual group $\widehat{(\mathbb{Z}/N\mathbb{Z})^\times}$ is isomorphic to $\mathbb{Z}/N\mathbb{Z})^\times$. In particular, $\#\widehat{(\mathbb{Z}/N\mathbb{Z})^\times} = \phi(N)$, where ϕ is the Euler's totient function as before.*

[13] A *character* is a group homomorphism from Γ to $\mathbb{C}^\times \cong \mathrm{GL}_1(\mathbb{C})$, and it is *unitary* if its image lies in the unit disc $\{z \in \mathbb{C} : |z| = 1\} = \mathrm{U}(\mathbb{C})$. In the terminology of representation, a (unitary) character is just a one-dimensional complex (unitary) representation of Γ.

Proof (sketch). First note that if H_1, H_2 are two finite abelian groups. Then there is a natural isomorphism from $\widehat{H_1} \times \widehat{H_2}$ to $\widehat{H_1 \times H_2}$ given by

$$(\chi_1, \chi_2) \mapsto \chi(h_1, h_2) := \chi_1(h_1)\chi_2(h_2)$$

with the inverse map given by

$$\chi \mapsto (\chi_1, \chi_2),$$

where $\chi_i \in \widehat{H_i}$ is defined such that $\chi_1(h_1) := \chi(h_1, 1)$ and $\chi_2(h_2) := \chi(1, h_2)$. Moreover, if $H = \langle h \rangle$ is a finite cyclic group of order n, then there is an isomorphism from $\mathbb{Z}/n\mathbb{Z}(\cong H)$ to $\widehat{H}$ given by

$$j \in \mathbb{Z}/n\mathbb{Z} \mapsto \chi_j(h^i) := \xi_n^{ij},$$

where $\xi_n \in \mathbb{C}^\times$ is a primitive n-th root of unity, that is, $\xi_n^n = 1$ and $\xi_n^i \neq 1$ for all $1 \leq i < n$. We now specify to the group $(\mathbb{Z}/N\mathbb{Z})^\times$. Note that $(\mathbb{Z}/N\mathbb{Z})^\times$ is isomorphic to a finite product of finite cyclic groups. Indeed, by the Chinese Remainder theorem we have $(\mathbb{Z}/N\mathbb{Z})^\times \cong \prod_i (\mathbb{Z}/p_i^{\alpha_i}\mathbb{Z})^\times$ with $N = \prod_i p_i^{\alpha_i}$ written in the prime decomposition form. Moreover, we know if p_i is odd or $\alpha_i = 1, 2$, $(\mathbb{Z}/p_i^{\alpha_i}\mathbb{Z})^\times$ is cyclic, otherwise it is isomorphic to a product of two cyclic groups with one factor being of order 2. In both cases, $(\mathbb{Z}/p_i^{\alpha_i}\mathbb{Z})^\times$ is a finite product of cyclic groups. Hence so is $(\mathbb{Z}/N\mathbb{Z})^\times$. Then we have $(\mathbb{Z}/N\mathbb{Z})^\times \cong \prod_i H_i$ with each H_i finite and cyclic. Thus

$$\widehat{(\mathbb{Z}/N\mathbb{Z})}^\times \cong \widehat{\prod_i H_i} \cong \prod_i \widehat{H_i} \cong \prod_i H_i \cong (\mathbb{Z}/N\mathbb{Z})^\times$$

as desired. $\square$

Proposition 11.2 (Orthogonality relations)**.** *We have for any $x \in (\mathbb{Z}/N\mathbb{Z})^\times$*

$$\sum_{\chi \pmod N} \chi(x) = \begin{cases} \phi(N) & \text{if } x \equiv 1 \pmod N, \\ 0 & \text{otherwise,} \end{cases}$$

and for any Dirichlet character χ of modulus N,

$$\sum_{x \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi(x) = \begin{cases} \phi(N) & \text{if } \chi = 1_N, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We only prove the first equation and the second follows by similar arguments. The case when $x \equiv 1 \pmod N$ is trivial; we thus assume $x \not\equiv 1 \pmod N$. Then there exists a Dirichlet character χ_0 of modulus N such that $\chi_0(x) \neq 1$. Multiplying the left hand side by $\chi_0(x)$ we get

$$\chi_0(x) \sum_{\chi \pmod N} \chi(x) = \sum_{\chi \pmod N} \chi_0\chi(x) = \sum_{\chi \pmod N} \chi(x),$$

where for the second identity we used the fact that as χ runs through all elements in $\widehat{(\mathbb{Z}/N\mathbb{Z})}^\times$, so does $\chi_0\chi$. Since $\chi_0(x) \neq 1$, the above identity implies that

$$\sum_{\chi \pmod N} \chi(x) = 0$$

as desired. $\square$

Definition 11.4. For any Dirichlet character χ of modulus N and any $\ell \in \mathbb{Z}/N\mathbb{Z}$, the corresponding *Gauss sum* is defined by

$$G(x, \chi) := \sum_{n=1}^{N} \chi(n) e\left(\tfrac{n\ell}{N}\right).$$

Exercise 20. *Let χ be a primitive Dirichlet character of modulus N. Show that*

(1) $G(\chi, \ell) = \overline{\chi}(\ell) G(\chi)$ *for any* $\ell \in \mathbb{Z}/N\mathbb{Z}$.
(2) $|G(\chi)| = \sqrt{N}$.

Given a Dirichlet character $\chi : \mathbb{Z} \to \mathbb{C}$ of modulus N, for later purpose, we extend χ to a function on $M_2(\mathbb{Z})$ (which we still denote by χ) as following

$$\chi : M_2(\mathbb{Z}) \to \mathbb{C}, \quad g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \overline{\chi}(a). \tag{11.5}$$

One can check that $\chi|_{\Gamma_0(N)} : \Gamma_0(N) \to \mathbb{C}^\times$ sending $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ to $\chi(d)$ is a unitary character of $\Gamma_0(N)$.

11.2. Modular forms with character.

As mentioned before, the main reason we study modular forms with character is the following decomposition which we leave as an exercise.

Proposition 11.3. *For any positive integers k, N we have*

$$\mathcal{M}_k(\Gamma_1(N)) = \bigoplus_{\chi} \mathcal{M}_k(\Gamma_0(N), \chi)$$

and

$$\mathcal{S}_k(\Gamma_1(N)) = \bigoplus_{\chi} \mathcal{S}_k(\Gamma_0(N), \chi),$$

where the direct sum is over all Dirichlet characters of modulus N (viewed as an unitary character of $\Gamma_0(N)$ defined as in previous subsection).

We now give the definition of Hecke operators with respect to a given Dirichlet character χ.

Definition 11.6. Let χ be a Dirichlet character of modulus N. The *n-th Hecke operator with character χ* is an operator on $\mathcal{M}_k(\Gamma_0(N), \chi)$ defined by

$$T_n^\chi f := n^{\frac{k}{2}-1} \sum_{\rho \in \Delta_n} \overline{\chi}(\rho) f[\rho]_k, \quad \forall\ f \in \mathcal{M}_k(\Gamma_0(N), \chi),$$

where Δ_n is the set of coset representatives of $\Gamma(1)\backslash G_n$ given as in Proposition 10.4 and $\chi : M_2(\mathbb{Z}) \to \mathbb{C}$ is the extension of χ defined as in (11.5).

Remark 11.7. Using the explicit description of Δ_n we can write

$$T_n^\chi f(z) = \frac{1}{n} \sum_{ad=n} \chi(a) a^k \sum_{0 \le b < d} f\left(\tfrac{az+b}{d}\right).$$

Moreover, since $\chi(a) = 0$ whenever $(a, N) > 1$, we have

$$T_n^\chi f := n^{\frac{k}{2}-1} \sum_{\rho \in \Delta_n^N} \overline{\chi}(\rho) f[\rho]_k, \tag{11.8}$$

where

$$\Delta_n^N := \left\{ \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \Delta_n : (a, N) = 1 \right\}.$$

The first result regarding T_n^χ is the assertion that T_n^χ sends $\mathcal{M}_k(\Gamma_0(N),\chi)$ and $\mathcal{S}_k(\Gamma_0(N),\chi)$ to themselves respectively.

Proposition 11.4. *Keep the notation and assumptions as above. Then we have*

$$T_n^\chi : \mathcal{M}_k(\Gamma_0(N),\chi) \to \mathcal{M}_k(\Gamma_0(N),\chi),$$

and

$$T_n^\chi : \mathcal{S}_k(\Gamma_0(N),\chi) \to \mathcal{S}_k(\Gamma_0(N),\chi).$$

We need the following technical lemma to prove this proposition.

Lemma 11.5. *For any $(\rho,\tau) \in \Delta_n^N \times \Gamma_0(N)$, there exists $(\rho',\tau') \in \Delta_n^N \times \Gamma_0(N)$ such that $\rho\tau = \tau'\rho'$. Moreover, as ρ runs through Δ_n^N, so does ρ'.*

Proof. Given $(\rho,\tau) \in \Delta_n^N \times \Gamma_0(N)$, note that $\rho\tau \in G_n = \bigsqcup_{\rho' \in \Delta_n} \Gamma(1)\rho'$. Hence there exists $(\rho',\tau') \in \Delta_n \times \Gamma(1)$ such that $\rho\tau = \tau'\rho'$. We will show this relation and the condition $(\rho,\tau) \in \Delta_n^N \times \Gamma_0(N)$ force (ρ',τ') to lie in the smaller set $\Delta_n^N \times \Gamma_0(N)$. Write $\rho = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \Delta_n^N$, $\tau = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma_0(N)$ and $\rho' = \left(\begin{smallmatrix} a' & b' \\ 0 & d' \end{smallmatrix}\right) \in \Delta_n$, $\tau' = \left(\begin{smallmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{smallmatrix}\right) \in \Gamma(1)$. We need to show $(a',N) = 1$ and $N \mid \gamma'$. By direct computation the relation $\rho\tau = \tau'\rho'$ implies that

$$\alpha a + \gamma b = \alpha' a' \quad \text{and} \quad \gamma d = \gamma' a'. \tag{11.9}$$

Note that $\tau \in \Gamma_0(N)$ implies that $N \mid \gamma$ and $(\alpha, N) = 1$. Moreover, $\rho \in \Delta_n^N$ implies that $(a,N) = 1$. Hence the first equality in (11.9) implies that $\alpha' a' \equiv \alpha a \pmod N$ which then implies that $(a',N) = 1$. The second equality in (11.9) then implies that $\gamma' a' = \gamma d \equiv 0 \pmod N$. This together with the condition $(a',N) = 1$ implies that $N \mid \gamma'$. We have thus proved the existence of the desired pair $(\rho',\tau') \in \Delta_n^N \times \Gamma_0(N)$.

For the moreover part, we note that for any $\tau \in \Gamma_0(N)$,

$$G_n = \bigsqcup_{\rho \in \Delta_n} \Gamma(1)\rho = \bigsqcup_{\rho \in \Delta_n} \Gamma(1)\rho\tau.$$

This shows that the set $\{\rho\tau\}_{\rho \in \Delta_n}$ is another set of coset representatives of $\Gamma(1)\backslash G_n$. Hence there exists a bijection $g : \Delta_n \to \Delta_n$ such that

$$\rho\tau = \tau_\rho g(\rho) \quad \text{for some } \tau_\rho \in \Gamma(1) \text{ and for all } \rho \in \Delta_n.$$

The previous argument then shows that when $\rho \in \Delta_n^N$, we must have $\tau_\rho \in \Gamma_0(N)$ and $g(\rho) \in \Delta_n^N$. Indeed, $(\tau_\rho, \gamma(\rho))$ is the pair (τ',ρ') above. We thus have $g(\Delta_n^N) \subset \Delta_n^N$. Since g itself is a bijection, so is $g|_{\Delta_n^N}$. This proves the moreover part. $\square$

We can now give the

Proof of Proposition 11.4. As argued in the proof of Proposition 9.3, for any $f \in \mathcal{M}_k(\Gamma_0(N),\chi)$ (resp. $f \in \mathcal{S}_k(\Gamma_0(N))$) and for any $\rho \in \mathrm{GL}_2^+(\mathbb{Q})$, the function $f[\rho]_k$ is holomorphic on $\mathbb{H}$ and holomorphic at cusps (resp. vanishes at cusps). Thus it suffices to show $T_n^\chi f$ satisfies the desired transformation rule. For this we use the expression (11.8) for T_n^χ. Take any $\tau \in \Gamma_0(N)$ we have

$$T_n^\chi f[\tau]_k = n^{\frac{k}{2}-1} \sum_{\rho \in \Delta_n^N} \overline{\chi}(\rho) f[\rho\tau]_k.$$

Now by Lemma 11.5 we have $\rho\tau = \tau'\rho'$ for some $\tau' \in \Gamma_0(N)$ and $\rho' \in \Delta_n^N$. Moreover, from (11.9) we have

$$\overline{\chi}(\rho)\chi(\tau') = \chi(a)\overline{\chi}(\alpha') = \overline{\chi}(\alpha)\chi(a') = \chi(\tau)\overline{\chi}(\rho'). \tag{11.10}$$

Hence

$$\begin{aligned} T_n^\chi f[\tau]_k &= n^{\frac{k}{2}-1} \sum_{\rho\in\Delta_n^N} \overline{\chi}(\rho) f[\tau'\rho']_k \\ &= \chi(\tau) n^{\frac{k}{2}-1} \sum_{\rho\in\Delta_n^N} \overline{\chi}(\rho') f[\rho']_k \\ &= \chi(\tau) T_n^\chi f, \end{aligned}$$

as desired. Here for the second equality we applied (11.10) and the assumption that $f \in \mathcal{M}_k(\Gamma_0(N), \chi)$ and for the last equality we used the moreover part of Lemma 11.5. □

We now state some of the properties of the Hecke operators T_n^χ. We note that the proofs are almost identical to that of Proposition 10.5 and Proposition 10.6 with obvious modifications to accommodate the character χ. We thus omit the details here.

Proposition 11.6. *We have*

1. $T_m^\chi T_n^\chi = T_{mn}^\chi$ *whenever* $(m, n) = 1$.
2. $T_{p^{r+1}}^\chi = T_p^\chi T_{p^r}^\chi - \chi(p)p^{k-1}T_{p^{r-1}}^\chi$.

To summarize,

$$T_m^\chi T_n^\chi = \sum_{d|(m,n)} \chi(d)d^{k-1}T_{mnd^{-2}}^\chi, \quad \forall\ m, n \geq 1.$$

In particular, $T_m^\chi T_n^\chi = T_n^\chi T_m^\chi$ *for any* $m, n \geq 1$.

We also have the following proposition describing effects of T_n^χ on Fourier coefficients.

Proposition 11.7. *Let* $f \in \mathcal{M}_k(\Gamma_0(N), \chi)$ *with a Fourier expansion* $f(z) = \sum_{n=0}^\infty \widehat{f}(m)e(mz)$. *Then we have for any* $n \geq 1$, $T_n^\chi f(z) = \sum_{m=0}^\infty \widehat{T_n^\chi f}(m)e(mz)$ *with*

$$\widehat{T_n^\chi f}(m) = \sum_{d|(m,n)} \chi(d)d^{k-1}\widehat{f}(mnd^{-2}).$$

Similar to the modular group case, there are some direct consequences of this proposition.

Corollary 11.8. *Keep the notation and assumptions as above. We have*

1. $\widehat{T_n^\chi f}(m) = \widehat{T_m^\chi f}(n)$ *for any* $m, n \geq 1$.
2. $\widehat{T_n^\chi f}(0) = \sigma_{k-1}^\chi(n)\widehat{f}(0)$ *with* $\sigma_{k-1}^\chi(n) := \sum_{d|n} \chi(d)d^{k-1}$.
3. $\widehat{T_n^\chi f}(1) = \widehat{f}(n)$.

11.3. **Normalized Petersson inner product.** Our next goal is to show that the Hecke operators are normal in order to apply the spectral theorem Theorem 10.9. However, in the case the previous strategy of studying the action of Hecke operators on Poincaré series is no longer enough. Indeed using similar arguments one can obtain

$$\langle T_n^\chi f, g\rangle = \chi(n)\langle f, T_n^\chi g\rangle, \quad \forall\ (n, N) = 1,$$

but for f, g from the subspace spanned by $\{P_m\}_{(m,N)=1}$[14]. However, it is not clear whethter this subspace is the whole $\mathcal{S}_k(\Gamma_0(N), \chi)$. To overcome this difficulty we will use the Petersson inner product in a more essential way. We first define the following normalized Petersson inner product which allows us to compare the "size" of modular forms of different levels.

Definition 11.11. Let Γ be a congruence subgroup with a unitary character ϑ. The *normalized Petersson inner product* is defined on the cusp form space $\mathcal{S}_k(\Gamma, \vartheta)$ by

$$\langle f, g\rangle_\Gamma := \frac{1}{V_\Gamma}\int_{\Gamma\backslash\mathbb{H}} y^k f(z)\overline{g(z)}\ d\mu(z), \quad \forall\ f, g \in \mathcal{S}_k(\Gamma, \vartheta).$$

where $V_\Gamma := [\mathrm{SL}_2(\mathbb{Z}) : \tilde{\Gamma}]$ with $\tilde{\Gamma} := \langle \pm I_2, \Gamma\rangle$.

One can easily check that for $f, g \in \mathcal{S}_k(\Gamma, \vartheta)$ the function $z \mapsto y^k f(z)\overline{g(z)}$ is left Γ-invariant. Hence the above definition is well-defined. The benefit of adding the normalizing factor V_Γ^{-1} is that we can now compare norm of cusp forms of different levels: Let $\Gamma_1 < \Gamma_2$ be two congruence subgroups with $\vartheta : \Gamma_2 \to \mathbb{C}^\times$ a unitary character of Γ_2. Then $f \in \mathcal{S}_k(\Gamma_2, \vartheta)$ can also be viewed as an element of $\mathcal{S}_k(\Gamma_1, \vartheta|_{\Gamma_1})$. Viewing f as elements of these two different vector spaces assigns two norms to f. We claim that these two norms are indeed the same. Let $\mathcal{F}_{\Gamma_2}$ be a fundamental domain of Γ_2. Then the disjoint union $\mathcal{F}_{\Gamma_1} := \bigsqcup_{\sigma\in\tilde{\Gamma}_1\backslash\tilde{\Gamma}_2} \sigma\mathcal{F}_{\Gamma_2}$ forms a fundamental domain of Γ_1. Taking this fundamental domain we have for any $f, g \in \mathcal{S}_k(\Gamma, \vartheta)$,

$$\begin{aligned}\langle f, g\rangle_{\Gamma_1} &= \frac{1}{V_{\Gamma_1}}\int_{\mathcal{F}_{\Gamma_1}} y^k f(z)\overline{g(z)}\ d\mu(z)\\ &= \frac{1}{V_{\Gamma_1}}\sum_{\sigma\in\tilde{\Gamma}_1\backslash\tilde{\Gamma}_2}\int_{\sigma\mathcal{F}_{\Gamma_2}} y^k f(z)\overline{g(z)}\ d\mu(z).\end{aligned}$$

Since the above integrand is left $\tilde{\Gamma}_2$-invariant[15], making a change of variable $\sigma z \mapsto z$ we have

$$\begin{aligned}\langle f, g\rangle_{\Gamma_1} &= \frac{1}{V_{\Gamma_1}}\sum_{\sigma\in\tilde{\Gamma}_1\backslash\tilde{\Gamma}_2}\int_{\mathcal{F}_{\Gamma_2}} y^k f(z)\overline{g(z)}\ d\mu(z)\\ &= \frac{[\tilde{\Gamma}_2 : \tilde{\Gamma}_1]}{V_{\Gamma_1}}\int_{\mathcal{F}_{\Gamma_2}} y^k f(z)\overline{g(z)}\ d\mu(z) = \langle f, g\rangle_{\Gamma_2}\end{aligned}$$

[14]Here the Poincare series is slightly different from the one defined before since we need to accommodate the character χ. More precisely, for any $m \geq 0$, $P_m(z) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\overline{\chi(\gamma)}j_\gamma(z)^{-k}e(m\gamma z)$ with $\Gamma = \Gamma_0(N)$.

[15]The assumption that $f \in \mathcal{S}_k(\Gamma_2, \vartheta)$ implies that it is left Γ_2-invariant. It is clearly also left $-I_2$-invariant.

as claimed. More generally, for any $f \in \mathcal{S}_k(\Gamma, \vartheta)$ and $\sigma \in \mathrm{GL}_2^+(\mathbb{Q})$, the function $f[\sigma]_k \in \mathcal{S}_k(\Gamma')$ for some sufficiently small congruence subgroup Γ' contained in the intersection $\sigma^{-1}\ker(\vartheta)\sigma \cap \Gamma$ (cf. Lemma 9.1 and Remark 9.1). The next proposition shows that the normalized norm of $f[\sigma]_k$ (with respect to Γ') is the same as that of f (with respect to Γ).

Proposition 11.9. *Let $\mathcal{S}_k(\Gamma, \vartheta)$ and $\sigma \in \mathrm{GL}_2^+(\mathbb{Q})$ be as above. Let Γ' be a congruence subgroup contained in $\sigma^{-1}\ker(\vartheta)\sigma \cap \Gamma$. Then for any $f, g \in \mathcal{S}_k(\Gamma, \vartheta)$ we have*

$$\langle f[\sigma]_k, g[\sigma]_k\rangle_{\Gamma'} = \langle f, g\rangle_{\Gamma}.$$

Proof. By definition we have

$$\begin{aligned}\langle f[\sigma]_k, g[\sigma]_k\rangle_{\Gamma'} &= \frac{1}{V_{\Gamma'}}\int_{\mathcal{F}_{\Gamma'}} y^k f[\sigma]_k(z)\overline{g[\sigma]_k(z)}\, d\mu(z)\\ &= \frac{1}{V_{\Gamma'}}\int_{\mathcal{F}_{\Gamma'}} \det(\sigma)^k y^k |j_\sigma(z)|^{-2k} f(\sigma z)\overline{g(\sigma z)}\, d\mu(z).\end{aligned}$$

Now making a change of variable $\sigma z \mapsto z$ and noting that $j_\sigma(z) = j_{\sigma^{-1}}(\sigma z)^{-1}$ we have

$$\begin{aligned}\langle f[\sigma]_k, g[\sigma]_k\rangle_{\Gamma'} &= \frac{1}{V_{\Gamma'}}\int_{\sigma\mathcal{F}_{\Gamma'}} \det(\sigma)^k \mathfrak{Im}(\sigma^{-1}z)^k |j_{\sigma^{-1}}(z)|^{2k} f(z)\overline{g(z)}\, d\mu(z)\\ &= \frac{1}{V_{\Gamma'}}\int_{\sigma\mathcal{F}_{\Gamma'}} y^k f(z)\overline{g(z)}\, d\mu(z) = \langle f, g\rangle_{\sigma\Gamma'\sigma^{-1}} = \langle f, g\rangle_{\Gamma},\end{aligned}$$

where for the second equality we used that $\mathfrak{Im}(\sigma^{-1}z) = \frac{\det(\sigma^{-1})\mathfrak{Im}(z)}{|j_{\sigma^{-1}}(z)|^2}$ and for the last equality we used that $\sigma\Gamma'\sigma^{-1} < \Gamma$ (since $\Gamma' < \sigma^{-1}\Gamma\sigma \cap \Gamma$). ☐

Remark 11.12. We note that the above computation shows that

$$\langle f[\sigma]_k, g\rangle_{\Gamma'} = \langle f, g[\sigma^{-1}]_k\rangle_{\sigma\Gamma'\sigma^{-1}}, \quad \forall\, f, g \in \mathcal{S}_k(\Gamma, \vartheta).$$

In fact by passing to sufficiently small subgroups, we can omit the subscripts. Moreover, recall that $f[\lambda\sigma]_k = f[\sigma]_k$ for any $\lambda > 0$. We have $g[\sigma^{-1}]_k = g[\sigma']_k$ with σ' satisfying $\sigma'\sigma = \det(\sigma)I_2$. Thus we have

$$\langle f[\sigma]_k, g\rangle = \langle f, g[\sigma']_k\rangle \tag{11.13}$$

with σ' as above.

Theorem 11.10. *We have for any $(n, N) = 1$ and for any $f, g \in \mathcal{S}_k(\Gamma_0(N), \chi)$,*

$$\langle T_n^\chi f, g\rangle = \chi(n)\langle f, T_n^\chi g\rangle. \tag{11.14}$$

In particular, T_n^χ is normal on $\mathcal{S}_k(\Gamma_0(N), \chi)$ with respect to the normalized Petersson inner product.

Proof. First note that in view of the relations in Proposition 11.6 it suffices to prove (11.14) for $n = p$ a prime number. We thus assume n is a prime. By definition and applying (11.13) we have

$$\langle T_n^\chi f, g\rangle = n^{\frac{k}{2}-1}\sum_{\rho\in\Delta_n}\overline{\chi}(\rho)\langle f[\rho]_k, g\rangle = n^{\frac{k}{2}-1}\sum_{\rho\in\Delta_n}\chi(n)\chi(\rho')\langle f, g[\rho']_k\rangle,$$

where for $\rho = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \Delta_n$, $\rho' = \left(\begin{smallmatrix} d & -b \\ 0 & a \end{smallmatrix}\right)$ satisfying that $\rho'\rho = nI_2$. Since $f \in \mathcal{S}_k(\Gamma_0(N), \chi)$, we have $f(z) = \overline{\chi}(\tau) f[\tau]_k$ for any $\tau \in \Gamma_0[N]$. We thus have

$$\begin{aligned} \langle T_n^\chi f, g\rangle &= \chi(n) n^{\frac{k}{2}-1} \sum_{\rho\in\Delta_n} \chi(\rho')\langle \overline{\chi}(\tau) f[\tau]_k, \overline{\chi}(\tau') g[\tau'\rho']_k\rangle \\ &= \chi(n) n^{\frac{k}{2}-1} \sum_{\rho\in\Delta_n} \langle f, \overline{\chi}(\tau')\overline{\chi}(\rho')\chi(\tau) g[\tau'\rho'\tau^{-1}]_k\rangle. \end{aligned}$$

Setting $\rho'' = \tau'\rho'\tau^{-1}$ one can show, using the explicit expression of ρ, ρ' that $\overline{\chi}(\tau')\overline{\chi}(\rho')\chi(\tau) = \overline{\chi}(\rho'')$. Hence

$$\langle T_n^\chi f, g\rangle = \chi(n) n^{\frac{k}{2}-1} \sum_{\rho\in\Delta_n} \langle f, \overline{\chi}(\rho'') g[\rho'']_k\rangle,$$

where $\rho'' = \tau'\rho'\tau^{-1}$ with $\tau, \tau' \in \Gamma_0(N)$ to be determined. Now by Exercise 21 we can choose $\tau, \tau' \in \Gamma_0(N)$ such that $\rho'' = \rho$. With the choice of these τ, τ' we get

$$\langle T_n^\chi f, g\rangle = \chi(n) n^{\frac{k}{2}-1} \sum_{\rho\in\Delta_n} \langle f, \overline{\chi}(\rho) g[\rho]_k\rangle = \chi(n)\langle f, T_n^\chi g\rangle$$

as desired.

The in particular part follows easily from (11.14) since it implies that $(T_n^\chi)^* = \overline{\chi}(n) T_n^\chi$ for any $(n, N) = 1$ which clearly commutes with T_n^χ. $\square$

Exercise 21. *Show that if n is squarefree and co-prime to N. Then there exist $\tau, \tau' \in \Gamma_0(N)$ such that $\tau'\rho'\tau^{-1} = \rho$ with $\rho = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \Delta_n$ and $\rho' = \left(\begin{smallmatrix} d & -b \\ 0 & a \end{smallmatrix}\right)$ as above.*

As a direct corollary we have the following

Corollary 11.11. *The space $\mathcal{S}_k(\Gamma_0(N), \chi)$ has a basis of joint eigenfunctions of $\{T_n^\chi : (n, N) = 1\}$.*

Let $f \in \mathcal{S}_k(\Gamma_0(N), \chi)$ be a joint eigenfunction as above, that is, for any $(n, N) = 1$, $T_n^\chi f = \lambda(n) f$ for some $\lambda(n) \in \mathbb{C}$. Then by (3) of Corollary 11.8 we can conclude that

$$\widehat{f}(n) = \widehat{T_n^\chi f}(1) = \lambda(n)\widehat{f}(1), \quad \forall\ (n, N) = 1.$$

The next natural question is whether there exists a multiplicity one theorem, i.e. whether the Hecke eigenvalues determine the eigenfunctions uniquely (up to scalars)? The following simple lemma gives equivalent characterizations of the multiplicity one theorem.

Lemma 11.12. *Let $\lambda : \mathbb{N} \to \mathbb{C}$ be such that*

$$V_\lambda = V_\lambda(\mathcal{S}(\Gamma_0(N), \chi)) := \{f \in \mathcal{S}(\Gamma_0(N), \chi) : T_n^\chi f = \lambda(n) f,\ \forall\ (n, N) = 1\}$$

is nonzero. The following are equivalent.

1. *$\dim V_\lambda = 1$.*
2. *Any $f \in V_\lambda$ is a joint eigenfunction for all Hecke operators.*
3. *If $f \in V_\lambda$ with $\widehat{f}(n) = 0$ for all $(n, N) = 1$, then $f = 0$.*

Proof. We first show (3) $\Rightarrow$ (1). Let f, g be two nonzero elements in V_λ. By (3) we have $\widehat{f}(1) \neq 0$ and $\widehat{g}(1) \neq 0$. Then the function $h = \widehat{f}(1)^{-1} f - \widehat{g}(1)^{-1} g \in V_\lambda$ with $\widehat{h}(1) = 0$ which again by (3) implies that $h = 0$. Hence $f = \lambda g$ for some $\lambda \neq 0$, implying that $\dim V_\lambda = 1$.

Next, we show (1) $\Rightarrow$ (2). Let $f \in V_\lambda$ be nonzero. For any $m \geq 1$ with $(m, N) > 1$, consider the function $g = T_m^\chi f$. For any $(n, N) = 1$, since T_n^χ commutes with T_m^χ we have

$$T_n^\chi g = T_n^\chi T_m^\chi f = T_m^\chi T_n^\chi f = T_m^\chi(\lambda(n) f) = \lambda(n) g.$$

This shows that $g \in V_\lambda$. Hence $g = \lambda(m) f$ for some $\lambda(m) \in \mathbb{C}$, that is $T_m^\chi f = \lambda(m) f$.

Finally, we show that (2) $\Rightarrow$ (3). Since $T_n^\chi = \lambda(n) f$ for any $n \geq 1$, we have by (3) of Corollary 11.8 that $\widehat{f}(n) = \lambda(n)\widehat{f}(1)$ for all $n \geq 1$. Hence $\widehat{f}(1) = 0$ implies that $f = 0$, proving (3). $\square$

The answer is negative in view of the following proposition.

Proposition 11.13. *Let $N, N_1, N_2 \in \mathbb{N}$ be such that $N_1 N_2 \mid N$ and let χ be a Dirichlet character of modulus N induced from some Dirichlet character χ_1 of modulus N_1. Then for any $f \in \mathcal{S}_k(\Gamma_0(N_1), \chi_1)$, the function $g = f[\alpha_{N_2}]_k$ lies in $\mathcal{S}_k(\Gamma_0(N), \chi)$ and satisfies $\widehat{g}(n) = 0$ for all $(n, N) = 1$. Here $\alpha_{N_2} = \left(\begin{smallmatrix} N_2 & \\ & 1 \end{smallmatrix}\right)$.*

Remark 11.15. Since $\chi_1|_{\Gamma_0(N)} = \chi|_{\Gamma_0(N)}$, $f \in \mathcal{S}_k(\Gamma_0(N_1), \chi_1) \subset \mathcal{S}_k(\Gamma_0(N), \chi)$. If f is a joint eigenfunction for $\{T_n^\chi : (n, N) = 1\}$. Then one can show $\tilde{f}$ is also a joint eigenfunction for $\{T_n^\chi : (n, N) = 1\}$ with the same eigenvalues.

Proof of Proposition 11.13. By definition, for $f(z) = \sum_{n=1}^\infty \widehat{f}(n) e(nz)$,

$$\begin{aligned} g(z) = N_2^{\frac{k}{2}} f(N_2 z) &= N_2^{\frac{k}{2}} \sum_{n=1}^\infty \widehat{f}(n) e(n N_2 z) \\ &= N_2^{\frac{k}{2}} \sum_{\substack{n \geq 1 \\ N_2 \mid n}} \widehat{f}\left(n/N_2\right) e(nz), \end{aligned}$$

implying that $\widehat{g}(n) = 0$ whenever $(n, N) = 1$. Next, we show $g \in \mathcal{S}_k(\Gamma_0(N), \chi)$. Since $f \in \mathcal{S}_k(\Gamma_0(N), \chi)$ is holomorphic on $\mathbb{H}$ and vanishes at $\mathbb{Q} \cup \{\infty\}$, so is $g = f[\alpha_{N_2}]_k$. Hence it suffices to prove the desired transformation rule for g. Take any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, note that $\alpha_{N_2}\gamma = \gamma' \alpha_{N_2}$ with $\gamma' = \left(\begin{smallmatrix} a & N_2 b \\ c/N_2 & d \end{smallmatrix}\right) \in \Gamma_0(N_1)$. Thus

$$g[\gamma]_k = f[\alpha_{N_2}\gamma]_k = f[\gamma'\alpha_{N_2}]_k = \chi_1(\gamma') f[\alpha_{N_2}]_k = \chi(\gamma) g.$$

Here for the last equality we used the assumption that χ is induced from χ_1 so that χ_1 and χ agree on the set $\{n \in \mathbb{N} : (n, N) = 1\}$ which in particular, implies that $\chi_1(\gamma') = \chi_1(d) = \chi(d) = \chi(\gamma)$ (since for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, $(d, N) = 1$). $\square$

Definition 11.16. Let $\mathcal{S}_k^{\mathrm{old}}(\Gamma_0(N), \chi)$ be the linear subspace of $\mathcal{S}_k(\Gamma_0(N), \chi)$ spanned by all forms of type $f[\alpha_{N_2}]_k$ with $f \in \mathcal{S}_k(\Gamma_0(N_1), \chi_1)$, $N_1 N_2 \mid N$ and χ_1 a Dirichlet character of modulus N_1 such that $\chi \pmod N$ is induced from χ_1. Let $\mathcal{S}_k^{\mathrm{new}}(\Gamma_0(N), \chi)$ be the orthogonal complement of $\mathcal{S}_k^{\mathrm{old}}(\Gamma_0(N), \chi)$ in $\mathcal{S}_k(\Gamma_0(N), \chi)$ with respect to the normalized Petersson inner product.

Proposition 11.14. *For any $(n, N) = 1$, the Hecke operator T_n^χ preserves the subspaces $\mathcal{S}_k^{\mathrm{old}}(\Gamma_0(N), \chi)$ and $\mathcal{S}_k^{\mathrm{new}}(\Gamma_0(N), \chi)$ respectively.*

Proof. Fix $n \in \mathbb{N}$ such that $(n,N)=1$. First we note that it suffices to prove that T_n^χ preserves $\mathcal{S}_k^{\mathrm{old}}(\Gamma_0(N),\chi)$. This is true since if T_n^χ preserves $\mathcal{S}_k^{\mathrm{old}}(\Gamma_0(N),\chi)$, then for any $f \in \mathcal{S}_k^{\mathrm{new}}(\Gamma_0(N),\chi)$ and $g \in \mathcal{S}_k^{\mathrm{old}}(\Gamma_0(N),\chi)$ we can apply Theorem 11.10 (and noting that by assumption $T_n^\chi g \in \mathcal{S}_k^{\mathrm{old}}(\Gamma_0(N),\chi)$) to get

$$\langle T_n^\chi f, g\rangle = \chi(n)\langle f, T_n^\chi g\rangle = 0.$$

Since g is arbitrary, this shows that $T_n^\chi f \in \mathcal{S}_k^{\mathrm{new}}(\Gamma_0(N),\chi)$.

Now we show $T_n^\chi g \in \mathcal{S}_k^{\mathrm{old}}(\Gamma_0(N),\chi)$ for any $g \in \mathcal{S}_k^{\mathrm{old}}(\Gamma_0(N),\chi)$. Since T_n^χ is linear, without loss of generality we may assume $g = f[\alpha_{N_2}]_k$ with $f \in \mathcal{S}_k(\Gamma_0(N_1),\chi_1)$ such that $N_1N_2 \mid N$ and $\chi_1 \pmod{N_1}$ induces $\chi \pmod{N}$. Then one easily sees that the conditions $(n,N)=1$ and χ_1 inducing χ imply that $T_n^\chi = T_n^{\chi_1}$. Hence $T_n^\chi g = T_n^{\chi_1} f[\alpha_{N_2}]_k = (T_n^{\chi_1} f)[\alpha_{N_2}]_k$. Since $T_n^{\chi_1} f \in \mathcal{S}_k(\Gamma_0(N_1),\chi_1)$, this implies that $T_n^\chi g \in \mathcal{S}_k^{\mathrm{old}}(\Gamma_0(N),\chi)$. $\square$

Theorem 11.15 (Multiplicity one theorem)**.** *The eigenfunction space*

$$V_\lambda^{\mathrm{new}} = \{f \in \mathcal{S}^{\mathrm{new}}(\Gamma_0(N),\chi) : T_n^\chi f = \lambda(n) f,\ \forall\ (n,N)=1\}$$

has dimension at most one.

Proof. We only give the proof for two special cases.

Case I: χ is primitive and N is squarefree. Since χ is assumed to be primitive, $\mathcal{S}_k^{\mathrm{old}}(\Gamma_0(N),\chi) = \{0\}$ and $\mathcal{S}_k^{\mathrm{new}}(\Gamma_0(N),\chi) = \mathcal{S}_k(\Gamma_0(N),\chi)$. In view of Lemma 11.12 take for any $f \in \mathcal{S}_k(\Gamma_0(N),\chi)$, it suffices to show that $\widehat{f}(n)=0$ for all $(n,N)=1$ implies $f=0$. We now assume $\widehat{f}(n)=0$ for all $(n,N)=1$ and we would like to show $f=0$. For any $d \in \mathbb{Z}$ with $(d,N)=1$, let $\gamma_d = \left(\begin{smallmatrix} a & b \\ N & d\end{smallmatrix}\right) \in \Gamma_0(N)$ with $a,b \in \mathbb{Z}$ so that $ad-bN=1$. Then we have

$$\begin{aligned}\chi(d)f(z) = f[\gamma_d]_k(z) &= (Nz+d)^{-k} f\left(\tfrac{az+b}{Nz+d}\right)\\ &= (Nz+d)^{-k} f\left(\tfrac{a}{N} - \tfrac{1}{N(Nz+d)}\right), \quad \forall\ z \in \mathbb{H}.\end{aligned}$$

Making a change of variable $Nz+d \mapsto z$ the above equation is equivalent to

$$\chi(d) f\left(\tfrac{z-d}{N}\right) = z^{-k} f\left(\tfrac{a}{N} - \tfrac{1}{Nz}\right), \quad \forall\ z \in \mathbb{H}.$$

Summing over $a \in (\mathbb{Z}/N\mathbb{Z})^\times$ and writing $f(z) = \sum_{n=1}^\infty \widehat{f}(n)e(nz)$ in Fourier expansion we get

$$\sum_{d \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi(d) \sum_{n=1}^\infty \widehat{f}(n) e\left(n\tfrac{z-d}{N}\right) = \sum_{d \in (\mathbb{Z}/N\mathbb{Z})^\times} z^{-k} \sum_{n=1}^\infty \widehat{f}(n) e\left(n\left(\tfrac{a}{N} - \tfrac{1}{Nz}\right)\right).$$

Further computing the left hand side we get

$$\mathrm{LHS} = \sum_{n=1}^\infty \widehat{f}(n) e\left(\tfrac{nz}{N}\right) \sum_{d \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi(d) e\left(-\tfrac{nd}{N}\right) = \sum_{n=1}^\infty \widehat{f}(n) e\left(\tfrac{nz}{N}\right) G(\chi,-n),$$

with $G(\chi,-n)$ the Gauss sum defined as before. Since χ is primitive, by Exercise 20 we have

$$\mathrm{LHS} = \sum_{n=1}^\infty \widehat{f}(n) e\left(\tfrac{nz}{N}\right) \overline{\chi(-n)} G(\chi) = 0,$$

where for the last equality we used the assumption that $\widehat{f}(n) = 0$ for $(n, N) = 1$ and the fact that $\chi(-n) = 0$ if $(n, N) > 1$. For the right hand side we have

$$\mathrm{RHS} = z^{-k} \sum_{n=1}^{\infty} \widehat{f}(n) e\left(-\tfrac{1}{Nz}\right) \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} e\left(\tfrac{an}{N}\right) = z^{-k} \sum_{n=1}^{\infty} \widehat{f}(n) e\left(-\tfrac{1}{Nz}\right) S(0, n; N).$$

By the multiplicity of the Kloosterman sum (cf. (3) of Exercise 12) and the assumption that N is square-free we have

$$S(0, n, N) = \prod_{p \mid N} S(0, n_p, p)$$

with n_p some integer depending on p and n. Now note that

$$S(0, n_p, p) = \sum_{a \in (\mathbb{Z}/p\mathbb{Z})^\times} e\left(\tfrac{n_p a}{p}\right) = \begin{cases} p-1 & p \mid n_p, \\ -1 & p \nmid n_p. \end{cases}$$

In paritcular, $S(0, n; N)$ never vanishes. Equating left and right hand side we get $\widehat{f}(n) = 0$ for all $n \in \mathbb{N}$, proving that $f = 0$.

Case II: $\chi = 1_N$ is trivial and $N = p$ is a prime. We assume $f \in \mathcal{S}_k^{\mathrm{new}}(\Gamma_0(p))$ with $\widehat{f}(n) = 0$ for all $(n, p) = 1$. We would like to show $f = 0$. To prove this we show $f \in \mathcal{S}_k^{\mathrm{old}}(\Gamma_0(p))$. Since p is a prime by assumption we have

$$f(z) = \sum_{n=1}^{\infty} \widehat{f}(np) e(npz).$$

Define

$$g(z) := \sum_{n=1}^{\infty} \widehat{f}(np) e(nz)$$

so that $f(z) = g(pz)$, or equivalently, $f = p^{-\frac{k}{2}} g[\alpha_p]_k$. To show $f \in \mathcal{S}_k^{\mathrm{old}}(\Gamma_0(p))$, it suffices to show $g \in \mathcal{S}_k(\Gamma_0(1))$. First since $f \in \mathcal{S}_k(\Gamma_0(p))$ we have $g = p^{\frac{k}{2}} f[\alpha_p^{-1}]_k \in \mathcal{S}_k(\alpha_p \Gamma_0(p) \alpha_p^{-1})$. By direct computation we have

$$\alpha_p \Gamma_0(p) \alpha_p^{-1} = \Gamma^0(p) := \left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \left(\begin{smallmatrix} * & 0 \\ * & * \end{smallmatrix}\right) \pmod{p}\right\}.$$

Moreover, in view of the expression $g(z) =$ we have $g(z+1) = g(z)$ for any $z \in \mathbb{H}$, or equivalently, $g[\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)]_k = g$. Hence $g \in \mathcal{S}_k(\Gamma)$ where $\Gamma = \left\langle \Gamma^0(p), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \right\rangle$. Hence we can conclude the proof by claiming that $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. In view of Theorem 3.1 it suffices to show $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \Gamma$. This is true since

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1-p & -p \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1-p & 0 \end{pmatrix}. \qquad \square$$

Remark 11.17. For the general case, define the operators on $\mathcal{S}_k(\Gamma_0(N), \chi)$

$$A_N f = \frac{1}{N} \sum_{b \in \mathbb{Z}/N\mathbb{Z}} f[\begin{pmatrix} 1 & \frac{b}{N} \\ 0 & 1 \end{pmatrix}]_k \quad \text{and} \quad K_N f = \sum_{d \mid N} \mu(d) A_d f.$$

Then one can check that

$$A_N f(z) = \sum_{N \mid n} \widehat{f}(n) e(nz)$$

and

$$K_N f(z) = \sum_{(n,N)=1} \widehat{f}(n)e(nz)$$

so that

$$\ker K_N = \left\{ f \in \mathcal{S}_k(\Gamma_0(N), \chi) : \widehat{f}(n) = 0 \ \forall \ (n, N) = 1 \right\}.$$

Then one show $\ker K_N \subset \mathcal{S}_k^{\mathrm{old}}(\Gamma_0(N), \chi)$. See also [DS05, Section 5.7] for an algebraic proof which is due to D. Carlton [Car01]

Definition 11.18. A nonzero joint eigenfunction $f \in \mathcal{S}_k^{\mathrm{new}}(\Gamma_0(N), \chi)$ for $\{T_n^\chi : (n, N) = 1\}$ is called a *Hecke new form.*

We have the following immediate corollary of Theorem 11.15 and Lemma 11.12.

Corollary 11.16. (1) *Any Hecke new form can be normalized.*
(2) *A normalized Hecke new form $f \in \mathcal{S}_k^{\mathrm{new}}(\Gamma_0(N), \chi)$ is a joint eigenfunction for all Hecke operators with the n-th Hecke eigenvalue being the n-th Fourier coefficient of f, i.e. $T_n^\chi f = \lambda(n)$ with $\lambda(n) = \widehat{f}(n)$ for any $n \geq 1$. In particular, $\widehat{n}$ satisfies*

$$\widehat{f}(m)\widehat{f}(n) = \sum_{d|(m,n)} \chi(d) d^{k-1} \widehat{f}(mnd^{-2}), \quad \forall \ m, n \geq 1. \tag{11.19}$$

12. Review on Riemann zeta function

In the next section we will study Hecke L-functions. Namely we attach an L-function to each cusp form via its Fourier coefficients and then study its analytic properties. Before doing so, we first give a brief review on the classical theory of Riemann zeta function which our later proof on Hecke L-functions resembles. Recall that the Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \mathfrak{Re}(s) > 1.$$

It has an Euler's product formula

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}, \quad \mathfrak{Re}(s) > 1 \tag{12.1}$$

which follows from the fundamental theorem of arithmetic. We note that based on this infinite product formula, Euler gave an alternative proof of Euclid's theorem on infinitude of primes.

We now sketch a proof of the analytic continuation and functional equation of $\zeta(s)$; see [SS03] for more details. Recall that the Gamma function is defined by the following integral

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt, \quad \mathfrak{Re}(s) > 0$$

with an analytic continuation to the whole s-plane. Consider also the theta function

$$\theta(t) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t}, \quad t > 0.$$

We also define

$$\psi(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 t}, \quad t > 0$$

so that $\theta(t) = 1 + 2\psi(t)$. Note that $\psi(t) \ll e^{-\pi t}$ has exponential decay in t. Then for $\mathfrak{Re}(s) > 1$ we can cmopute

$$\begin{aligned}
\int_0^{\infty} \psi(t) t^{\frac{s}{2}-1}\, dt &= \int_0^{\infty} \sum_{n=1}^{\infty} e^{-\pi n^2 t} t^{\frac{s}{2}-1}\, dt \\
&= \sum_{n=1}^{\infty} \int_0^{\infty} e^{-\pi n^2 t} t^{\frac{s}{2}-1}\, dt \\
&\overset{\pi n^2 t \mapsto t}{=} \sum_{n=1}^{\infty} (\pi n^2)^{-\frac{s}{2}} \int_0^{\infty} e^{-t} t^{\frac{s}{2}-1}\, dt \\
&= \pi^{-\frac{s}{2}} \Gamma(\tfrac{s}{2}) \zeta(s) =: \xi(s).
\end{aligned}$$

Here $\xi(s)$ is called the *completed Riemann zeta function*. We split the above integral into integration over $(0,1)$ and $(1,\infty)$ respectively to get

$$\xi(s) = \int_0^1 \psi(t) t^{\frac{s}{2}-1}\, dt + \int_1^{\infty} \psi(t) t^{\frac{s}{2}-1}\, dt.$$

Now by the Poisson summation formula one can show θ satisfies the following inversion formula

$$\theta(t) = t^{-\frac{1}{2}} \theta(\tfrac{1}{t}), \quad t > 0,$$

which in terms of ψ is equivalent to

$$\psi(t) = t^{-\frac{1}{2}} \psi(\tfrac{1}{t}) + \frac{1}{2t^{1/2}} - \frac{1}{2}, \quad t > 0.$$

Applying this formula to the above first integral and then making a change of variable $1/t \mapsto t$ we get

$$\xi(s) = -\frac{1}{1-s} - \frac{1}{s} + \int_1^{\infty} \psi(t) \left(t^{\frac{1-s}{2}} + t^{\frac{s}{2}} \right) \frac{dt}{t}. \tag{12.2}$$

Since ψ decays exponentially as $t \to \infty$, the above integral is absolutely convergent for any $s \in \mathbb{C}$ and defines an entire function. We thus get an analytic continuation of $\xi(s)$ (hence also $\zeta(s)$) to the whole s-plane with two simple poles at $s = 0, 1$ (while $\zeta(s) = \frac{\pi^{\frac{s}{2}} \xi(s)}{\Gamma(s/2)}$ has only one simple pole at $s = 1$ with the simple pole coming from $\xi(s)$ at $s = 0$ cancelled out by the simple zero of $1/\Gamma(s/2)$ at $s = 0$). Moreover, inspecting the above expression one sees that it is invariant after changing s to $1-s$. We thus get the following functional equation

$$\xi(1-s) = \xi(s).$$

13. Hecke L-functions

For each cusp form we can associate an L-function via its Fourier coefficients.

Definition 13.1. For any $f \in \mathcal{S}_k(\Gamma_0(N), \chi)$ the associated *Hecke L-function* is given by

$$L_f(s) = \sum_{n=1}^{\infty} \frac{\widehat{f}(n)}{n^s}$$

for $s \in \mathbb{C}$ as long as the defining series is absolutely convergent. Here $\widehat{f}(n)$ is the n-th Fourier coefficient of f. We also denote by

$$\Lambda_f(s) := \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s) L_f(s), \quad s \in \mathbb{C}$$

the *completed L-function associated to f*.

Remark 13.2. We have shown in Corollary 8.11 that $|\widehat{f}(n)| \ll n^{\frac{k}{2}}$, hence the above defining series for $L_f(s)$ is absolutely convergent for $\mathfrak{Re}(s) > \frac{k}{2} + 1$.

Below we establish some general analytic properties of Hecke L-functions.

13.1. **Analytic continutation.** Using similar arguments as for $\zeta(s)$ we can prove the analytic continuation of $L_f(s)$ via an integral representation of it.

Proposition 13.1. *For any $f \in \mathcal{S}_k(\Gamma_0(N), \chi)$, $L_f(s)$ has an analytic continuation to an entire function.*

Proof. Consider the integral

$$I_f(s) = \int_0^{\infty} f(iy) y^{s-1}\, dy, \quad s \in \mathbb{C}.$$

Note that since f is a cusp form, it decays exponentially at cusps. More precisely, the exponential decay at ∞ and 0 means respectively that $|f(iy)| \ll e^{-cy}$ as $y \to \infty$ and $|f(iy)| \ll e^{-c/y}$ as $y \to 0^+$ for some positive constant c. In particular, this implies that the defining integral for $I_f(s)$ is absolutely convergent for any $s \in \mathbb{C}$, thus $I_f(s)$ is an entire function. Next, we relate this integral with $L_f(s)$. Writing $f(iy)$ in Fourier expansion we have

$$\begin{aligned} I_f(s) &= \int_0^{\infty} \sum_{n=1}^{\infty} \widehat{f}(n) e^{-2\pi n y} y^{s-1}\, dy \\ &\overset{2\pi n y \mapsto y}{=} \sum_{n=1}^{\infty} \widehat{f}(n)(2\pi n)^{-s} \int_0^{\infty} e^{-y} y^{s-1}\, dy \\ &= (2\pi)^{-s} \Gamma(s) L_f(s). \end{aligned}$$

Thus

$$L_f(s) = \frac{(2\pi)^s}{\Gamma(s)} I_f(s).$$

Recall that $1/\Gamma(s)$ is entire, see e.g. [SS03, p. 165, Theorem 1.6], thus the above right hand side is also entire, giving the analytic continuation of $L_f(s)$ to an entire function. $\square$

Remark 13.3. We can also attach an L-function to a modular form $f \in \mathcal{M}_k(\Gamma_0(N), \chi)$. It can be shown that its Fourier coefficients also satisfy a bound $|\widehat{f}(n)| \ll n^{\alpha}$ for some $\alpha > 0$. Thus $L_f(s)$ is absolutely convergents for $\mathfrak{Re}(s) > \alpha + 1$. Using similar argument one can show $L_f(s)$ also has an analytic continuation but potentially with simple poles (due to the lack of exponential decay of f at cusps).

Remark 13.4. It is also clear from the above proof that $\Lambda_f(s) = N^{\frac{s}{2}} I_f(s)$ has an analytic continuation to an entire function. Moreover, using the exponential decay of $f(iy)$ as $y \to 0^+, \infty$, we see that $\Lambda_f(s)$ is bounded on every vertical strip $\sigma_1 < \mathfrak{Re}(s) < \sigma_2$ with the bounding constant depending on σ_1 and σ_2.

13.2. **Functional equation.** In this section we prove the functional equation satisfied by $\Lambda_f(s)$. We first prove it for the case when $N = 1$.

Theorem 13.2. *For $f \in \mathcal{S}_k(\Gamma_0(1))$ the completed L-function $\Lambda_f(s)$ satisfies*

$$\Lambda_f(s) = i^k \Lambda_f(k-s). \tag{13.5}$$

Proof. As before we have for $\mathfrak{Re}(s) > \frac{k}{2} + 1$,

$$\begin{aligned}\Lambda_f(s) = (2\pi)^{-s}\Gamma(s)L_f(s) &= \int_0^\infty f(iy)y^{s-1}\,dy\\ &= \int_0^1 f(iy)y^{s-1}\,dy + \int_1^\infty f(iy)y^{s-1}\,dy.\end{aligned}$$

Let $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Since $f \in \mathcal{S}_k(\Gamma_0(1))$ we have $f[S]_k = f$ which is equivalent to $f(z) = (-z)^{-k} f(-1/z)$ for any $z \in \mathbb{H}$. Taking $z = iy$ we get

$$f(iy) = (-iy)^{-k} f\left(-1/iy\right) = i^k y^{-k} f(i/y).$$

Hence

$$\begin{aligned}\Lambda_f(s) &= \int_0^1 i^k y^{-k} f(i/y) y^{s-1}\,dy + \int_1^\infty f(iy)y^{s-1}\,dy\\ &= \int_1^\infty f(iy)\left(y^s + i^k y^{k-s}\right)\frac{dy}{y},\end{aligned}$$

from which the functional equation (13.5) follows easily. Here in the second line we made a change of variable $\frac{1}{y} \mapsto y$ in the first integral of the first line. □

The case for general N is slightly more involved since the transformation $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ does not lie in $\Gamma_0(N)$ when $N > 1$. Instead, we use the transformation $w_N := \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$. Note that w_N also does not lie in $\Gamma_0(N)$, but it normalizes $\Gamma_0(N)$, i.e. $w_N^{-1}\Gamma_0(N)w_N = \Gamma_0(N)$. We will see later (Lemma 15.1) that for $f \in \mathcal{S}_k(\Gamma_0(N),\chi)$, $g := f[w_N]_k \in \mathcal{S}_k(\Gamma_0(N),\bar{\chi})$ is also a cusp form. Hence we can also associate to it an L-function $L_g(s)$ and its completion $\Lambda_g(s)$ which both can be analytically continued to the whole s-plane. We now state the functional equation for cusp forms of a general level.

Theorem 13.3. *Let $f \in \mathcal{S}_k(\Gamma_0(N),\chi)$ and $g = f[w_N]_k \in \mathcal{S}_k(\Gamma_0(N),\bar{\chi})$ be as above. Then we have*

$$\Lambda_f(s) = i^k \Lambda_g(k-s). \tag{13.6}$$

Proof. Similar as the $N = 1$ case we have

$$\Lambda_f(s) = N^{\frac{s}{2}} \int_0^\infty f(iy)y^{s-1}\,dy, \quad s \in \mathbb{C},$$

and

$$\Lambda_g(s) = N^{\frac{s}{2}} \int_0^\infty g(iy)y^{s-1}\,dy, \quad s \in \mathbb{C}.$$

Moreover, note that $w_N = -NI_2$, thus the relation $g = f[w_N]_k$ is equivalent to $(-1)^k f = f[w_N^2]_k = g[w_N]_k$, or equivalently, $f = (-1)^k g[w_N]_k$. For $z = iy \in \mathbb{H}$ we have

$$f(iy) = (-1)^k N^{\frac{k}{2}} (Niy)^{-k} g\left(\tfrac{i}{Ny}\right) = i^k N^{-\frac{k}{2}} y^{-k} g\left(\tfrac{i}{Ny}\right).$$

Plugging in this relation into the above expression of $\Lambda_f(s)$ we have

$$\begin{aligned}\Lambda_f(s) &= i^k N^{\frac{s-k}{2}} \int_0^\infty g\left(\tfrac{i}{Ny}\right) y^{s-k-1}\, dy \\ &\overset{1/Ny \mapsto y}{=} i^k N^{\frac{k-s}{2}} \int_0^\infty g(iy)\, y^{k-s-1}\, dy = i^k \Lambda_g(k-s).\end{aligned}$$

This finishes the proof. □

13.3. **Euler's product.** In this section we prove the Euler product formula of $L_f(s)$. For this we need to further assume f to be a normalized Hecke new form. We first give a general criterion on when an L-function has an Euler product.

Proposition 13.4. *Let $L(s) = \sum_{n=1}^\infty \frac{a(n)}{n^s}$ be absolute convergent for $\mathfrak{Re}(s) > \sigma$ for some $\sigma > 0$ and satisfying $a(mn) = a(m)a(n)$ for any $(m,n) = 1$. Then we have*

$$L(s) = \prod_p \left(\sum_{j=0}^\infty \frac{a(p^j)}{p^{js}}\right), \quad \forall\, \mathfrak{Re}(s) > \sigma.$$

Proof. For any integer $M \geq 2$ define

$$P_M := \{p \in \mathbb{N} : p \text{ is a prime and } p \leq M\}$$

and

$$A_M := \{p_1^{\alpha_1} \cdots p_k^{\alpha_k} \in \mathbb{N} : p_i \in P_M,\ 0 \leq \alpha_i \leq M\}.$$

Then by the fundamental theorem of arithmetic and the multiplicity of $\{a(n)\}_{n\in\mathbb{N}}$ we have

$$\prod_{p\in P_M} \left(\sum_{j=0}^M \frac{a(p^j)}{p^{js}}\right) = \sum_{n\in A_M} \frac{a(n)}{n^s}.$$

By Exercise 22 we have $\{1, 2, \ldots, M\} \subset A_M$. Hence for $\mathfrak{Re}(s) > \sigma$

$$\left| L(s) - \prod_{p\in P_M} \left(\sum_{j=0}^M \frac{a(p^j)}{p^{js}}\right)\right| \leq \sum_{n=M+1}^\infty \frac{|a(n)|}{n^{\mathfrak{Re}(s)}} \to 0, \quad \text{as } M \to \infty.$$

Similarly, one can show

$$\lim_{M\to\infty} \prod_{p\in P_M} \left(\sum_{j=0}^M \frac{a(p^j)}{p^{js}}\right) = \prod_p \left(\sum_{j=0}^\infty \frac{a(p^j)}{p^{js}}\right), \quad \forall\, \mathfrak{Re}(s) > \sigma.$$

Combining these two limiting equations we get the desired identity. □

Exercise 22. *Let P_M and A_M be as above. Show that $\{1, 2, \ldots, M\} \subset A_M$.*

With this proposition we can now prove the Euler's product formula for $L_f(s)$ when f is a normalized Hecke new form.

Theorem 13.5. *Suppose* $f \in \mathcal{S}_k^{\mathrm{new}}(\Gamma_0(N), \chi)$ *is a normalized Hecke new form. Then for any* $\mathfrak{Re}(s) > \frac{k}{2} + 1$

$$L_f(s) = \prod_p \left(1 - \widehat{f}(p)p^{-s} + \chi(p)p^{k-1-2s}\right)^{-1}. \tag{13.7}$$

Proof. Note that since f is a normalized Hecke new form, its Fourier coefficients, being the Hecke eigenvalues, satisfy the relation (11.19) which is equivalent to the following two relations

$$\widehat{f}(mn) = \widehat{f}(m)\widehat{f}(n), \quad \forall\ (m,n) = 1$$

and

$$\widehat{f}(p^{r+1}) = \widehat{f}(p)\widehat{f}(p^r) - \chi(p)p^{k-1}\widehat{f}(p^{r-1}), \quad p \text{ a prime, } r \geq 1.$$

In view of Remark 13.2, Proposition 13.4 and the above first relation we have for $\mathfrak{Re}(s) > \frac{k}{2} + 1$,

$$L_f(s) = \prod_p \phi_p(f,s),$$

where $\phi_p(f,s) := \sum_{r=0}^{\infty} \frac{\widehat{f}(p^r)}{p^{rs}}$. It thus remains to compute $\phi_p(f,s)$. For this we apply the above second relation. We have

$$\begin{aligned}
\phi_p(f,s) &= 1 + \widehat{f}(p)p^{-s} + \sum_{r=2}^{\infty} \widehat{f}(p^r)p^{-rs} \\
&= 1 + \widehat{f}(p)p^{-s} + \sum_{r=1}^{\infty} \widehat{f}(p^{r+1})p^{-(r+1)s} \\
&= 1 + \widehat{f}(p)p^{-s} + \sum_{r=1}^{\infty} \left(\widehat{f}(p)\widehat{f}(p^r) - \chi(p)p^{k-1}\widehat{f}(p^{r-1})\right)p^{-(r+1)s} \\
&= 1 + \widehat{f}(p)p^{-s} + \widehat{f}(p)p^{-s}\sum_{r=1}^{\infty} \widehat{f}(p^r)p^{-rs} - \chi(p)p^{k-1-2s}\sum_{r=1}^{\infty} \widehat{f}(p^{r-1})p^{-(r-1)s} \\
&= 1 + \widehat{f}(p)p^{-s}\phi_p(f,s) - \chi(p)p^{k-1-2s}\phi_p(f,s).
\end{aligned}$$

Solving this equation for $\phi_p(f,s)$ one easily gets the desired formula for $\phi_p(f,s)$. □

14. Hecke's converse theorem

The functional equation (13.6) satisfied by Hecke L-functions is a consequence of the modularity of the corresponding cusp form. The following theorem of Hecke shows that when $N = 1$ the converse is also true, that is the functional equation (13.5) actually also encodes the modularity.

Theorem 14.1 (Hecke)**.** *Let* $L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ *with* $|a_n| \ll n^{\alpha}$ *for some* $\alpha > 0$ *(so that* $L(s)$ *converges absolutely for* $\mathfrak{Re}(s) > \alpha + 1$*). Assume* $L(s)$ *has an analytic continuation to an entire function and* $\Lambda(s) := (2\pi)^{-s}\Gamma(s)L(s)$ *is bounded on every vertical strip and satisfies* $\Lambda(s) = i^k\Lambda(k-s)$*. Then* $f(z) := \sum_{n=1}^{\infty} a_n e(nz) \in \mathcal{S}_k(\Gamma_0(1))$*.*

To prove this theorem we need some preliminary results. We first recall the Stirling's approximation formula for gamma functions (see e.g. Corollary 16 of this online note) that for any fixed $\sigma \in \mathbb{R}$,

$$|\Gamma(\sigma + it)| \sim \sqrt{2\pi}|t|^{\sigma - \frac{1}{2}} e^{-\pi|t|/2}, \quad \text{as } |t| \to \infty. \tag{14.1}$$

In particular, this implies that $|\Gamma(s)|$ decays exponentially on any vertical line with $|\mathfrak{Im}(s)| \to \infty$. Next, we recall the Phragmén-Lindelöf principle; see e.g. [Lan70, p. 262].

Proposition 14.2. *Let $f(s)$ be a function that is holomorphic in a strip $\sigma_1 \leq \mathfrak{Re}(s) \leq \sigma_2$ and satisfying $|f(s)| \ll e^{|s|^A}$ for all $\sigma_1 \leq \mathfrak{Re}(s) \leq \sigma_2$ and for some $A > 0$. Suppose that $|f(s)| \ll |s|^M$ for $\mathfrak{Re}(s) = \sigma_1, \sigma_2$ and for some $M \in \mathbb{R}$, then $|f(s)| \ll |s|^M$ uniformly for all $\sigma_1 \leq \mathfrak{Re}(s) \leq \sigma_2$.*

Remark 14.2. Let f be satisfying above conditions. When $M = 0$ Phragmén-Lindelöf principle asserts that if f is bounded on the two edges of a vertical strip then it is also bounded on this strip. This is a generalization of the maximum modulus principle (see e.g. [SS03, p. 92]) except here the region is no longer assumed to be bounded. As a compensation of this relax of condition, the growth condition $|f(s)| \ll e^{|s|^A}$ is necessary. For example, consider the function $f(s) = e^{e^{is}}$ which is holomorphic on the strip $\frac{\pi}{2} \leq \mathfrak{Re}(s) \leq \frac{5\pi}{2}$ with absolute value bounded by 1 for $\mathfrak{Re}(s) = \frac{\pi}{2}, \frac{2\pi}{2}$ but $|f(2\pi + it)| = e^{e^{-t}} \to \infty$ as $t \to -\infty$.

Finally, we introduce the Mellin transform of a function defined on the set of positive real numbers. Given a continuous function $\phi : (0, \infty) \to \mathbb{C}$, its *Mellin transform* is defined by

$$\mathcal{M}(\phi)(s) = \int_0^\infty \phi(y) y^s \, \frac{dy}{y}$$

whenever this integral is absolutely convergent. One can show that there exist constants $-\infty \leq \sigma_1 < \sigma_2 \leq \infty$ such that the above integral is absolutely convergent for any $\sigma_1 < \mathfrak{Re}(s) < \sigma_2$ and is divergent for $\mathfrak{Re}(s) < \sigma_1$ or $\mathfrak{Re}(s) > \sigma_2$. We have the *Mellin inversion formula* which states that for any $\sigma \in (\sigma_1, \sigma_2)$

$$\phi(y) = \frac{1}{2\pi i} \int_{(\sigma)} \mathcal{M}(\phi)(s) y^{-s} \, ds, \quad y > 0. \tag{14.3}$$

We note that $\mathcal{M}(\phi)(s)$ can be interpreted as a Fourier transform of the function $\psi(r) := \phi(e^r) : \mathbb{R} \to \mathbb{C}$ and Mellin inversion formula follows from the more classical Fourier inversion formula; see e.g. [Bum97, p. 55-56]. We can now give the

Proof of Theorem 14.1. First we show that f is holomorphic and vanishes at ∞. For any $z = x + iy \in \mathbb{H}$ using the bound $|a_n| \ll n^\alpha$ we have

$$\begin{aligned} |f(x+iy)| &\ll \sum_{n=1}^\infty n^\alpha e^{-2\pi n y} \leq \sum_{n=1}^\infty \int_n^{n+1} t^\alpha e^{-\pi t y} \, dt \\ &= \int_1^\infty t^\alpha e^{-\pi t y} \, dt \overset{ty \mapsto t}{=} y^{-(\alpha+1)} \int_y^\infty t^\alpha e^{-\pi t} \, dt \ll y^{-(\alpha+1)}. \end{aligned}$$

In particular, this shows that the defining series of f converges absolutely and uniformly on compact sets, hence f is holomorphic on $\mathbb{H}$. Moreover, $f(iy) \to 0$ as $y \to \infty$.

Next, we establish the modularity of f. The relation $f(z+1) = f(z)$ is clear. We thus only need to show $f[S]_k = f$ with $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, i.e. $f(z) = (-z)^{-k} f(-1/z)$ for any $z \in \mathbb{H}$. In view of analytic continuation, it suffices to prove this identity for $z = iy$ for all $y > 0$, that is,

$$f(iy) = (-iy)^{-k} f(i/y) = i^k y^{-k} f(i/y), \quad \forall\, y > 0. \tag{14.4}$$

For this note that for $\mathfrak{Re}(s) > \alpha + k$, using the arguments as in Proposition 13.1 we have

$$\Lambda(s) = \int_0^\infty f(iy) y^{s-1}\, dy.$$

Note that $\Lambda(s)$ is the Mellin transform of the function $y \mapsto f(iy)$. Moreover, using the Stirling's approximation formula for the gamma function (14.1) and the fact that $L(s)$ is uniformly bounded for $\mathfrak{Re}(s) \geq \alpha + k$, we have

$$|\Lambda(s)| \ll |s|^{-2}, \quad \forall\, \mathfrak{Re}(s) \geq \alpha + k. \tag{14.5}$$

Hence we can apply the Mellin inversion formula (14.3) to get for any $\sigma > \alpha + k$,

$$f(iy) = \frac{1}{2\pi i} \int_{(\sigma)} \Lambda(s) y^{-s}\, ds, \quad y > 0.$$

Next, we want to shift the contour from $\mathfrak{Re}(s) = \sigma$ to $\mathfrak{Re}(s) = \frac{k}{2}$ for which we need to control the growth of $\Lambda(s)$ on vertical strips. To achieve this we apply the Phragmén-Lindelöf principle. By the functional equation and the bound (14.5) we have

$$|\Lambda(s)| = |\Lambda(k-s)| \ll |k-s|^{-2} \asymp_k |s|^{-2}, \quad \forall\, \mathfrak{Re}(s) \leq -\alpha.$$

Thus by the Phragmén-Lindelöf principle we can conclude that for any $-\alpha < \sigma < k + \alpha$,

$$|\Lambda(s)| \ll |s|^{-2}, \quad \forall\, \mathfrak{Re}(s) = \sigma.$$

Thus we can shift the contour and apply the functional equation to get that for any $y > 0$,

$$\begin{aligned}
f(iy) &= \frac{1}{2\pi i} \int_{(\frac{k}{2})} \Lambda(s) y^{-s}\, ds = \frac{1}{2\pi i} \int_{(\frac{k}{2})} i^k \Lambda(k-s) y^{-s}\, ds \\
&\overset{k-s \mapsto s}{=} \frac{1}{2\pi i} \int_{(\frac{k}{2})} i^k \Lambda(s) y^{-(k-s)}\, ds = \frac{i^k y^{-k}}{2\pi i} \int_{(\frac{k}{2})} \Lambda(s) (1/y)^{-s}\, ds \\
&= i^k y^{-k} f(i/y).
\end{aligned}$$

This proves the identity (14.4) and hence also the theorem. □

Remark 14.6. If in Theorem 14.1 we replace the assumption that $\Lambda(s)$ has an analytic continuation to an entire function with the weaker assumption that $\Lambda(s)$ has a meromorphic continuation with two potential simple poles at $s = 0, 1$ and $a_0 \in \mathbb{C}$ is such that

$$\Lambda(s) + a_0(s^{-1} + i^k (k-s)^{-1})$$

entire, then one can similarly show $f(z) := \sum_{n=0}^\infty a_0 e(nz) \in \mathcal{M}_k(\Gamma_0(1))$, cf. Theorem 17.1 below.

15. Fricke involution

In Theorem 13.3 in order to state the functional equation of the L-function of a cusp form $f \in \mathcal{S}_k(\Gamma_0(N), \chi)$, we need to define a new function $g := f[w_N]_k$ with $w_N = \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$. In this section we study this map more closely which in turn would give us a more refined functional equation than (13.6) when f is a normalized Hecke new form.

Definition 15.1. The *Fricke involution* W_N is an operator on $\mathcal{S}_k(\Gamma_0(N), \chi)$ defined by

$$W_N f = f[w_N]_k, \quad \forall\ f \in \mathcal{S}_k(\Gamma_0(N), \chi),$$

where $w_N = \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$ is as above.

Lemma 15.1. *For any $f \in \mathcal{S}_k(\Gamma_0(N), \chi)$, $W_N f \in \mathcal{S}_k(\Gamma_0(N), \overline{\chi})$.*

Proof. The proof uses the fact that w_N normalizes $\Gamma_0(N)$. Indeed, by direction computation for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, we have

$$w_N \gamma w_N^{-1} = \begin{pmatrix} d & c/N \\ -bN & a \end{pmatrix} =: \gamma' \in \Gamma_0(N),$$

or equivalently, $w_N\gamma = \gamma' w_N$. Now take $f \in \mathcal{S}_k(\Gamma_0(N), \chi)$, let $g = W_N f$. Then

$$g[\gamma]_k = f[w_N\gamma]_k = f[\gamma' w_N]_k = \chi(a) f[w_N]_k = \bar{\chi}(\gamma) g.$$

The fact that g is holomorphic and vanishes at cusps follows from the same arguments as in Proposition 9.3. $\square$

Next, we explore the relations between W_N and Hecke operators. Below when there is no ambiguity we abbreviate W_N and w_N by W and w respectively.

Proposition 15.2. *For $(n, N) = 1$ we have*

$$W_N T_n^{\chi} = \chi(n) T_n^{\bar{\chi}} W_N.$$

Proof. Note that by definition for $f \in \mathcal{S}_k(\Gamma_0(N), \chi)$,

$$W T_n^{\chi} f = n^{\frac{k}{2}-1} \sum_{\rho \in \Delta_n} \bar{\chi}(\rho) f[\rho w]_k$$

and

$$T_n^{\bar{\chi}} W f = n^{\frac{k}{2}-1} \sum_{\rho \in \Delta_n} \chi(\rho) f[w\rho]_k.$$

Note that for $\rho = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \Delta_n$, $w\rho w^{-1} = \left(\begin{smallmatrix} d & 0 \\ -Nb & a \end{smallmatrix}\right) \notin \Delta_n$. To transform it into Δ_n we left multiply an element in $\Gamma_0(N)$ and use the modularity of f with respect to $\Gamma_0(N)$. More precisely, let $\tau = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(N)$ to be determined, we have by direct computation

$$\rho' := \tau^{-1} w\rho w^{-1} = \begin{pmatrix} \delta d + \beta bN & -\beta a \\ -\gamma d - \alpha bN & \alpha a \end{pmatrix}.$$

To make $\rho' \in \Delta_n$ we need $-\gamma d - \alpha bN = 0$ and $0 \le -\beta a < \alpha a$, i.e. $-\alpha < \beta \le 0$. For this we take $\alpha = \frac{d}{(b,d)}$ and $\gamma = -\frac{bN}{(b,d)}$. One easily sees that $N \mid \gamma$ and $(\alpha, \gamma) = 1$ (since $ad = n$ and $(n, N) = 1$). Next, we choose β to make $\tau \in \Gamma_0(N)$, i.e. $\alpha\delta - \beta\gamma = 1$. One necessary condition is that $\beta\gamma \equiv -1 \pmod{\alpha}$. We choose β to be the unique integer $-\alpha < \beta \le 0$ satisfying this congruence condition. Once β is chosen we can find the unique $\delta \in \mathbb{Z}$ such that $\tau \in \Gamma_0(N)$. With the choice of this τ we have $\rho' = \begin{pmatrix} (b,d) & -\beta a \\ 0 & n/(b,d) \end{pmatrix} \in \Delta_n$ with $-\alpha < \beta \leq 0$ the unique integer such that $\beta \frac{bN}{(b,d)} \equiv 1 \pmod{\frac{d}{(b,d)}}$. This map from $\rho\Delta_n$ to $\rho' \in \Delta_n$ is a bijection as one can easily write down the inverse map. Thus we have

$$f[w\rho]_k = f[\tau\rho' w]_k = \chi(\tau) f[\rho' w]_k$$

and

$$\chi(\rho)\chi(\tau) = \overline{\chi}(a)\overline{\chi}(\alpha) = \overline{\chi}(n)\chi((b,d)) = \overline{\chi}(n)\overline{\chi}(\rho').$$

Hence

$$\begin{aligned} T_n^{\bar{\chi}} W f &= n^{\frac{k}{2}-1} \sum_{\rho \in \Delta_n} \chi(\rho)\chi(\tau) f[\rho' w]_k \\ &= \overline{\chi}(n) n^{\frac{k}{2}-1} \sum_{\rho' \in \Delta_n} \overline{\chi}(\rho') f[\rho' w]_k = \overline{\chi}(n) W T_n^{\chi} f \end{aligned}$$

as desired. □

We also need another operator which brings $W_N f$ back to $\mathcal{S}_k(\Gamma_0(N), \chi)$. Let K be such that $Kf(z) = \overline{f(-\overline{z})}$. We list some simple properties of K.

Proposition 15.3. *Let K be as above. We have*

1. $K^2 = 1$ *and* $K\lambda f = \bar{\lambda} K f$.
2. $WK = (-1)^k KW$.
3. $K : \mathcal{S}_k(\Gamma_0(N), \chi) \to \mathcal{S}_k(\Gamma_0(N), \overline{\chi})$.
4. $Kf(z) = \sum \overline{\widehat{f}(n)} e(nz)$.
5. $KT_n^{\chi} = T_n^{\bar{\chi}} K$ *for all* $n \geq 1$.

Proof. (1) is clear. For (2) note that by definition $Wf(z) = N^{\frac{k}{2}}(Nz)^{-k} f(-1/Nz)$. Hence

$$KWf(z) = N^{\frac{k}{2}} \overline{(-N\overline{z})}^{-k} K(f(-1/Nz)) = (-1)^k N^{\frac{k}{2}} (Nz)^{-k} \overline{f(-1/N\overline{z})}.$$

Similarly,

$$WKf(z) = N^{\frac{k}{2}}(Nz)^{-k} Kf(-1/Nz) = N^{\frac{k}{2}}(Nz)^{-k} \overline{f(-1/N\overline{z})}.$$

Then (2) follows by comparing these equations.

For (3), take any $f \in \mathcal{S}_k(\Gamma_0(N), \chi)$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, we have

$$(Kf)[\gamma]_k = j_\gamma(z)^{-k} Kf(\gamma z) = j_\gamma(z)^{-k} \overline{f(-\gamma\overline{z})}.$$

Note that

$$-\gamma\overline{z} = -\frac{a\overline{z} + b}{c\overline{z} + d} = \frac{a(-\overline{z}) - b}{-c(-\overline{z}) + d} = \tilde{\gamma}(-\overline{z}),$$

where $\tilde{\gamma} = \left(\begin{smallmatrix} a & -b \\ -c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Moreover, one sees that

$$\overline{j_{\tilde{\gamma}}(-\overline{z})} = \overline{-c(-\overline{z}) + d} = cz + d = j_\gamma(z).$$

Hence

$$(Kf)[\gamma]_k = \overline{j_{\tilde{\gamma}}(-\overline{z})}^{-k} \overline{f(\tilde{\gamma}(-\overline{z}))} = \overline{f[\tilde{\gamma}]_k(-\overline{z})} = \overline{\chi}(\tilde{\gamma}) \overline{f(-\overline{z})} = \overline{\chi}(\gamma) Kf(z),$$

finishing the proof of (3). Here for the last equality we used the simple identity $\chi(\tilde{\gamma}) = \chi(\gamma) = \chi(d)$.

Property (4) follows easily by noting that $K(e(nz)) = e(nz)$ which can be checked easily using the identity $\overline{e(w)} = \overline{e^{2\pi i w}} = e^{-2\pi i \bar{w}} = e(-\bar{w})$ for any $w \in \mathbb{C}$.

For (5), note that for any $f \in \mathcal{S}_k(\Gamma_0(N), \chi)$ and $n \geq 1$,

$$\begin{aligned} KT_n^\chi f(z) &= n^{\frac{k}{2}-1} \sum_{\rho \in \Delta_n} \chi(\rho) K f[\rho]_k(z) \\ &= n^{\frac{k}{2}-1} \sum_{\rho \in \Delta_n} \overline{\overline{\chi}(\rho)} K f[\rho]_k(z) = T_n^{\bar{\chi}} K f(z). \end{aligned}$$

□

Let $\bar{W} = KW$. We have the following properties of $\bar{W}$ which are direct consequences of above properties of W and K.

Corollary 15.4. *We have*

(1) $\bar{W}^2 = 1$ *and* $\bar{W}\lambda f = \bar{\lambda}\bar{W} f$.
(2) $\bar{W} : \mathcal{S}_k(\Gamma_0(N), \chi) \to \mathcal{S}_k(\Gamma_0(N), \chi)$.
(3) $T_n^\chi \bar{W} = \chi(n)\bar{W} T_n^\chi$ *for any* $(n, N) = 1$.

With these properties we can show that Hecke new forms are eigenfunctions of $\bar{W}$.

Proposition 15.5. *Let* $f \in \mathcal{S}_k^{\text{new}}(\Gamma_0(N), \chi)$ *be a normalized Hecke new form. Then* $\bar{W} f = \eta f$ *for some* $\eta \in \mathbb{C}$ *with* $|\eta| = 1$.

Proof. Since f is a Hecke new form, we have $T_n^\chi f = \lambda(n) f$ for any $n \geq 1$ with $\lambda(n) = \widehat{f}(n)$. By Corollary 15.4 we have for any $(n, N) = 1$,

$$T_n^\chi \bar{W} f = \chi(n)\bar{W} T_n^\chi f = \chi(n)\bar{W}\lambda(n) f = \chi(n)\overline{\lambda(n)}\bar{W} f.$$

We claim that $\chi(n)\overline{\lambda(n)} = \lambda(n)$ for $(n, N) = 1$. This is true since by Theorem 11.10 we have

$$\lambda(n)\langle f, f\rangle = \langle T_n^\chi f, f\rangle = \chi(n)\langle f, T_n^\chi f\rangle = \chi(n)\overline{\lambda(n)}\langle f, f\rangle.$$

Hence we have

$$T_n^\chi \bar{W} f = \lambda(n)\bar{W} f, \quad \forall\ (n, N) = 1,$$

i,e. $\bar{W} f$ lies in the same eigenspace (for $\{T_n^\chi : (n, N) = 1\}$) as f. Thus by the multiplicity one theorem we have $\bar{W} f = \eta f$ for some $\eta \in \mathbb{C}$. Moreover, $f = \bar{W}^2 f = \bar{W}\eta f = \bar{\eta}\bar{W} f = |\eta|^2 f$, implying that $|\eta| = 1$. □

We now state the functional equation satisfied by eigenfunctions of $\bar{W}$ which in particular includes Hecke new forms in view of Proposition 15.5.

Theorem 15.6. *If* $f \in \mathcal{S}_k(\Gamma_0(N), \chi)$ *satisfies* $\bar{W} f = \eta f$ *for some* $\eta \in \mathbb{C}$. *Then*

$$\Lambda_f(s) = \overline{i^k \eta \Lambda_f(k - \bar{s})}. \tag{15.2}$$

In particular, (15.2) *holds for normalized Hecke new forms.*

Proof. Let $g = W f$ so that $\Lambda_f(s) = i^k \Lambda_g(k - s)$ in view of Theorem 13.3. For this g we also have

$$\eta f = \bar{W} f = K W f = K g = \sum_{n=1}^{\infty} \overline{\widehat{g}(n)} e(nz).$$

This shows that $\overline{\widehat{g}(n)} = \eta \widehat{f}(n)$, or equivalently, $\widehat{g}(n) = \overline{\eta}\overline{\widehat{f}(n)}$. Hence

$$L_g(s) = \sum_{n=1}^{\infty} \frac{\widehat{g}(n)}{n^s} = \overline{\eta}\overline{L_f(\bar{s})}$$

which a priori holds for $\mathfrak{Re}(s) > \frac{k}{2} + 1$ and can be extended to the whole s-plane by analytic continuation. Thus we have

$$\begin{aligned}\Lambda_f(s) &= i^k \Lambda_g(k-s) = i^k \left(\tfrac{\sqrt{N}}{2\pi}\right)^{k-s} \Gamma(k-s)\overline{\eta}\overline{L_f(k-\bar{s})} \\ &= \overline{i^{-k}\eta \left(\tfrac{\sqrt{N}}{2\pi}\right)^{k-\bar{s}} \Gamma(k-\bar{s}) L_f(k-\bar{s})} \\ &= \overline{i^{-k}\eta\Lambda_f(k-\bar{s})}.\end{aligned}$$

Here for the last equality we used that $\overline{\Gamma(s)} = \Gamma(\bar{s})$ which can be easily seen from its integral representation. $\square$

Remark 15.3. If we further assume χ is real and f has real Fourier coefficients, then $\widehat{g}(n) = \overline{\eta}\widehat{f}(n)$, implying that

$$\Lambda_g(s) = \overline{\eta}\Lambda_f(s).$$

Moreover,

$$Wf = K\overline{W}f = K\eta f = \overline{\eta}Kf = \overline{\eta}f,$$

implying that $(-1)^k f = W^2 f = \overline{\eta}^2 f$. Hence $\overline{\eta} = \pm i^k$. Thus

$$\Lambda_f(s) = i^k\Lambda_g(k-s) = i^k\overline{\eta}\Lambda_f(k-s) = \pm\Lambda_f(k-s).$$

16. Twisting automorphic forms and L-functions

The main goal of this section is to study how modular forms behave under twisting by characters. This operation produces more cusp forms from a fixed cusp form and hence also more functional equations associated to this cusp form.

Given $f \in \mathcal{M}_k(\Gamma_0(N), \chi)$ and a primitive Dirichlet character χ_1 of modulus N_1, we define

$$f_{\chi_1}(z) := \sum_{n=0}^{\infty} \widehat{f}(n)\chi_1(n)e(nz), \quad z \in \mathbb{H},$$

where $\widehat{f}(n)$ is the n-th Fourier coefficient of f as usual.

Theorem 16.1. *Let $f \in \mathcal{M}_k(\Gamma_0(N), \chi)$ and $\chi_1 \pmod{N_1}$ be as above. We have $f_{\chi_1} \in \mathcal{M}_k(\Gamma_0(NN_1^2), \chi\chi_1^2)$. If f is a cusp form, then so is f_{χ_1}.*

To prove this theorem we need the following lemma giving an alternative expression of f_{χ_1}.

Lemma 16.2. *Keep the notation and assumptions as in Theorem 16.1. We have*

$$f_{\chi_1} = \frac{G(\chi_1)}{N_1} \sum_{\ell=1}^{N_1} \overline{\chi}_1(\ell) f[\alpha_\ell]_k, \tag{16.1}$$

where $\alpha_\ell = \left(\begin{smallmatrix} 1 & -\frac{\ell}{N_1} \\ 0 & 1 \end{smallmatrix}\right)$.

Proof. Using the fact that χ_1 is a periodic function of period N_1 we can rewrite

$$f_{\chi_1}(z) = \sum_{r=1}^{N_1} \chi_1(r) \sum_{\substack{n \geq 0 \\ n \equiv r \pmod{N_1}}} \widehat{f}(n) e(nz).$$

Note that

$$\frac{1}{N_1} \sum_{\ell=1}^{N_1} e\left(\tfrac{(r-n)\ell}{N_1}\right) = I(n \equiv r \pmod{N_1}),$$

where $I(n \equiv r \pmod{N_1})$ is the indicator function of the condition $n \equiv r \pmod{N_1}$. Hence we have

$$\begin{aligned}
f_{\chi_1}(z) &= \sum_{r=1}^{N_1} \chi_1(r) \sum_{n=0}^{\infty} \frac{1}{N_1} \sum_{\ell=1}^{N_1} e\left(\tfrac{(r-n)\ell}{N_1}\right) \widehat{f}(n) e(nz) \\
&= \frac{1}{N_1} \sum_{\ell=1}^{N_1} \sum_{r=1}^{N_1} \chi_1(r) e\left(\frac{r\ell}{N_1}\right) \sum_{n=0}^{\infty} \widehat{f}(n) e\left(n(z - \tfrac{\ell}{N_1})\right) \\
&= \frac{1}{N_1} \sum_{\ell=1}^{N_1} G(\chi_1, \ell) f\left(z - \tfrac{\ell}{N_1}\right) \\
&= \frac{G(\chi_1)}{N_1} \sum_{\ell=1}^{\infty} \overline{\chi}_1(\ell) f[\alpha_\ell]_k(z),
\end{aligned}$$

finishing the proof, where for the second equality we used the identity $G(\chi_1, \ell) = \overline{\chi}_1(\ell) G(\chi_1)$ which is where we need the primitivity assumption on χ_1. $\square$

We can now give the

Proof of Theorem 16.1. Given any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(NN_1^2)$ we need to show $f_{\chi_1}[\gamma]_k = \chi\chi_1^2(\gamma) f_{\chi_1}$. By Lemma 16.2 we have

$$f_{\chi_1}[\gamma]_k = \frac{G(\chi_1)}{N_1} \sum_{\ell=1}^{N_1} \overline{\chi}_1(\ell) f[\alpha_\ell \gamma]_k$$

with $\alpha_\ell = \begin{pmatrix} 1 & -\frac{\ell}{N_1} \\ 0 & 1 \end{pmatrix}$ as in Lemma 16.2. Let $\ell' \in \mathbb{Z}/N_1\mathbb{Z}$ to be determined. Then by direct computation we have

$$\alpha_\ell \gamma \alpha_{\ell'}^{-1} = \begin{pmatrix} a - \frac{c\ell}{N_1} & b + \frac{a\ell' - d\ell}{N_1} - \frac{cd^2\ell^2}{N_1^2} \\ c & d + \frac{cd^2\ell}{N_1} \end{pmatrix} =: \tau.$$

We wish to choose ℓ' so that $\tau \in \Gamma_0(NN_1^2)$. Since $NN_1^2 \mid c$ one easily sees that for this we only need the term $\frac{a\ell' - d\ell}{N_1}$ in the top right entry of τ to be an integer. We choose $\ell' = d^2\ell$ so that

$$\frac{a\ell' - d\ell}{N_1} = \frac{ad^2\ell - d\ell}{N_1} = \frac{(ad-1)d\ell}{N_1} = \frac{bcd\ell}{N_1}$$

is indeed an integer. With this choice we have $\tau \in \Gamma_0(NN_1^2)$ and $\tau \equiv \left(\begin{smallmatrix} a & * \\ 0 & d \end{smallmatrix}\right) \pmod N$. Thus

$$\begin{aligned} f_{\chi_1}[\gamma]_k &= \frac{G(\chi_1)}{N_1}\sum_{\ell=1}^{N_1}\overline{\chi}_1(\ell)f[\tau\alpha_{d^2\ell}]_k \\ &= \frac{G(\chi_1)}{N_1}\sum_{\ell=1}^{N_1}\overline{\chi}_1(\ell)\chi_1^2(d)\chi(d)f[\alpha_\ell]_k \\ &= \chi\chi_1^2(\gamma)f_{\chi_1} \end{aligned}$$

as desired. Here in the second line we made a change of variable $d^2\ell \mapsto \ell$ which can be done since $(d, N_1) = 1$. $\square$

Let f and χ_1 be as above. Define the L-function and completed L-function associated to f_{χ_1} by for $\mathfrak{Re}(s) > \frac{k}{2}+1$

$$L_f(s,\chi_1) := \sum_{n=1}^{\infty}\widehat{f}(n)\chi_1(n)n^{-s}$$

and

$$\Lambda_f(s,\chi_1) := \left(\frac{\sqrt{NN_1^2}}{2\pi}\right)^s\Gamma(s)L_f(s,\chi_1).$$

We have the following functional equation satisfied $\Lambda_f(s,\chi_1)$.

Theorem 16.3. *Assume further $f \in \mathcal{S}_k(\Gamma_0(N),\chi)$ and $(N,N_1) = 1$. Then $\Lambda_f(s,\chi_1)$ has an analytic continuation to an entire function, is bounded in vertical strips and satisfies the functional equation*

$$\Lambda_f(s,\chi_1) = i^k\omega(\chi_1)\Lambda_g(k-s,\bar{\chi}_1), \tag{16.2}$$

where $g = f[w_N]_k \in \mathcal{S}_k(\Gamma_0(N),\bar{\chi})$ and $\omega(\chi_1) := \chi(N_1)\chi_1(N)G(\chi_1)^2N_1^{-1}$.

Proof. Let $\tilde{N} = NN_1^2$. Since $f_{\chi_1} \in \mathcal{S}_k(\Gamma_0(\tilde{N}),\chi\chi_1^2)$ and $\Lambda_f(s,\chi_1) = \Lambda_{f_{\chi_1}}(s)$, by Proposition 13.1 and Remark 13.4 it has the analytic continuation and is bounded on vertical strips and satisfies the functional equation

$$\Lambda_f(s,\chi_1) = i^k\Lambda_{f_{\chi_1}[w_{\tilde{N}}]_k}(k-s).$$

This theorem then follows by the following proposition which asserts $f_{\chi_1}[w_{\tilde{N}}]_k = \omega(\chi_1)g_{\bar{\chi}_1}$. Indeed assuming this identity, we have

$$\Lambda_f(s,\chi_1) = i^k\omega(\chi_1)\Lambda_{g_{\bar{\chi}_1}}(k-s) = i^k\omega(\chi_1)\Lambda_g(s,\bar{\chi}_1)$$

as desired. $\square$

Proposition 16.4. *Keep the assumptions and notation as in Theorem 16.3. Then we have*

$$f_{\chi_1}[w_{NN_1^2}]_k = \omega(\chi_1)g_{\bar{\chi}_1}. \tag{16.3}$$

Proof. Let $\tilde{N} = NN_1^2$ be as above. By Lemma 16.2 and noting that $\chi_1(\ell) = 0$ whenever $(\ell, N_1) > 1$ we have

$$f_{\chi_1}[w_{\tilde{N}}]_k = \frac{G(\chi_1)}{N_1}\sum_{\ell\in(\mathbb{Z}/N_1\mathbb{Z})^\times}\overline{\chi}_1(\ell)f[\alpha_\ell w_{\tilde{N}}]_k,$$

where $\alpha_\ell = \begin{pmatrix} 1 & -\frac{\ell}{N_1} \\ 0 & 1 \end{pmatrix}$ is above. By direct computation we have for any $v \in \mathbb{Z}$

$$\alpha_\ell w_{\tilde{N}} = N_1 w_N \begin{pmatrix} N_1 & v \\ \ell N & \frac{1+\ell v N}{N_1} \end{pmatrix} \alpha_v.$$

Let τ be the second matrix in the right hand side of the above equation. Take $v \in \mathbb{Z}$ so that $\ell v N \equiv -1 \pmod{N_1}$. With this choice of v we have $\tau \in \Gamma_0(N)$. Moreover, since $(N, N_1) = 1$, this congruence relation gives a one-to-one correspondence between $\ell \in (\mathbb{Z}/N_1\mathbb{Z})^\times$ and $v \in (\mathbb{Z}/N_1\mathbb{Z})^\times$. Hence we have

$$\begin{aligned}
f_{\chi_1}[w_{\tilde{N}}]_k &= \frac{G(\chi_1)}{N_1} \sum_{\ell \in (\mathbb{Z}/N_1\mathbb{Z})^\times} \bar{\chi}_1(\ell) f[N_1 w_N \tau \alpha_v]_k \\
&= \frac{G(\chi_1)}{N_1} \sum_{v \in (\mathbb{Z}/N_1\mathbb{Z})^\times} \chi_1(-vN) g[\tau\alpha_v]_k \\
&= \frac{G(\chi_1)}{N_1} \sum_{v \in (\mathbb{Z}/N_1\mathbb{Z})^\times} \chi_1(-vN)\chi(N_1) g[\alpha_v]_k \\
&= \chi_1(-N)\chi(N_1) \frac{G(\chi_1)}{G(\bar{\chi}_1)} \frac{G(\bar{\chi}_1)}{N_1} \sum_{v \in (\mathbb{Z}/N_1\mathbb{Z})^\times} \chi_1(v) g[\alpha_v]_k \\
&= \chi_1(N)\chi(N_1) G(\chi_1)^2 N_1^{-1} g_{\bar{\chi}_1}.
\end{aligned}$$

Here in the last line we used that $\chi_1(-1)G(\bar{\chi}_1) = \overline{G(\chi_1)}$, $|G(\chi_1)|^2 = N_1$ and applied Lemma 16.2 for $g \in \mathcal{S}_k(\Gamma_0(N), \bar{\chi})$. □

17. Weil's converse theorem

With the new functional equations proved in the previous section via twisting, we can now state the converse theorem for general levels, generalizing Theorem 14.1.

Theorem 17.1 (Weil)**.** *Let $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ be two sequence of complex numbers satisfying $\max\{|a_n|, |b_n|\} \ll n^\alpha$ for all $n \geq 1$ and for some $\alpha > 0$. Let $f(z) = \sum_{n=0}^\infty a_n e(nz)$ and $g(z) = \sum_{n=0}^\infty b_n e(nz)$. Let k be a positive integer and let χ be a Dirichlet character of modulus $N \in \mathbb{N}$. Let $L_f(s), L_g(s)$, $\Lambda_f(s)$ and $\Lambda_g(s)$ be defined as before. Similarly, for any primitive Dirichlet character χ_1 of modulus N_1, let $L_f(s, \chi_1)$, $L_g(s, \chi_1)$, $\Lambda_f(s, \chi_1)$ and $\Lambda(g(s, \chi_1)$ be given as before. Suppose*

1. *$\Lambda_f(s)$ and $\Lambda_g(s)$ both have meromorphic continuation to the whole s-plane with*
$$\Lambda_f(s) + a_0 s^{-1} + b_0 i^k (k-s)^{-1}$$
and
$$\Lambda_g(s) + b_0 s^{-1} + a_0 i^{-k} (k-s)^{-1}$$
both entire and bounded on vertical strips and satisfy the functional equation (13.6).
2. *Let $\mathcal{R}$ be a finite set of primes co-prime to N and attaining every primitive residue class modulo N. Suppose for any $N_1 \in \mathcal{R}$ and for any primitive χ_1 modulo N_1, $\Lambda_f(s, \chi_1)$ and $\Lambda_g(s, \bar{\chi}_1)$ have analytic continuation to entire functions and are bounded on vertical strips and satisfy the functional*

equation (16.2). *Then we have* $f \in \mathcal{M}_k(\Gamma_0(N),\chi)$ *and* $g = f[w_N]_k \in \mathcal{M}_k(\Gamma_0(N),\bar{\chi})$.

Below we give a sketch of this theorem. For more details see [Iwa97, Theorem 7.8] or [Bum97, Theorem 1.5.1].

Proof (Sketch). The general idea is to show that assumption (1) implies the relation $g = f[w_N]_k$ while assumption (2) implies the modularity. We first show the above first assertion. Similar as before we have for $\mathfrak{Re}(s) \gg 1$,

$$\Lambda_f(s) = N^{\frac{s}{2}}(2\pi)^{-s}\Gamma(s)L_f(s) = N^{\frac{s}{2}}\int_0^\infty (f(iy)-a_0)y^{s-1}\,dy.$$

Then by the Mellin inversion formula we have for $\sigma \gg 1$,

$$f(iy)-a_0 = \frac{1}{2\pi i}\int_{(\sigma)} N^{-\frac{s}{2}}\Lambda_f(s)y^{-s}\,ds.$$

Now by the Phragmén-Lindelöf principle we can shift the contour from $\mathfrak{Re}(s)=\sigma$ to $\mathfrak{Re}(s)=\frac{k}{2}$ and picking up the simple pole of $\Lambda_f(s)$ at $s=k$ to get

$$f(iy)-a_0 = \frac{1}{2\pi i}\int_{(\frac{k}{2})} N^{-\frac{s}{2}}\Lambda_f(s)y^{-s}\,ds + b_0 i^k N^{-\frac{k}{2}}y^{-k}. \tag{17.1}$$

Similarly, we have

$$g(iy)-b_0 = \frac{1}{2\pi i}\int_{(\frac{k}{2})} N^{-\frac{s}{2}}\Lambda_f(s)y^{-s}\,ds + a_0 i^{-k} N^{-\frac{k}{2}}y^{-k}. \tag{17.2}$$

Starting from (17.1) and applying the functional equation (13.6) we have for any $y>0$,

$$\begin{aligned}
f(iy)-a_0 &= \frac{1}{2\pi i}\int_{(\frac{k}{2})} N^{-\frac{s}{2}} i^k \Lambda_g(k-s)y^{-s}\,ds + b_0 i^k N^{-\frac{k}{2}}y^{-k}\\
&\overset{k-s\mapsto s}{=} \frac{1}{2\pi i}\int_{(\frac{k}{2})} N^{-\frac{k-s}{2}} i^k \Lambda_g(s)y^{-(k-s)}\,ds + b_0 i^k N^{-\frac{k}{2}}y^{-k}\\
&= \frac{i^k N^{-\frac{k}{2}}y^{-k}}{2\pi i}\int_{(\frac{k}{2})} N^{-\frac{s}{2}}\Lambda_g(s)(1/Ny)^{-s}\,ds + b_0 i^k N^{-\frac{k}{2}}y^{-k}\\
&\overset{(17.2)}{=} i^k N^{-\frac{k}{2}}y^{-k}\left(g(i/Ny)-b_0-a_0 i^{-k}N^{-\frac{k}{2}}(1/Ny)^{-k}\right) + b_0 i^k N^{-\frac{k}{2}}y^{-k}\\
&= i^k N^{-\frac{k}{2}}y^{-k}g(i/Ny)-a_0.
\end{aligned}$$

This identity, together with an analytic continuation implies the desired identity $g = f[w_N]_k$.

Now we sketch how assumption (2) implies modularity. First, the condition $\max\{|a_n|,|b_n|\} \ll n^\alpha$ implies that f,g are holomorphic functions on $\mathbb{H}$ (via similar estimates as in proof of Theorem 14.1). Now using similar arguments as above one can use assumption (2) to show that relation (16.3) holds for any primitive Dirichlet character χ_1 of modulus N_1 wtih $N_1 \in \mathcal{R}$. Next, use relation (16.3) and apply identity (16.1) for the pairs (f,χ_1) and $(g,\bar{\chi}_1)$ (wtih χ_1 to varying) [16] to

[16] We proved (16.1) under the assumption that $f \in \mathcal{S}_k(\Gamma_0(N),\chi)$, however inspecting the proof we see that the only assumption we used for f is the condition that f has a Fourier expansion of the form $f(z) = \sum_{n=0}^\infty \widehat{f}(n)e(nz)$ which is clearly satisfied by the two functions here.

show $g[\gamma]_k = \bar{\chi}(\gamma)g$ for any

$$\gamma = \begin{pmatrix} N_1 & u \\ Nv & N_2 \end{pmatrix} \quad \text{with } N_1, N_2 \in \mathcal{R} \text{ and } u, v \in \mathbb{Z} \text{ such that } \gamma \in \Gamma_0(N). \tag{17.3}$$

Finally, for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, there exists $N_1, N_2 \in \mathcal{R}$ such that

$$\gamma = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} N_1 & u \\ Nv & N_2 \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

for some $m, n \in \mathbb{Z}$ and $\left(\begin{smallmatrix} N_1 & u \\ Nv & N_2 \end{smallmatrix}\right)$ of the form as in (17.3). From this one can easily show that $g[\gamma]_k = \bar{\chi}(\gamma)g$ for any $\gamma \in \Gamma_0(N)$. Hence $g \in \mathcal{M}_k(\Gamma_0(N), \bar{\chi})$. This then implies that $f = (-1)^k g[w_N]_k \in \mathcal{S}_k(\Gamma_0(N), \chi)$, finishing the proof $\square$

17.1. **An applications of the converse theorem.** In this section we illustrate an application of the converse theorem, namely we construct explicit modular forms via L-functions. The general strategy is that if for a given L-function $L(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ one can verify that it satisfies all the assumptions in the converse theorem, then one can conclude that the resulting function $f(z) := \sum_{n=0}^{\infty} a_n e(nz)$ is a modular form with $a_0 \in \mathbb{C}$ determined by the assumption (1) in (17.1).

We first consider the most classical L-function, namely the Riemann zeta function $\zeta(s)$. In view of the Euler's product formulas (12.1) and (13.7), in order to construct a Hecke L-function from the Riemann zeta function, one needs two copies of Riemann zeta functions. More precisely, for a given even integer $k \geq 4$, consider the L-function

$$L(s) := \zeta(s)\zeta(s-k+1),$$

and its completion

$$\Lambda(s) := (2\pi)^{-s}\Gamma(s)L(s).$$

We note that both $L(s)$ and $\Lambda(s)$ have a moromorphic continuation in view of the meromorphic continuation of $\zeta(s)$. The following proposition confirms that $\Lambda(s)$ satisfies the assumptions in Hecke's converse theorem Theorem 14.1.

Proposition 17.2. *The L-function $\Lambda(s)$ has simple poles at $s = 0, k$ and is holomorphic elsewhere. Moreover, it satisfies the functional equation*

$$\Lambda(s) = i^k \Lambda(k-s).$$

This proposition follows easily the following lemma which we leave as an exercise.

Exercise 23. *Show that*

$$\Lambda(s) = 2^{-\frac{k}{2}-1}\pi^{-\frac{k}{2}}\xi(s)\xi(s-k+1)\prod_{j=1}^{k/2}(s-2j+1), \tag{17.4}$$

where $\xi(s) = \pi^{-\frac{s}{2}}\Gamma(s/2)\zeta(s)$ is the completed Riemann zeta function.

We can now give the

Proof of Proposition 17.2. The meoromorphic continuation of $\Lambda(s)$ follows easily from the expression (17.4) and the meromorphic continuation of $\xi(s)$. Moreover, we see from (17.4) that $\Lambda(s)$ has 4 potential simple poles at $s = 0, 1, k-1, k$. However, the two potential poles $s = 1, k-1$ are canceled out by the simple zeros from the factors $s-1$ and $s-k+1$ respectively from the product in (17.4). Thus

$\Lambda(s)$ only has two simple poles at $s = 0$ and k as claimed. For the functional equation we again use (17.4) to get

$$\begin{aligned}\Lambda(k-s) &= 2^{-\frac{k}{2}-1}\pi^{-\frac{k}{2}}\xi(k-s)\xi(1-s)\prod_{j=1}^{k/2}(k-s-2j+1)\\ &= 2^{-\frac{k}{2}-1}\pi^{-\frac{k}{2}}\xi(s-k+1)\xi(s)(-1)^{\frac{k}{2}}\prod_{i=1}^{k/2}(s-2i+1)\\ &= i^k\Lambda(s),\end{aligned}$$

as desired. Here for the second equality we used the functional equation $\xi(1-s) = \xi(s)$ and made a change of variable $k-2j+1 \mapsto 2i-1$. □

Applying Hecke's converse theorem Theorem 14.1 (see also Remark 14.6) we immediately get the following.

Corollary 17.3. *Let $k \geq 4$ be even and let $L(s)$ and $\Lambda(s)$ be as above. Let $\{a_n\}_{n\in\mathbb{N}}$ be such that $L(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ for $\mathfrak{Re}(s) \gg 1$. Then $f(z) := \sum_{n=0}^{\infty} a_n e(nz) \in \mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z}))$, where $a_0 := -\operatorname{Res}_{s=0}\Lambda(s)$.*

Remark 17.5. Indeed one can see that the modular form f from the above corollary is an Eisenstein series: By direct computation we have $a_0 = -\zeta(0)\zeta(-k+1)$ and $a_n = \sigma_{k-1}(n)$ for any $n \in \mathbb{N}$. Using the functional equation $\xi(s) = \xi(1-s)$ and the formula $\zeta(k) = -\frac{(2\pi i)^k}{2k!}B_k$ with B_k the k-th Bernoulli number (cf. Homework 1) one sees that $\zeta(0) = -\frac{1}{2}$ and $\zeta(-k+1) = (-1)^{k-1}\frac{B_k}{k}$. Hence $a_0 = -\frac{B_k}{2k}$, implying that $f = -\frac{B_k}{2k}E_k$, where E_k is the normalized weight-k Eisenstein series given in (2.23).

Next, we state (without proof) a theorem constructing modular forms of general level via Dirichlet L-functions.

Theorem 17.4. *For $i = 1, 2$, let χ_i be a primitive Dirichlet character of modulus $N_i \in \mathbb{N}$. Let $\chi = \chi_1\chi_2$ which is a Dirichlet character of modulus $N := N_1N_2$. Assume $N > 1$ and let $k \in \mathbb{N}$ be such that $\chi(-1) = (-1)^k$. Consider the L-function*

$$L(s) := L(s,\chi_1)L(s-k+1,\chi_2) = \sum_{n=1}^{\infty} a_n n^{-s}, \quad \mathfrak{Re}(s) \gg 1,$$

where for any $n \in \mathbb{N}$, $a_n = a_n(\chi_1,\chi_2) := \sum_{ad=n}\chi_1(a)\chi_2(d)d^{k-1}$. Let $a_0 = 0$ unless $k = 1$ and $N_2 = 1$ in which case we set $a_0 = \frac{1}{2}L(0,\chi)$. Define

$$f(z) = f_{\chi_1,\chi_2}(z) := \sum_{n=0}^{\infty} a_n e(nz).$$

Then $f \in \mathcal{M}_k(\Gamma_0(N),\chi)$.

18. Rankin-Selberg L-function

In the previous section we have constructed a Hecke L-function (degree two) from two Riemann zeta functions or Dirichlet L-functions functions (degree one). In this section, we discuss a construction where one forms a degree four L-function from two Hecke L-functions and establish some of its analytic properties that one would expect from an L-function. This construction is due to Rankin [Ran39] and

Selberg [Sel40] independently, and we mainly follow the treatments in [Bum97, Chapter 1.6].

18.1. **Real analytic Eisenstein series.** To describe this construction, we need to introduce a new object, namely the real analytic Eisenstein series. For simplicity of presentation throughout this section we assume $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Recall that the stabilizer subgroup of the cusp ∞ is given by $\Gamma_\infty = \langle \pm \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle$. The *real analytic Eisenstein series* is defined by

$$E(z,s) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathfrak{Im}(\gamma z)^s, \quad z \in \mathbb{H},\ \mathfrak{Re}(s) > 1.$$

Here the assumption $\mathfrak{Re}(s) > 1$ is to ensure absolute convergence. Indeed, recall that there is a bijection between $\Gamma_\infty \backslash \Gamma$ and $\mathbb{Z}^2_{\mathrm{pr}}/\pm$ sending $\Gamma_\infty \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to $\pm(c,d)$; see Lemma 7.1 and Remark 7.1. This implies the following alternative expression of $E(z,s)$ that

$$E(z,s) = \frac{1}{2} \sum_{(c,d) \in \mathbb{Z}^2_{\mathrm{pr}}} \frac{y^s}{|cz+d|^{2s}}, \tag{18.1}$$

from which one sees that the defining series of $E(z,s)$ is absolutely convergent as long as $\mathfrak{Re}(s) > 1$.

Since left multiplying an element in Γ_∞ does change imaginary part of a complex number, the above definition does not depend on the choice of the coset representatives of $\Gamma_\infty \backslash \Gamma$. In particular, we get that $E(z,s)$ is Γ-invariant in the variable $z \in \mathbb{H}$, that is $E(\gamma z, s) = E(z,s)$ for any $\gamma \in \Gamma$ and $z \in \mathbb{H}$. Moreover, unlike the holomorphic Eisenstein series defined in (2.13), $E(z,s)$ is not holomorphic in $z \in \mathbb{H}$, rather, it is only smooth in the two real variables x, y with $z = x + iy$.

The analytic properties of $E(z,s)$ are better described in terms of its *completion* which is defined by

$$E^*(z,s) := \xi(2s)E(z,s) = \pi^{-s}\Gamma(s)\zeta(2s)E(z,s).$$

Theorem 18.1. *The completed Eisenstein series $E^*(z,s)$ has a meromorphic continuation to all of $s \in \mathbb{C}$ which is analytic except with two simple poles at $s = 0, 1$. Moreover, it satisfies the functional equation*

$$E^*(z,s) = E^*(z, 1-s) \tag{18.2}$$

and the growth condition that for any $s \neq 0, 1$,

$$|E^*(z,s)| \ll_s y^\sigma \quad \text{as } y \to \infty \text{ with } \sigma = \max\{\mathfrak{Re}(s), 1 - \mathfrak{Re}(s)\}. \tag{18.3}$$

Proof (sketch). Using similar arguments as in the proof of Theorem 7.6 one can prove the following Fourier expansion formula for $E^*(z,s)$ that

$$E^*(z,s) = \xi(2s)y^s + \xi(2-2s)y^{1-s} + 4\sqrt{y}\sum_{n=1}^{\infty} \eta_{s-1/2}(n) K_{s-1/2}(2\pi n y)\cos(2\pi n x),$$

where $\eta_{s-1/2}(n) := \sum_{ad=n} (\frac{a}{d})^{s-\frac{1}{2}}$ and

$$K_s(y) := \frac{1}{2}\int_0^\infty e^{-y(t+t^{-1})/2} t^s \,\frac{dt}{t}, \quad s \in \mathbb{C},\ y > 0$$

is the K-Bessel function of the second kind; see [Bum97, p. 66-69] for more details. It is not difficult to see from the definition that both $K_s(y)$ and $\eta_s(n)$ are invariant after chaning s to $-s$, and $K_s(y)$ has an exponential decay in y in the sense that for any $s \in \mathbb{C}$,

$$|K_s(y)| \ll_s e^{-y/2} \quad \text{as } y \to \infty.$$

Using this exponential decay we can see that the right hand side of the above Fourier expansion formula is absolutely convergent for any $s \neq 0, 1$. When $s = 0, 1$ it has a simple pole coming from the terms $\xi(2s)$ and $\xi(2-2s)$ respectively. This proves the meromorphic continuation of $E^*(z,s)$.

Next, using the functional equations $\xi(1-s) = \xi(s)$, $K_s(y) = K_{-s}(y)$ and $\eta_s(n) = \eta_{-s}(n)$ we see that the right hand side is invariant changing s to $1-s$, proving (18.2).

The growth condition follows by noting that the contribution of the non constant terms is uniformly bounded due to the exponential decay of $K_s(y)$. Thus for any $s \neq 0, 1$ we have

$$|E^*(z,s)| \ll |\xi(2s)y^s| + |\xi(2-2s)y^{1-s}| + 1 \ll_s y^{\sigma} \quad \text{as } y \to \infty,$$

with $\sigma = \max\{\mathfrak{Re}(s), 1-\mathfrak{Re}(s)\}$ as in this theorem. This finishes the proof. □

Remark 18.4. From the above Fourier expansion formula and the integral expression (12.2) for $\xi(s)$ we see that

$$\operatorname{Res}_{s=1} E^*(z,s) = \operatorname{Res}_{s=1} \xi(2-2s) = \frac{1}{2}.$$

Equivalently, we have

$$\operatorname{Res}_{s=1} E(z,s) = \operatorname{Res}_{s=1} \frac{1}{\xi(2s)} E^*(z,s) = \frac{1}{2\xi(2)} = \frac{3}{\pi} = \frac{1}{\mu(\Gamma\backslash\mathbb{H})}.$$

We note that this identity (that $\operatorname{Res}_{s=1} E(z,s)$ equals the of the covolume of the corresponding lattice) is not a coincidence and indeed holds for general lattices[17].

18.2. **Rankin-Selberg L-function.** Assume $\phi : \mathbb{H} \to \mathbb{C}$ is smooth, Γ-invariant and has *super-polynomial decay* at ∞, that is

$$|\phi(x+iy)| \ll y^{-M} \quad \text{for all } M > 0 \text{ as } y \to \infty. \tag{18.5}$$

Since ϕ is assumed to be Γ-invariant, it has a Fourier expansion

$$\phi(x+iy) = \sum_{n\in\mathbb{Z}} \phi_n(y)e(nx)$$

with

$$\phi_n(y) := \int_0^1 \phi(x+iy)e(-nx)\, dx.$$

The 0-th Fourier coefficient $\phi_0(y)$ is usually called the *constant term* of ϕ. Given ϕ as above, define

$$I_\phi(s) := \pi^{-s}\Gamma(s)\zeta(2s)\mathcal{M}(\phi_0)(s-1),$$

where $\mathcal{M}(\phi_0)(s-1) := \int_0^\infty \phi_0(y)y^{s-1}\,\frac{dy}{y}$ is the Mellin transform of ϕ evaluated at $s-1$. Note that since ϕ has super-polynomial decay in y at ∞, so is its constant

[17] For a general non-uniform lattice $\Gamma < \mathrm{SL}_2(\mathbb{R})$, the (real-analytic) Eisenstein series of Γ at a cusp $\mathfrak{a}$ is defined by $E_{\mathfrak{a}}(z,s) := \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma}(\mathfrak{Im}\sigma_{\mathfrak{a}}^{-1}\gamma z)^s$, where $\Gamma_{\mathfrak{a}} < \Gamma$ is the stabilizer group of $\mathfrak{a}$ in Γ and $\sigma_{\mathfrak{a}}$ is the corresponding scaling matrix defined as before.

term ϕ_0. Thus $\mathcal{M}(\phi_0)(s-1)$ is absolutely convergent as long as $\mathfrak{Re}(s) > 1$. We have the following proposition showing that $I_\phi(s)$ has a meromorphic continuation.

Proposition 18.2. *Let ϕ be as above. Then we have for any $\mathfrak{Re}(s) > 1$,*

$$I_\phi(s) = \int_{\Gamma\backslash\mathbb{H}} E^*(z,s)\phi(z)\ d\mu(z), \tag{18.6}$$

where $d\mu(z) = \frac{dxdy}{y^2}$ is the hyperbolic measure (restricted on a fundamental domain for $\Gamma\backslash\mathbb{H}$). In particular, $I_\phi(s)$ has a meromorphic continuation to all of $s \in \mathbb{C}$ with at most simple poles at $s = 0, 1$ with

$$\mathrm{Res}_{s=1} I_\phi(s) = \frac{1}{2}\int_{\Gamma\backslash\mathbb{H}} \phi(z)\ d\mu(z).$$

Moreover, $I_\phi(s)$ satisfies the functional equation $I_\phi(1-s) = I_\phi(s)$.

Proof. The proof uses the unfolding argument which is the essence of the Rankin-Selberg method. We first note that the in particular parts are easy consequences of the identity (18.6) together with the properties of the Eisenstein series $E^*(z.s)$ stated above. For example, assuming (18.6) and using the super-polynomial decay of ϕ at ∞ and the growth condition (18.3) of $E^*(z,s)$ and taking the standard fundamental domain (3.3) for $\Gamma\backslash\mathbb{H}$ we see that the integral in (18.6) is absolutely convergent for any $s \neq 0, 1$, proving the meromorphic continuation of $I_\phi(s)$. The other assertions follow similarly. We thus only prove (18.6). For $\mathfrak{Re}(s) > 1$, we can compute the integral

$$\begin{aligned}\int_{\Gamma\backslash\mathbb{H}} E(z,s)\phi(z)\ d\mu(z) &= \int_{\mathcal{F}_\Gamma} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathfrak{Im}(\gamma z)^s \phi(z)\ d\mu(z)\\ &\overset{\gamma z\mapsto z}{=} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \int_{\gamma\mathcal{F}_\Gamma} \mathfrak{Im}(z)^s\phi(\gamma^{-1}z)\ d\mu(\gamma^{-1}z).\end{aligned}$$

Now using the Γ-invariance of ϕ and the hyperbolic measure μ we have

$$\begin{aligned}\int_{\Gamma\backslash\mathbb{H}} E(z,s)\phi(z)\ d\mu(z) &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \int_{\gamma\mathcal{F}_\Gamma} y^s\phi(z)\ d\mu(z)\\ &= \int_{\bigsqcup_{\gamma\in\Gamma_\infty\backslash\Gamma}\gamma\mathcal{F}_\Gamma} y^s\phi(x+iy)\ \frac{dxdy}{y^2}.\end{aligned}$$

Here the first equality can be justified by the super-polynomial decay of ϕ in y and the Γ-invariance of ϕ. The disjoint union $\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\gamma\mathcal{F}_\Gamma$ is a fundamental domain for $\Gamma_\infty\backslash\mathbb{H}$ which we can take to be $\{x+iy \in \mathbb{H} : 0 \le x < 1\}$. Hence

$$\begin{aligned}\int_{\Gamma\backslash\mathbb{H}} E(z,s)\phi(z)\ d\mu(z) &= \int_0^\infty \left(\int_0^1 \phi(x+iy)\ dx\right) y^{s-1}\ dy\\ &= \int_0^\infty \phi_0(y)y^{s-2}\ dy.\end{aligned}$$

Multiplying both sides by $\xi(2s)$ we get (18.6), finishing the proof. $\square$

Let $f, g \in \mathcal{M}_k$ with one of them cuspidal. For $\mathfrak{Re}(s) \gg 1$ sufficiently large, the *Rankin-Selberg L-function* of f and g is defined by

$$L(s, f \times g) := \zeta(2s-2k+2) \sum_{n=1}^{\infty} \widehat{f}(n)\overline{\widehat{g}(n)} n^{-s},$$

and its completion is defined by

$$\Lambda(s, f \times g) := (2\pi)^{-s}\Gamma(s)\Gamma(s-k+1)L(s, f \times g).$$

We now apply Proposition 18.2 to $\phi(z) := y^k f(z)\overline{g(z)}$ to deduce the meromorphic continuation and functional equation of $\Lambda(s, f \times g)$.

Theorem 18.3. *Let $f, g \in \mathcal{M}_k$ with one of them cuspidal. The completed L-function $\Lambda(s, f \times g)$ has a meromorphic continuation to all of $s \in \mathbb{C}$ with at most simple poles at $s = k, k-1$. Moreover, it satisfies the functional equation*

$$\Lambda(s, f \times g) = \Lambda(2k-1-s, f \times g). \tag{18.7}$$

Proof. Take $\phi(z) = y^k f(z)\overline{g(z)}$ and as before one can check that ϕ is Γ-invariant. Since one of f and g is cuspidal, we have ϕ decays exponentially at ∞. Moreover, writing f and g in Fourier expansion we get

$$\begin{aligned}
\phi_0(y) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \int_0^1 y^k \widehat{f}(n)\overline{\widehat{g}(m)} e(nz)\overline{e(mz)}\, dx \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} y^k e^{-2\pi(m+n)y} \widehat{f}(n)\overline{\widehat{g}(m)} \int_0^1 e((n-m)x)\, dx \\
&= \sum_{n=1}^{\infty} \widehat{f}(n)\overline{\widehat{g}(n)} y^k e^{-4\pi n y},
\end{aligned}$$

where for the last equality we used that $\widehat{f}(0)\overline{\widehat{g}(0)} = 0$. Hence we have for $\mathfrak{Re}(s) > 1$,

$$\begin{aligned}
\int_0^{\infty} \phi_0(y) y^{s-2}\, dy &= \int_0^{\infty} \sum_{n=1}^{\infty} \widehat{f}(n)\overline{\widehat{g}(n)} y^{k+s-2} e^{-4\pi n y}\, dy \\
&\overset{4\pi n y \mapsto y}{=} \sum_{n=1}^{\infty} \widehat{f}(n)\overline{\widehat{g}(n)} \int_0^{\infty} \left(\frac{y}{4\pi n}\right)^{k+s-2} e^{-y}\, \frac{dy}{4\pi n} \\
&= (4\pi)^{-s-k+1}\Gamma(s+k-1) \sum_{n=1}^{\infty} \widehat{f}(n)\overline{\widehat{g}(n)} n^{-(k+s-1)}.
\end{aligned}$$

Now from the last equation in the proof of Proposition 18.2 we have

$$\begin{aligned}
I_\phi(s) &= \xi(2s) \int_0^{\infty} \phi_0(y) y^{s-2}\, dy \\
&= 4^{-s-k+1}\pi^{-2s-k+1}\Gamma(s)\Gamma(s+k-1)\zeta(2s) \sum_{n=1}^{\infty} \widehat{f}(n)\overline{\widehat{g}(n)} n^{-(k+s-1)},
\end{aligned}$$

where $I_\phi(s)$ is the meormorphic function defined in (18.6). This shows that

$$\Lambda(s, f \times g) = \pi^{1-k} I_\phi(s-k+1),$$

giving the desired meromorphic continuation of $\Lambda(s, f \times g)$. Moreover, the functional equation (18.7) also follows from this relation and the functional equation $I_\phi(1-s) = I_\phi(s)$ satisfied by $I_\phi(s)$. ∎

Remark 18.8. If f and g are further assumed to be normalized Hecke eigen-cusp forms, the corresponding Rankin-Selberg L-function also has an Euler product formula

$$L(s, f \times g) = \prod_p \prod_{i=1}^{2} \prod_{j=1}^{2} (1 - \alpha_i(p)\beta_j(p)p^{-s})^{-1}, \quad \mathfrak{Re}(s) \gg 1.$$

Here $\alpha_i(p), \beta_j(p) \in \mathbb{C}$ are such that

$$L_f(s) = \prod_p (1 - \alpha_1(p)p^{-s})^{-1}(1 - \alpha_2(p)p^{-s})^{-1},$$

and

$$L_g(s) = \prod_p (1 - \beta_1(p)p^{-s})^{-1}(1 - \beta_2(p)p^{-s})^{-1}.$$

See [Bum97, Theorem 1.6.3] for more details.

19. Quadratic forms, lattices and theta series

A real quadratic form in n variables is a function $Q : \mathbb{R}^n \to \mathbb{R}$ of the form

$$Q(x) = \sum_{1 \le i \le j \le n} b_{ij} x_i x_j,$$

with $b_{ij} \in \mathbb{R}$. Equivalently, one can use a real symmetric matrix $A \in M_n(\mathbb{R})$ to represent Q. Explicitly, let $A = (a_{ij})_{1 \le i,j \le n}$ with

$$a_{ji} = a_{ij} = \begin{cases} b_{ij} & \text{if } i = j, \\ \frac{1}{2} b_{ij} & \text{if } 1 \le i < j \le n. \end{cases}$$

Then we have $Q(x) = x^t A x$. Here x^t denotes the transpose of the column vector $x \in \mathbb{R}^n$. We introduce the following definitions regarding a quadratic form.

Definition 19.1. A quadratic form Q is called

1. *positive definite* if $Q(x) \ge 0$ for any $x \in \mathbb{R}^n$ and $Q(x) = 0$ if and only if $x = 0$.
2. *integral* if $Q(\mathbb{Z}^n) \subset \mathbb{Z}$.
3. *even* if $Q(\mathbb{Z}^n) \subset 2\mathbb{Z}$.

Alternatively, these definitions can be rephrased in terms of the symmetric matrix A. Indeed, a quadratic form $Q(x) = x^t A x$ is

1. positive definite if and only if all eigenvalues of A are positive.
2. integral if and only if $a_{ij} \in \frac{1}{2}\mathbb{Z}$ for all $1 \le i \ne j \le n$ and $a_{ii} \in \mathbb{Z}$ for all $1 \le i \le n$.
3. even if and only if $a_{ij} \in \mathbb{Z}$ for all $1 \le i \ne j \le n$ and $a_{ii} \in 2\mathbb{Z}$ for all $1 \le i \le n$.

Given a positive definite, integral quadratic form Q we are interested in understanding what integers that it can represent, that is, for what $m \in \mathbb{Z}$ there exists $v \in \mathbb{Z}^n$ such that $Q(v) = m$. Equivalent, define the counting function

$$r_Q(m) := \# \{v \in \mathbb{Z}^n : Q(v) = m\},$$

then we would like to know when $r_Q(m)$ is positive. If such a representation exists, the next next natural question is to count the number of such representations, i.e. to find a formula for $r_Q(m)$. In the next two sections we study these two questions using modular forms. The bridge for this approach is the following *theta function associated to Q*

$$\Theta_Q(z) := \sum_{v\in\mathbb{Z}^n} e^{\pi i Q(v)z} = 1 + \sum_{m=1}^{\infty} r_Q(m)e^{\pi i m z}, \quad z \in \mathbb{H}. \tag{19.2}$$

Indeed we will show that Θ_Q is a modular form of weight $\frac{n}{2}$ and the main ingredient is the Jacobi's inversion formula (see Theorem 19.4 below). To state this formula, it is more convenient to represent quadratic forms by lattices.

19.1. **Backgrouds on lattices.** Quadratic forms can also be defined in terms of lattices in $\mathbb{R}^n$. A *lattice* L in $\mathbb{R}^n$ is a discrete subgroup with full rank, that is, there exist $w_1, \ldots, w_n \in L$ such that $L = \mathbb{Z}w_1 + \cdots + \mathbb{Z}w_n$ and $\mathrm{Span}_{\mathbb{R}}(L) = \mathbb{R}^n$. The set $\{w_1, \ldots, w_n\}$ is called a *basis* of L. Given a lattice L with a basis $\{w_1, \ldots, w_n\}$, the *Gram matrix* of this basis is defined by $A = (a_{ij})_{1\le i,j\le n}$ with $a_{ij} = w_i^t w_j$, or equivalently, $A = g^t g$ where $g = (w_1, \cdots, w_n)$ is the matrix with the i-th column given by w_i. It is clear that the Gram matrix A is real symmetric and thus defines a quadratic form

$$Q_L(x) = x^t Ax = x^t g^t gx = \|gx\|^2.$$

where $\|\cdot\|$ is the usual Euclidean norm on $\mathbb{R}^n$. We note that a priori Q_L also depends on the choice of a basis of L, however, since we are interested in its values at integer points and $L = g\mathbb{Z}^n$, we have

$$Q_L(\mathbb{Z}^n) = \left\{\|w\|^2 : w \in L\right\} \tag{19.3}$$

is independent of the choice of basis, we thus only use the subscript L for this quadratic form. Moreover, it is easy to see from the definition that Q_L is positive definite. On the other hand, the following lemma shows that every positive definite quadratic form comes from some lattice.

Lemma 19.1. *For every positive definite quadratic form Q in n variables, there exists some lattice $L \subset \mathbb{R}^n$ such that $Q = Q_L$.*

Proof of Lemma 19.1. Let A be the real symmetric matrix representing Q, i.e. $Q(x) = x^t Ax$. Since Q is positive definite, the bilinear form $\langle\cdot,\cdot\rangle_Q$ sending $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ to $x^t Ay$ defines an inner product in $\mathbb{R}^n$. Let $\{u_1, \ldots, u_n\}$ be an orthonormal basis of $\mathbb{R}^n$ with respect to this inner product, i.e. $\langle u_i, u_j\rangle_Q = \delta_{ij}$ for any $1 \le i, j \le n$. Let $g \in \mathrm{GL}_n(\mathbb{R})$ be such that $gu_i = e_i$ for each $1 \le i \le n$, where $e_i \in \mathbb{R}^n$ is the vector with the i-th coordinate 1 and 0 elsewhere. Now let $L = g\mathbb{Z}^n$; it has a basis $w_1 = ge_1, \ldots, w_n = ge_n$. Then we have

$$u_i^t Au_j = \langle u_i, u_j\rangle_Q = \delta_{ij} = e_i^t e_j = (gu_i)^t gu_j = u_i^t g^t gu_j, \quad \forall\, 1 \le i, j \le n.$$

This implies that $A = g^t g$, or equivalently, $a_{ij} = e_i^t g^t ge_j = w_i^t w_j$ for any $1 \le i, j \le n$. Hence A is the Gram matrix of the basis $\{w_1, \ldots, w_n\}$, i.e. $Q = Q_L$ as desired. □

We introduce several notions regarding a lattice: We say a lattice $L \subset \mathbb{R}^n$ is

(1) *integral* (resp. *even*) if Q_L is integral (resp. even) as a quadratic form.
(2) *unimodular* if the volume of its fundamental domain, denoted by $\mathrm{vol}(\mathbb{R}^n/L)$, is 1.
(3) *self dual* if it equals its *dual lattice* which is defined by

$$L^\# := \left\{u \in \mathbb{R}^n : u^t w \in \mathbb{Z} \text{ for any } w \in L\right\}.$$

Remark 19.4. In view of the relation (19.3), we see that L is even if and only if $\|w\|^2 \in 2\mathbb{Z}$ for any $w \in L$.

Lemma 19.2. *Let $L \subset \mathbb{R}^n$ be a lattice with a basis $\{w_1, \dots, w_n\}$. Let $\{u_1, \dots, u_n\}$ be the dual basis, i.e. $u_i^t w_j = \delta_{ij}$ for any $1 \le i, j \le n$. Then $L^\# = \mathbb{Z}u_1 + \dots + \mathbb{Z}u_n$. In particular,*

$$\mathrm{vol}(\mathbb{R}^n/L)\,\mathrm{vol}(\mathbb{R}^n/L^\#) = 1. \tag{19.5}$$

Proof. Let $L' = \mathbb{Z}u_1 + \dots + \mathbb{Z}u_n$ be the lattice generated by $\{u_1, \dots, u_n\}$. We need to show $L' = L^\#$. The relation $L' \subset L^\#$ is clear in view of the definition of $L^\#$. For the other containment, take any $u \in L^\#$ and consider the vector

$$\tilde{u} := \sum_{i=1}^{n} (u^t w_i) u_i \in L'.$$

Then we have $\tilde{u}^t w_i = u^t w_i$ for any $1 \le i \le n$, implying that $u = \tilde{u} \in L'$. This proves that $L' = L^\#$. For the moreover part, let $g = (w_1, \cdots, w_n)$ and $\tilde{g} = (u_1, \cdots, u_n)$ so that $L = g\mathbb{Z}^n$ and $L^\# = \tilde{g}\mathbb{Z}^n$. Then the relations $u_i^t w_j = \delta_{ij}$ $(1 \le i, j \le n)$ imply that $\tilde{g}g = I_n$, i.e. $\tilde{g} = (g^t)^{-1}$. Thus we have

$$\mathrm{vol}(\mathbb{R}^n/L)\,\mathrm{vol}(\mathbb{R}^n/L^\#) = |\det(g)\det(\tilde{g})| = |\det(g)\det((g^t)^{-1})| = 1,$$

finishing the proof. □

In view of the relation (19.5) it is clear that a self dual lattice is also unimodular. The converse in general is not true. However, when L is even, these two notions are indeed equivalent.

Lemma 19.3. *An even lattice is self dual if and only if it is unimodular.*

Proof. One direction is trivial. We only need to show an even, unimodular lattice is also self dual. Assume $L \subset \mathbb{R}^n$ is an even, unimodular lattice. Then by (19.3) we have $\|w\|^2 \in 2\mathbb{Z}$ for any $w \in L$. In particular, for any $u, w \in L$, we have $\|u\|^2, \|w\|^2, \|u+w\|^2 = \|u\|^2 + \|w\|^2 + 2u^t w$ are all even, implying that $u^t w \in \mathbb{Z}$. Since u, w are arbitrarily chosen, by definition of the dual lattice, this implies $L \subset L^\#$. But then the assumption that L is unimodular forces $L = L^\#$. This finishes the proof. □

19.2. **Jacobi's inversion formula.** Let Q be a positive definite quadratic form in n variables. In view of Lemma 19.1 we may assume $Q = Q_L$ for some lattice $L \subset \mathbb{R}^n$. The main goal of this section is to prove Jacobi's inversion formula for the theta series Θ_{Q_L} defined in (19.2). Below we will use the slightly simplified notation Θ_L for Θ_{Q_L}.

Theorem 19.4 (Jacobi)**.** *Let L be a lattice in $\mathbb{R}^n$ with $L^\#$ its dual lattice. Then we have for any $z \in \mathbb{H}$,*

$$\Theta_L(z) = \frac{(-iz)^{-\frac{n}{2}}}{\mathrm{vol}(\mathbb{R}^n/L)} \Theta_{L^\#}(-1/z). \tag{19.6}$$

Remark 19.7. Here when n is odd, $i^{\frac{n}{2}}$ is defined as following: For any nonzero $z \in \mathbb{C}$ we choose $\arg(z) \in (-\pi, \pi]$ and define $\log z := \log|z| + i\arg(z)$ and for any $s \in \mathbb{C}$, define $z^s := e^{s \log z}$. For instance, following this convention, we have $i^{\frac{1}{2}} = e^{\frac{\pi}{4}i}$ while $(-i)^{\frac{1}{2}} = e^{-\frac{\pi}{4}i}$.

To prove this theorem, we need the following Poisson summation formula for lattices.

Proposition 19.5. *Let L and $L^{\#}$ be as above and let $\mathcal{S}(\mathbb{R}^n)$ be the space of Schwartz function on $\mathbb{R}^n$. Then for any $f \in \mathcal{S}(\mathbb{R}^n)$ we have*

$$\sum_{w\in L} f(w) = \frac{1}{\mathrm{vol}(\mathbb{R}^n/L)} \sum_{u\in L^{\#}} \widehat{f}(u), \tag{19.8}$$

where $\widehat{f}(y) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x^t y}\, dx$ is the Fourier transform of f.

Proof. The case when $L = \mathbb{Z}^n$ follows from exact same arguments as in section 1.1 (which handle the one dimensional case); we omit the proof. For a general lattice $L = g\mathbb{Z}^n$, we have

$$\sum_{w\in L} f(w) = \sum_{v\in\mathbb{Z}^n} f(gv) = \sum_{v\in\mathbb{Z}^n} f_g(v),$$

where $f_g(x) := f(gx)$ for any $x \in \mathbb{R}^n$. By direct computation we have

$$\begin{aligned}\widehat{f_g}(y) &= \int_{\mathbb{R}^n} f_g(x)e^{-2\pi i x^t y}\, dx = \int_{\mathbb{R}^n} f(gx)e^{-2\pi i x^t y}\, dx \\ &\overset{gx\mapsto x}{=} \frac{1}{|\det(g)|}\int_{\mathbb{R}^n} f(x)e^{-2\pi i (g^{-1}x)^t y}\, dx \\ &= \frac{1}{\mathrm{vol}(\mathbb{R}^n/L)}\int_{\mathbb{R}^n} f(x)e^{-2\pi i x^t (g^t)^{-1} y}\, dx \\ &= \frac{1}{\mathrm{vol}(\mathbb{R}^n/L)} \widehat{f}((g^t)^{-1}y).\end{aligned} \tag{19.9}$$

Hence applying the Poisson summation formula for the integer lattice and noting that $L^{\#} = (g^t)^{-1}\mathbb{Z}^n$ (see the proof of Lemma 19.2) we get

$$\begin{aligned}\sum_{w\in L} f(w) = \sum_{v\in\mathbb{Z}^n} \widehat{f_g}(v) &= \frac{1}{\mathrm{vol}(\mathbb{R}^n/L)}\sum_{v\in\mathbb{Z}^n} \widehat{f}((g^t)^{-1}v) \\ &= \frac{1}{\mathrm{vol}(\mathbb{R}^n/L)}\sum_{w\in L^{\#}} \widehat{f}(w),\end{aligned}$$

finishing the proof. □

Remark 19.10. We note that the arguments in section 1.1 indeed imply the following more general formula that

$$\sum_{v\in\mathbb{Z}^n} f(x+v) = \sum_{v\in\mathbb{Z}^n} \widehat{f}(v)e^{2\pi i v^t x}, \quad \forall\, x \in \mathbb{R}^n, \tag{19.11}$$

and (19.8) (for the case of $L = \mathbb{Z}^n$) follows by taking $x = 0$.

We can now give the

Proof of Theorem 19.4. First note that in view of the relation (19.3) we have

$$\Theta_L(z) = \sum_{w\in L} e^{\pi i\|w\|^2 z}.$$

Next, using analytic continuation we may prove (19.6) for $z = iy \in \mathbb{H}$. Let $f(x) := e^{-\pi\|x\|^2}$ and for any $\lambda > 0$ denote by $f_\lambda(x) = f(\lambda x)$. It is well known that the Fourier transform of this function equals itself, i.e. $\widehat{f} = f$. Moreover, applying

(19.9) (for $g = \lambda I_n$) we have $\widehat{f_\lambda}(\xi) = \lambda^{-n}\widehat{f}(\frac{\xi}{\lambda})$ for any $\xi \in \mathbb{R}^n$ and $\lambda > 0$. Now by definition

$$\begin{aligned}\Theta_L(iy) &= \sum_{w\in L} e^{-\pi\|w\|^2 y} = \sum_{w\in L} f_{\sqrt{y}}(w) \overset{(19.8)}{=} \frac{1}{\mathrm{vol}(\mathbb{R}^n/L)}\sum_{w\in L^\#}\widehat{f_{\sqrt{y}}}(w)\\ &= \frac{y^{-\frac{n}{2}}}{\mathrm{vol}(\mathbb{R}^n/L)}\sum_{w\in L^\#} f\left(\tfrac{w}{\sqrt{y}}\right) = \frac{y^{-\frac{n}{2}}}{\mathrm{vol}(\mathbb{R}^n/L)}\sum_{w\in L^\#} e^{-\pi\|w\|^2/y}\\ &= \frac{(-i\times iy)^{-\frac{n}{2}}}{\mathrm{vol}(\mathbb{R}^n/L)}\sum_{w\in L^\#} e^{\pi i\|w\|^2\times(-1/iy)} = \frac{(-i\times iy)^{-\frac{n}{2}}}{\mathrm{vol}(\mathbb{R}^n/L)}\Theta_{L^\#}(-1/iy),\end{aligned}$$

finishing the proof. Here for the fourth equality we used the identities that

$$\widehat{f_{\sqrt{y}}}(w) = y^{-\frac{n}{2}}\widehat{f}\left(\tfrac{w}{\sqrt{y}}\right) = y^{-\frac{n}{2}} f\left(\tfrac{w}{\sqrt{y}}\right). \qquad \square$$

19.3. **Even unimodular lattices.** A direct consequence of Jacobi's inversion formula is that the theta series of an even, unimodular lattice is a modular form with respect to the modular group.

Corollary 19.6. *Suppose L is an even unimodular lattice in $\mathbb{R}^n$ and further assume $n \equiv 0 \pmod 8$, then $\Theta_L \in \mathcal{M}_{n/2}(\mathrm{SL}_2(\mathbb{Z}))$.*

Proof. Using the expression

$$\Theta_L(x+iy) = 1 + \sum_{w\in L\setminus\{0\}} e^{\pi i\|w\|^2 x}e^{-\pi\|w\|^2 y},$$

it is easy to see that the above defining series is absolutely convergent for any $x+iy \in \mathbb{H}$ and uniformly convergent on any compact sets in $\mathbb{H}$. Hence it defines a holomorphic function. Moreover, one can also show that $\Theta_L(\infty) = 1$. It thus remains to show that $\Theta_L[T]_k = \Theta_L$ and $\Theta_L[S]_k = \Theta_L$, where $T = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ are the two generators of $\mathrm{SL}_2(\mathbb{Z})$ as before. The first equation is equivalent to $\Theta_L(z+1) = \Theta_L(z)$ for any $z \in \mathbb{H}$ which follows easily from the assumption that L is even. The second equation is just Jacobi's inversion formula (19.6) after noting that $(-i)^{\frac{n}{2}} = 1$ since $n \equiv 0 \pmod 8$. $\square$

The next proposition shows that the assumption that $n \equiv 0 \pmod 8$ is in fact redundant.

Proposition 19.7. *Suppose L is an even unimodular lattice in $\mathbb{R}^n$, then $n \equiv 0 \pmod 8$.*

Proof. Let $U = TS = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right) = \left(\begin{smallmatrix}1&-1\\1&0\end{smallmatrix}\right)$ with T, S as above. Note that $U^2 = \left(\begin{smallmatrix}0&-1\\1&-1\end{smallmatrix}\right)$ and $U^3 = \left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)$. By Lemma 19.3 L is also self dual. Since L is even we have $\Theta_L(z+1) = \Theta_L(z)$ for any $z \in \mathbb{H}$. Thus

$$\Theta_L(Uz) = \Theta_L(TSz) = \Theta_L(Sz+1) = \Theta_L(Sz) = (-iz)^{\frac{n}{2}}\Theta_L(z),$$

where the last equality is just Jacobi's inversion formula (19.6). From this and noting that $U^3 = -I_2$ we have for any $z \in \mathbb{H}$,

$$\begin{aligned}\Theta_L(z) &= \Theta_L(U^3z) = (-iU^2z)^{\frac{n}{2}}\Theta_L(U^2z) = (-iU^2z)^{\frac{n}{2}}(-iUz)^{\frac{n}{2}}\Theta_L(Uz)\\ &= (-iU^2z)^{\frac{n}{2}}(-iUz)^{\frac{n}{2}}(-iz)^{\frac{n}{2}}\Theta_L(z).\end{aligned}$$

This implies that

$$(-iU^2z)^{\frac{n}{2}}(-iUz)^{\frac{n}{2}}(-iz)^{\frac{n}{2}} = 1 \quad \forall\, z \in \mathbb{H}.$$

By direct computation we have

$$\begin{aligned}(-iU^2z)^{\frac{n}{2}}(-iUz)^{\frac{n}{2}}(-iz)^{\frac{n}{2}} &= (-i)^{\frac{3n}{2}}\left(U^2z \times Uz \times z\right)^{\frac{n}{2}} \\ &= (-i)^{\frac{3n}{2}} \times \left(\frac{-1}{z-1} \times \frac{z-1}{z} \times z\right)^{\frac{n}{2}} = i^{\frac{3n}{2}},\end{aligned}$$

implying that $i^{\frac{3n}{2}} = 1$ which further implies that $n \equiv 0 \pmod 8$. □

The next proposition shows that $n \equiv 0 \pmod 8$ is not just a necessary condition for the existence of even unimodular lattices, it is also sufficient.

Proposition 19.8. *There exists an even unimodular lattice in $\mathbb{R}^n$ if and only if $n \equiv 0 \pmod 8$.*

Proof. The direction "$\Rightarrow$" is just Proposition 19.7. For the other direction, we assume $n \equiv 0 \pmod 8$ and we construct an even unimodular lattice as following: Let

$$F_n := \left\{ v \in \mathbb{Z}^n : \sum_{i=1}^{n} v_i \equiv 0 \pmod 2 \right\}$$

be the set consisting of integer vectors with sum of its entries even. Next, let $\delta = (\frac{1}{2}, \cdots, \frac{1}{2}) \in \mathbb{R}^n$ and let $E_n = \langle \delta, F_n \rangle$ be the group generated by δ and F_n. We claim that E_n is an even unimodular lattice. First, note that F_n is kernel of the group homomorphism form $\mathbb{Z}^n$ to $\mathbb{Z}/2\mathbb{Z}$ sending v to $\sum_{i=1}^n v_i \pmod 2$. Hence F_n is an index 2 subgroup of $\mathbb{Z}^n$. Next, using the fact that $2\delta \in F_n$ (since n is even) we see that E_n is also a lattice in $\mathbb{R}^n$ containing F_n as an index 2 subgroup. Hence $\mathrm{vol}(\mathbb{R}^n/E_n) = \mathrm{vol}(\mathbb{R}^n/\mathbb{Z}^n) = 1$, i.e. E_n is unimodular. Next we show E_n is even. It suffices to show $\|w\|^2 \in 2\mathbb{Z}$ for any $w \in E_n$. By definition we can write $w = v + k\delta$ for some $v \in F_n$ and $k \in \mathbb{Z}$. Then we have (noting that $x^2 \equiv x \pmod 2$ for any $x \in \mathbb{Z}$)

$$\|w\|^2 = \|v\|^2 + kv^t(2\delta) + k^2\|\delta\|^2 \equiv (k+1)\sum_i v_i + \frac{k^2 n}{4} \equiv 0 \pmod 2,$$

where for the second congruence equation we used the assumptions that $v \in F_n$ and $n \equiv 0 \pmod 8$ (so that $\frac{n}{4}$ is even). This finishes the proof. □

We have the following direct corollary regarding the counting function $r_Q(m)$ for quadratic forms coming from an even, unimodular lattice.

Corollary 19.9. *Let L be an even unimodular lattice in $\mathbb{R}^n$ and let $Q = Q_L$ be the quadratic form coming from L. Then we have for any $m \in \mathbb{N}$,*

$$r_Q(2m) = -\frac{2k}{B_k}\sigma_{k-1}(m) + O_\epsilon(m^{\frac{k-1}{2}+\epsilon}),$$

where $k = \frac{n}{2}$, B_k is the k-th Bernoulli number and $\sigma_s(m) = \sum_{d|m} d^s$ is the s-divisor function. Moreover, for $n = 8$ or 16 the above error term does not exist.

Remark 19.12. For $n = 8$, one can check that $\{\delta, e_1 - e_2, e_2 - e_3, \cdots, e_6 - e_7, e_6 + e_7\}$ is a basis for E_8. The Gram matrix of this basis is

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 & 2 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 1 & 2 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix},$$

which corresponds to the quadratic form

$$Q(x) = 2\sum_{i=1}^{8} x_i^2 - 2\sum_{i=1}^{7} x_i x_{i+1} + 2x_1 x_8.$$

For this quadratic form Corollary 19.9 implies that $r_Q(2m) = 240\sigma_3(m)$ (noting that $B_4 = -\frac{1}{30}$.) Similarly, since $B_8 = -\frac{1}{30}$ we have $r_Q(2m) = 480\sigma_7(m)$ for any quadratic form coming from an even unimodular lattice in dimension 16.

Proof. By Corollary 19.6 we have $\Theta_L \in \mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}E_k \oplus \mathcal{S}_k(\mathrm{SL}_2(\mathbb{Z}))$ where E_k is the normalized weight-k Eisenstein series. Hence $\Theta_L = \lambda E_k + f$ for some $\lambda \in \mathbb{C}$ and $f \in \mathcal{S}_k(\mathrm{SL}_2(\mathbb{Z}))$. By comparing the constant term in both sides we get that $\lambda = 1$. Noting that $r_Q(2m)$ is the m-th Fourier coefficients of Θ_L (cf. (19.2)) and using the Fourier expansion of E_k (see (2.23)) we get that

$$r_Q(2m) = -\frac{2k}{B_k}\sigma_{k-1}(m) + R(m),$$

where $R(m)$ is the m-th Fourier coefficient of the cusp form f. The formula then follows from Deligne's bound that $|R(m)| \ll_\epsilon m^{\frac{k-1}{2}+\epsilon}$ on the Ramanujan's conjecture. The moreover part is true since for $n = 8$ or 16, $\mathcal{S}_k(\mathrm{SL}_2(\mathbb{Z})) = \{0\}$ by the dimension formula (4.1). $\square$

Finally, we mention that in general even unimodular lattices are not unique. Indeed, there are 1 (E_8), 2 (E_{16} and $E_8 \oplus E_8$[18]) and 24 even unimodular lattices in dimension 8, 16 and 24 respectively, and the number grows rapidly after dimension 24. For instance there are more than 1 billion even unimodular lattices in dimensional 32; see [Kin03, Corollary 17] and the references therein for more details.

In the next section we study quadratic forms whose theta series are no longer modular forms with respect to the modular group. For simplicity of presentation we do not treat the most general case, instead we consider the most classical case, namely the sum-of-squares quadratic form.

20. Sum of squares functions

Let $n \in \mathbb{N}$ and let

$$Q_n(x) = x_1^2 + \cdots + x_n^2 \tag{20.1}$$

be the sum-of-n-squares quadratic form. It is a classical question to represent a positive integer by Q_n. For small n it is not always the case that such a representation

[18] This is the direct sum of two E_8 lattices given by $E_8 \oplus E_8 := \{(u, v) \in \mathbb{R}^{16} : u, v \in E_8\}$.

exists. Indeed, a positive integer m is a sum of two squares if and only if m has the prime decomposition $m = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ with α_i even whenever $p_i \equiv 3 \pmod 4$ (see e.g. [HW08, Theorem 366]), while Legendre's theorem (see e.g. [Shi20]) states that m is a sum of three squares if and only if $m \neq 4^a(8b+7)$ for some non-negative integers a, b. Finally, Lagrange's four-square theorem (see e.g. [HW08, Theorem 369]) confirms that every positive integer is a sum of four squares. This, in particular, also implies that every positive integer is a sum of $n \geq 4$ squares.

In this section we derive explicit formulas for the number of such representations, (i.e. for the counting function $r_{Q_n}(m)$) for the special case when $n \equiv 0 \pmod 4$. We note that the other cases are similar but slightly more involved. For simplicity of notation we abbreviate $r_{Q_n}(m)$ and Θ_{Q_n} by $r_n(m)$ and Θ_n respectively. As before we use modular forms and the main ingredient for our approach is the Jacobi's inversion formula (19.6).

Proposition 20.1. *Assume $n \equiv 0 \pmod 4$ and let $k = \frac{n}{2}$. We have $\Theta_n \in \mathcal{M}_k(\Gamma_\theta)$ if $n \equiv 0 \pmod 8$ and $\Theta_n \in \mathcal{M}_k(\Gamma(2))$ if $n \equiv 4 \pmod 8$, where $\Gamma_\theta = \langle T^2, S\rangle$ is the congruence subgroup generated by T^2 and S as discussed in Exercise 6.8.*

Proof. The facts that Θ_n is holomorphic on $\mathbb{H}$ and at cusps follow from similar arguments as in Corollary 19.6. Next, note that $\Theta_n(z+2) = \Theta_n(z)$ for any $z \in \mathbb{H}$, i.e. $\Theta_n[T^2]_k = \Theta_n$. Moreover, applying Jacobi's inversion formula (19.6) to $L = \mathbb{Z}^n$ we get

$$\Theta_n(Sz) = (-iz)^k \Theta_n(z), \quad \forall\, z \in \mathbb{H}.$$

When $n \equiv 0 \pmod 8$, $k = \frac{n}{2} \equiv 0 \pmod 4$ and $(-i)^k = 1$, thus $\Theta_n(Sz) = z^k\Theta_n(z)$, i.e. $\Theta_n[S]_k = \Theta_n$. This implies that $\Theta_n \in \mathcal{M}_k(\Gamma_\theta)$.

When $n \equiv 4 \pmod 8$ we have $\Theta_n[Sz] = -\Theta_n(z)$ for any $z \in \mathbb{H}$, i.e. $\Theta_n[S]_k = -\Theta_n$. Then let $U = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$ and note that $U = -ST^2S$. Thus

$$\Theta_n[U]_k = \Theta_n[-ST^2S] = -\Theta_n[T^2S]_k = -\Theta_n[S]_k = \Theta_n.$$

This shows that Θ_n is weakly modular with respect to $\langle \pm I_2, T^2, U\rangle$, which by Exercise 24 below is the principle congruence subgroup $\Gamma(2)$. This finishes the proof. $\square$

Exercise 24. *Show that $\Gamma(2) = \langle \pm I_2, \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)\rangle$.*

As before we will proceed by writing Θ_n as a linear combination of functions from $\mathcal{M}_k(\Gamma_\theta)$ or $\mathcal{M}_k(\Gamma(2))$ and then compare the Fourier coefficients in both sides. The main term would come from Eisenstein series. In the next two sections we first study the Eisenstein series of a principle congruence subgroup $\Gamma(N)$ and then specify to the special case when $N = 2$ where certain things can be simplified.

20.1. **Eisenstein series of principle congruence subgroups.** Let $N \in \mathbb{N}$. Recall from Lemma 6.10 that the cusps of $\Gamma(N)$ are in one-one-one correspondence with order N elements in $(\mathbb{Z}/N\mathbb{Z})^2$. For any integer $k \geq 3$ and and any order N element $\overline{v} \in (\mathbb{Z}/N\mathbb{Z})^2$ the *weight-k Eisenstein series associated with $\overline{v}$* is defined by

$$G_k^{\overline{v}}(z) := \sum_{\substack{m \in \mathbb{Z}^2 \setminus \{0\} \\ m \equiv \overline{v} \pmod N}} \frac{1}{(m_1 z + m_2)^k}, \quad z \in \mathbb{H}. \tag{20.2}$$

The following lemma shows that $G_k^{\overline{v}}$ is indeed a weight-k modular form with respect to $\Gamma(N)$.

Lemma 20.2. *For any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ we have $G_k^{\overline{v}}[\gamma]_k = G_k^{\overline{v\gamma}}$. In particular, $G_k^{\overline{v}} \in \mathcal{M}_k(\Gamma(N))$.*

Proof. The in particular part follows from the first statement and the fact that $\Gamma(N)$ acts trivially on the group $(\mathbb{Z}/N\mathbb{Z})^2$. We thus only need to prove the first statement. For this take $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we have by definition

$$\begin{aligned} G_k^{\overline{v}}[\gamma]_k(z) &= (cz+d)^{-k} \sum_{\substack{m\in\mathbb{Z}^2\setminus\{0\} \\ m\equiv\overline{v} \pmod N}} \frac{1}{(m_1\frac{az+b}{cz+d}+m_2)^k} \\ &= \sum_{\substack{m\in\mathbb{Z}^2\setminus\{0\} \\ m\equiv\overline{v} \pmod N}} \frac{1}{((m_1a+m_2c)z+(m_1b+m_2d))^k}. \end{aligned}$$

Making a change of variable $n = (n_1, n_2) := (m_1a + m_2c, m_1b + m_2d)$ and noting that $n = m\gamma$ runs over all nonzero integer pairs satisfying $n \equiv \overline{v\gamma} \pmod N$ we get

$$G_k^{\overline{v}}[\gamma]_k(z) = \sum_{\substack{n\in\mathbb{Z}^2\setminus\{0\} \\ n\equiv\overline{v\gamma} \pmod N}} \frac{1}{(n_1z+n_2)^k} = G_k^{\overline{v\gamma}}(z),$$

as desired. $\square$

Next, we compute the values of $G_k^{\overline{v}}$ at cusps. First, using similar computation as in Lemma 20.2 one can easily compute

$$G_k^{\overline{v}}(\infty) := \lim_{y\to\infty} G_k^{\overline{v}}(iy) = \sum_{\substack{0=m_1\equiv v_1 \pmod N \\ 0\neq m_2\equiv v_2 \pmod N}} \frac{1}{m_2^k}$$

is exactly the contribution of the terms with $m_1 = 0$. For a general cusp $r = \gamma\infty \in \mathbb{Q} \cup \{\infty\}$ we can use Lemma 20.2 to compute

$$G_k^{\overline{v}}(r) := G_k^{\overline{v}}[\gamma]_k(\infty) = \begin{cases} 0 & \text{if } (v\gamma)_1 \not\equiv 0 \pmod N, \\ \sum_{m\equiv(v\gamma)_2 \pmod N} \frac{1}{m^k} & \text{if } (v\gamma)_1 \equiv 0 \pmod N. \end{cases}$$

Next, we compute its Fourier expansion (at the cusp ∞). Note that since $\Gamma(N)_\infty = \langle \text{“}\pm\text{”} \left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right) \rangle$, $G_k^{\overline{v}}$ has the following Fourier expansion

$$G_k^{\overline{v}}(z) = \sum_{n=0}^{\infty} c_{k,\overline{v}}(n) e\left(\tfrac{nz}{N}\right)$$

for some $c_{k,\overline{v}}(n) \in \mathbb{C}$. Since $e(\frac{nz}{N})$ decays exponentially in $\mathfrak{Im}(z)$ as $\mathfrak{Im}(z) \to \infty$, we see that the constant term $c_{k,\overline{v}}(0)$ is exactly $\lim_{y\to\infty} G_k^{\overline{v}}(iy)$ which by definition is $G_k^{\overline{v}}(\infty)$. Non-constant Fourier coefficients are computed in the following proposition.

Proposition 20.3. *We have for any $n \geq 1$,*

$$c_{k,\overline{v}}(n) = N^{-k} C_k \left(\sum_{\substack{d\mid n \\ \frac{n}{d}\equiv v_1 \pmod N}} d^{k-1} e\left(\tfrac{dv_2}{N}\right) + (-1)^k \sum_{\substack{d\mid n \\ \frac{n}{d}\equiv -v_1 \pmod N}} d^{k-1} e\left(\tfrac{-dv_2}{N}\right) \right),$$

where $C_k = \frac{(-2\pi i)^k}{(k-1)!}$.

Proof. As mentioned above $c_{k,\overline{v}}(0) = G_k^{\overline{v}}(\infty)$ is the sum over all terms with $m_1 = 0$. Thus we have

$$\begin{aligned} G_k^{\overline{v}}(z) &= c_{k,\overline{v}}(0) + \sum_{\substack{m_1>0 \\ m \equiv \overline{v} \pmod N}} \frac{1}{(m_1 z + m_2)^k} + \sum_{\substack{m_1<0 \\ m \equiv \overline{v} \pmod N}} \frac{1}{(m_1 z + m_2)^k} \\ &= c_{k,\overline{v}}(0) + \sum_{\substack{m_1>0 \\ m \equiv \overline{v} \pmod N}} \frac{1}{(m_1 z + m_2)^k} + (-1)^k \sum_{\substack{m_1>0 \\ m \equiv -\overline{v} \pmod N}} \frac{1}{(m_1 z + m_2)^k} \\ &=: c_{k,\overline{v}}(0) + I + II. \end{aligned}$$

Here for the second equality we made a change of variable $m \mapsto -m$. Now to compute I we rewrite m_2 as $m_2 = v_2 + \ell N$ with $\ell \in \mathbb{Z}$ and apply the identity (2.19) to get that

$$\begin{aligned} \sum_{m_2 \equiv v_2 \pmod N} \frac{1}{(m_1 z + m_2)^k} &= \sum_{\ell \in \mathbb{Z}} \frac{1}{N^k (\frac{m_1 z + v_2}{N} + \ell)^k} \\ &= N^{-k} C_k \sum_{d=1}^{\infty} d^{k-1} e\left(d \frac{m_1 z + v_2}{N}\right). \end{aligned}$$

Hence

$$\begin{aligned} I &= N^{-k} C_k \sum_{\substack{m_1 \geq 1 \\ m_1 \equiv v_1 \pmod N}} \sum_{d=1}^{\infty} d^{k-1} e\left(d \frac{m_1 z + v_2}{N}\right) \\ &\overset{dm_1 \mapsto n}{=} N^{-k} C_k \sum_{n=1}^{\infty} \sum_{\substack{d \mid n \\ \frac{n}{d} \equiv v_1 \pmod N}} d^{k-1} e\left(\frac{d v_2}{N}\right) e\left(\frac{nz}{N}\right). \end{aligned}$$

The formula for II is identical except with $-\overline{v}$ in place of $\overline{v}$. Plugging these formulas for I and II into the previous expression for $G_k^{\overline{v}}$ we get the desired formula for $c_{k,\overline{v}}(n)$. □

20.2. **Specifying to $N = 2$.** We now specify our discussion to the case when $N = 2$ where the formulas can be further simplified mainly due to the simple fact that $x \equiv -x \pmod 2$ for any $x \in \mathbb{Z}$. First note that since $-I_2 \in \Gamma(2)$, $\mathcal{M}_k(\Gamma(2)) = \{0\}$ if k is odd, we thus assume $k \geq 4$ is even. Note that there are three vectors in $(\mathbb{Z}/2\mathbb{Z})^2$ with order 2, namely $(1,0), (0,1)$ and $(1,1)$. They correspond to cusps $\infty, 0 = S\infty$ and $1 = TS\infty$ respectively. For the values at cusps we have $G_k^{(1,0)}(\infty) = 0$,

$$\begin{aligned} G_k^{(1,0)}(0) = G_k^{(1,0)}[S]_k(\infty) &= G_k^{(1,0)S}(\infty) = G_k^{(0,1)}(\infty) \\ &= \sum_{m \text{ odd}} \frac{1}{m^k} = 2(1 - 2^{-k})\zeta(k), \end{aligned}$$

and $G_k^{(1,0)}(1) = G_k^{(1,0)TS}(\infty) = G_k^{(1,1)}(\infty) = 0$. The computations for the other two Eisenstein series are similar and are summarized in the following chart.

(20.3)

	∞	0	1
$E_k^{(1,0)}$	0	1	0
$E_k^{(0,1)}$	1	0	0
$E_k^{(1,1)}$	0	0	1

Here $E_k^{\overline{v}} := \frac{1}{2(1-2^{-k})\zeta(k)} G_k^{\overline{v}}$ is the normalized Eisenstein series at $\overline{v}$.

Next, for the non-constant Fourier coefficients, since $v_1 \equiv -v_1 \pmod 2$, k is even and $e\left(\frac{dv_2}{N}\right) = e\left(\frac{-dv_2}{N}\right) = (-1)^{dv_2}$ we have for $n \neq 0$,

$$c_{k,\overline{v}}(n) = 2^{1-k} C_k \sum_{\substack{d \mid n \\ \frac{n}{d} \equiv v_1 \pmod N}} (-1)^{dv_2} d^{k-1} =: 2^{1-k} C_k \sigma_{k-1}^{\overline{v}}(n).$$

Explicitly, we have

$$\sigma_{k-1}^{(1,0)}(n) = \sum_{\substack{d \mid n \\ \frac{n}{d} \text{ odd}}} d^{k-1}, \quad \sigma_{k-1}^{(0,1)}(n) = \sum_{\substack{d \mid n \\ \frac{n}{d} \text{ even}}} (-1)^d d^{k-1},$$

and

$$\sigma_{k-1}^{(1,1)}(n) = \sum_{\substack{d \mid n \\ \frac{n}{d} \text{ odd}}} (-1)^d d^{k-1}.$$

Remark 20.4. For later reference we note that $\sigma_{k-1}^{(1,0)}(n)$ and $\sigma_{k-1}^{(0,1)}(n)$ have the following alternative expressions which can be checked directly:

$$\sigma_{k-1}^{(1,0)}(n) = \sum_{\substack{d \mid n \\ \frac{n}{d} \text{ odd}}} (-1)^{n+d} d^{k-1} \quad \text{and} \quad \sigma_{k-1}^{(0,1)}(n) = \sum_{\substack{d \mid n \\ \frac{n}{d} \text{ even}}} (-1)^{n+d} d^{k-1}. \tag{20.5}$$

20.3. **Explicit formulas for $r_n(m)$.** With the results on Eisenstein series for $\Gamma(2)$ obtained in the previous two sections, we derive in this section explicit formulas for the counting function $r_n(m)$. We first treat the case when $n \equiv 0 \pmod 8$. Recall that $\Theta_n \in \mathcal{M}_k(\Gamma_\theta)$ with $k = \frac{n}{2}$ and $\Gamma_\theta = \langle T^2, S \rangle$. Recall Γ_θ has two cusps represented by ∞ and 1 (see Example 6.8 and case (2) of Example 6.11) and thus $\mathcal{M}_k(\Gamma_\theta)$ contains two linearly independent Eisenstein series. We can use Eisenstein series of $\Gamma(2)$ to represent these two Eisenstein series.

Lemma 20.4. *Assume $n \equiv 0 \pmod 8$ and let $k = \frac{n}{2}$. Then we have $G_k^{(1,1)}, G_k^{(1,0)} + G_k^{(0,1)} \in \mathcal{M}_k(\Gamma_\theta)$.*

Proof. It suffices to verify $F[S]_k = F$ for $F = G_k^{(1,1)}$ or $G_k^{(1,0)} + G_k^{(0,1)}$. For $F = G_k^{(1,1)}$ we have

$$G_k^{(1,1)}[S]_k = G_k^{(1,1)S} = G_k^{(1,1)},$$

as desired. Here for the first equality we applied Lemma 20.2 and for last equality we used that $(1,1)S = (1,-1) \equiv (1,1) \pmod 2$. Similarly, for $F = G_k^{(1,0)} + G_k^{(0,1)}$

we have

$$\left(G_k^{(1,0)}+G_k^{(0,1)}\right)[S]_k = G_k^{(1,0)S}+G_k^{(0,1)S} = G_k^{(0,1)}+G_k^{(1,0)},$$

as desired. This finishes the proof. $\square$

Now in view of this lemma and Proposition 20.1 we have that for $n \equiv 0 \pmod 8$,

$$\Theta_n = \lambda_1 G_k^{(1,1)} + \lambda_2\left(G_k^{(1,0)}+G_k^{(0,1)}\right)+f$$

for some $\lambda_1,\lambda_2\in\mathbb{C}$ and $f\in\mathcal{S}_k(\Gamma_\theta)$. The next goal is to determine the coefficients λ_1 and λ_2. Since we have computed the values of these Eisenstein series at cusps, it suffices to compute the values of Θ_n at cusps. For the cusp ∞, it is easy to see that

$$\Theta_n(\infty) = \lim_{y\to\infty}\Theta_n(iy) = 1.$$

We thus only need to compute $\Theta_n(1)$.

Lemma 20.5. *Assume* $n\equiv 0 \pmod 4$ *we have* $\Theta_n(1)=0$.

Proof. Since $1 = TS\infty$, by definition $\Theta_n(1)=\Theta_n[TS]_k(\infty)$ and (noting that $TS = \left(\begin{smallmatrix}1 & -1\\ 1 & 0\end{smallmatrix}\right)$)

$$\begin{aligned}\Theta_n[TS]_k(z) &= z^{-k}\Theta_n\left(1-\tfrac{1}{z}\right) = z^{-k}\sum_{v\in\mathbb{Z}^n} e^{\pi i\|v\|^2(1-\frac{1}{z})}\\ &= z^{-k}\sum_{v\in\mathbb{Z}^n} e^{-\frac{\pi i\|v\|^2}{z}+2\pi i v^t\delta}\end{aligned}$$

where $\delta=(\frac12,\cdots,\frac12)\in\mathbb{R}^n$ and for the last equality we used that $\|v\|^2\equiv\sum_{i=1}^n v_i = 2v^t\delta \pmod 2$ for any $v\in\mathbb{Z}^n$. Plugging in $z=iy$ we get

$$\Theta_n[TS]_k(iy) = (iy)^{-k}\sum_{v\in\mathbb{Z}^n} e^{-\frac{\pi\|v\|^2}{y}+2\pi i v^t\delta}.$$

Let $f(x)=e^{-\pi\|x\|^2}$ and apply the generalized Poisson summation formula (19.11) for $f_{\sqrt{y}}$ we get for any $x\in\mathbb{R}^n$,

$$\begin{aligned}\sum_{v\in\mathbb{Z}^n} f_{\sqrt{y}}(x+v) &= \sum_{v\in\mathbb{Z}^n}\widehat{f_{\sqrt{y}}}(v)e^{2\pi i v^t x}\\ &= y^{-k}\sum_{v\in\mathbb{Z}^n} f\left(\tfrac{v}{\sqrt{y}}\right)e^{2\pi i v^t x}\\ &= y^{-k}\sum_{v\in\mathbb{Z}^n} e^{-\frac{\pi\|v\|^2}{y}+2\pi i v^t x},\end{aligned}$$

where for the second equality we used that $\widehat{f_{\sqrt{y}}}(x) = y^{-k}\widehat{f}(\frac{x}{\sqrt{y}}) = y^{-k}f(\frac{x}{\sqrt{y}})$. Hence taking $x=\delta$ we get

$$\Theta_n[TS]_k(iy) = i^{-k}\sum_{v\in\mathbb{Z}^n} f_{\sqrt{y}}(\delta+v) = i^{-k}\sum_{v\in\mathbb{Z}^n} e^{-\pi\|\delta+v\|^2 y}.$$

Taking $y\to\infty$ one can show that the above rightmost sum vanishes, implying that $\Theta_n(1)=0$. This concludes the proof. $\square$

We can now determine the coefficients λ_1 and λ_2 to compute $r_n(m)$.

Theorem 20.6. *Assume $n \equiv 0 \pmod 8$. Then for any $m \in \mathbb{N}$,*

$$r_n(m) = \frac{-2k}{(2^k-1)B_k}\sum_{d|m}(-1)^{m+d}d^{k-1} + O(m^{\frac{k}{2}}), \tag{20.6}$$

where B_k is the k-th Bernoulli number.

Proof. In view of the chart (20.3) and the computations that $\Theta_n(\infty) = 1$ and $\Theta_n(1) = 0$ we see that

$$\Theta_n = \frac{1}{2(1-2^{-k})\zeta(k)}\left(G_k^{(1,0)} + G_k^{(0,1)}\right) + f$$

for some $f \in \mathcal{S}_k(\Gamma_\theta)$. Taking the m-th Fourier coefficient in both sides we get

$$r_n(m) = \frac{2^{1-k}C_k}{2(1-2^{-k})\zeta(k)}\left(\sigma_{k-1}^{(1,0)}(m) + \sigma_{k-1}^{(0,1)}(m)\right) + R(m)$$

where $C_k = \frac{(-2\pi i)^k}{(k-1)!}$ and $R(m)$ is the m-th Fourier coefficient of the cusp form f. Using the formula $\zeta(k) = -\frac{(2\pi i)^k}{2k!}B_k$ (cf. Homework 1) we get

$$\frac{2^{1-k}C_k}{2(1-2^{-k})\zeta(k)} = \frac{-2k}{(2^k-1)B_k} \quad \text{for even } k \geq 2. \tag{20.7}$$

Moreover, using the alternative expression (20.5) for $\sigma_{k-1}^{(1,0)}(m)$ and $\sigma_{k-1}^{(0,1)}(m)$ we get

$$\sigma_{k-1}^{(1,0)}(m) + \sigma_{k-1}^{(0,1)}(m) = \sum_{d|m}(-1)^{m+d}d^{k-1}.$$

Finally we can finish the proof by plugging all these terms into the above expression for $r_n(m)$ and applying Hecke's bound Proposition 8.10 on Fourier coefficients of cusp forms to get $|R(m)| \ll m^{\frac{k}{2}}$. □

Remark 20.8. When $n = 8$ one can show using the pole/zero theorem Theorem 4.3 that $\mathcal{S}_4(\Gamma_\theta) = \{0\}$. This implies that (using also that $B_4 = -\frac{1}{30}$)

$$r_8(m) = 16\sum_{d|m}(-1)^{m+d}d^3.$$

Next, we treat the case when $n \equiv 4 \pmod 8$. Recall from Proposition 20.1 that in this case $\Theta_n \in \mathcal{M}_k(\Gamma(2))$ with $k = \frac{n}{2}$ as before. The case when $n = 4$ (i.e. when $k = 2$) is a little different since the construction of Eisenstein series discussed in the previous section is no longer valid. (Recall that we need the weight parameter k to be greater than 2 to ensure absolute convergence.) We thus treat this case separately later. For now we assume $n \geq 12$ and in this case we similarly have

$$\Theta_n = \lambda_1 G_k^{(1,1)} + \lambda_2 G_k^{(1,0)} + \lambda_3 G_k^{(0,1)} + f$$

for some $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ and $f \in \mathcal{S}_k(\Gamma(2))$. To determine these coefficients we need to evaluate values of Θ_n at the three cusps of $\Gamma(2)$ which can be represented by $\infty, 0, 1$ respectively. We similarly have $\Theta_n(\infty) = 1$ and $\Theta_n(1) = 0$ (cf. Proposition 20.5). It thus remains to compute $\Theta_n(0)$. For this we note that Jacobi's inversion formula implies that in this case (noting that $i^k = -1$) $\Theta_n[S]_k = -\Theta_n$ and hence

$$\Theta_n(0) = \Theta_n[S]_k(\infty) = -\Theta_n(\infty) = -1.$$

We can then determine these coefficients to give the following formula for $r_n(m)$.

Theorem 20.7. *Assume $n \equiv 4 \pmod 8$ with $n \geq 12$. Then we have*

$$r_n(m) = \frac{2k}{(2^k-1)B_k}(-1)^{m-1}\sum_{d|m}(-1)^{d+\frac{m}{d}}d^{k-1} + O(m^{\frac{k}{2}}).$$

Proof. Comparing the chart (20.3) and the above values of Θ_n at cusps, we get that

$$\Theta_n = \frac{1}{2(1-2^{-k})\zeta(k)}\left(G_k^{(0,1)} - G_k^{(1,0)}\right) + f$$

with $f \in \mathcal{S}_k(\Gamma(2))$. Taking the m-th Fourier coefficient and applying Hecke's bound on the Fourier coefficient of f and the identity (20.7) we get

$$r_n(m) = \frac{-2k}{(2^k-1)B_k}\left(\sigma_{k-1}^{(0,1)}(m) - \sigma_{k-1}^{(1,0)}(m)\right) + O(m^{\frac{k}{2}}).$$

Finally we can apply (20.5) to get

$$\begin{aligned}
\sigma_{k-1}^{(0,1)}(m) - \sigma_{k-1}^{(1,0)}(m) &= \sum_{\substack{d|m \\ \frac{m}{d}\text{ even}}}(-1)^{m+d}d^{k-1} - \sum_{\substack{d|m \\ \frac{m}{d}\text{ odd}}}(-1)^{m+d}d^{k-1} \\
&= (-1)^m \sum_{\substack{d|m \\ \frac{m}{d}\text{ even}}}(-1)^{d+\frac{m}{d}}d^{k-1} + \sum_{\substack{d|m \\ \frac{m}{d}\text{ odd}}}(-1)^{d+\frac{m}{d}}d^{k-1} \\
&= (-1)^m \sum_{d|m}(-1)^{d+\frac{m}{d}}d^{k-1}.
\end{aligned}$$

This finishes the proof. $\square$

Finally we treat the case when $n = 4$. In this case $\Theta_n \in \mathcal{M}_2(\Gamma(2))$. For this we need more backgrounds on weight-2 Eisenstein series. We refer the reader to [DS05, Chapter 4.6] for more details and here we only give a sketch of the necessary ingredients to compute $r_4(m)$. As mentioned before the problem is that the defining series for $k = 2$ in (20.2) is no longer absolutely convergent. Instead we define $G_2^{\bar{v}}$ with $\bar{v} \in (\mathbb{Z}/N\mathbb{Z})^2$ of order N by fixing the order of summation as following:

$$G_2^{\bar{v}}(z) := \sum_{\substack{m_1 \in \mathbb{Z} \\ m_1 \equiv v_1 \pmod N}} \sum_{\substack{m_2 \equiv v_2 \pmod N \\ m_2 \neq 0 \text{ if } m_1 = 0}} \frac{1}{(m_1 z + m_2)^2}, \quad z \in \mathbb{H}.$$

Then inspecting the proof of the Fourier expansion in sections 20.1 and 20.2 we see that $G_2^{\overline{v}}$ shares the same Fourier expansion formula as $G_k^{\overline{v}}$ for $k \geq 3$. However, due the the lack of absolute convergence $G_2^{\overline{v}}$ no longer satisfies the conclusion in Lemma 20.2, instead it satisfies the following transformation rule that

$$F[\gamma]_2 = F, \quad \forall\, \gamma \in \Gamma(N),$$

where $F(z) := G_2^{\overline{v}}(z) - \frac{\pi}{N^2 \mathfrak{Im} z}$. This function F is however not holomorphic anymore, to resolve this problem we consider *differences* of Eisenstein series. Indeed, for any two $\overline{v}, \overline{v}' \in (\mathbb{Z}/N\mathbb{Z})^2$ of order N, the difference

$$G_2^{\overline{v}}(z) - G_2^{\overline{v}'}(z) = \left(G_2^{\overline{v}}(z) - \frac{\pi}{N^2\mathfrak{Im} z}\right) - \left(G_2^{\overline{v}'}(z) - \frac{\pi}{N^2\mathfrak{Im} z}\right)$$

satisfies the same transformation rule as F, while is still holomorphic. Specifying to $N = 2$ we have the following more explicit description of the space $\mathcal{M}_2(\Gamma(2))$; see [DS05, p. 108 and Theorem 4.6.1].

Theorem 20.8. *We have* $\mathcal{M}_2(\Gamma(2)) = \mathrm{Span}_{\mathbb{C}}\left\{G_2^{(0,1)} - G_2^{(1,0)}, G_2^{(1,0)} - G_2^{(1,1)}\right\}$.

This, together with our previous computations of values of Θ_n and Eisenstein series at cusps implies that

$$\Theta_4 = \frac{1}{2(1-2^{-k})\zeta(k)}\left(G_k^{(0,1)} - G_k^{(1,0)}\right)$$

from which we can derive the famous Jacobi's four-squares formula.

Theorem 20.9 (Jacobi). *We have for any* $m \in \mathbb{N}$,

$$r_4(m) = 8\sum_{\substack{d|m \\ 4\nmid d}} d.$$

Proof. By the same arguments as above and using Theorem 20.8 we have

$$\Theta_4 = \frac{1}{2(1-2^{-2})\zeta(2)}\left(G_2^{(0,1)} - G_2^{(1,0)}\right),$$

which implies that (noting also that $B_2 = \frac{1}{6}$)

$$r_4(m) = 8(-1)^{m-1}\sum_{d|m}(-1)^{d+\frac{m}{d}}d.$$

If $m \not\equiv 0 \pmod 4$ (so that the condition $4 \nmid d$ for any divisor $d \mid m$ is automatically satisfied) one can easily check that

$$r_4(m) = 8(-1)^{m-1}\sum_{d|m}(-1)^{d+\frac{m}{d}}d = 8\sum_{d|m} d = 8\sum_{\substack{d|m \\ 4\nmid d}} d.$$

We thus assume $m \equiv 0 \pmod 4$. For such m we can write it as $m = 2^\alpha m_0$ with $\alpha \geq 2$ and m_0 odd. Then every divisor $d \mid m$ is of the form $d = 2^\beta d_0$ with $0 \leq \beta \leq \alpha$ and $d_0 \mid m_0$. We then have

$$\begin{aligned} r_4(m) &= -8\left(-\sum_{d_0|m_0} d_0 + \sum_{\beta=1}^{\alpha-1}\sum_{d_0|m_0} 2^\beta d_0 - \sum_{d_0|m_0} 2^\alpha d_0\right) \\ &= 8\left(2^\alpha - (2^\alpha - 2) + 1\right)\sum_{d_0|m_0} d_0 = 24\sum_{d_0|m_0} d_0, \end{aligned}$$

while

$$8\sum_{\substack{d|m \\ 4\nmid d}} d = 8\sum_{\beta=0}^{1}\sum_{d_0|m_0} 2^\beta d_0 = 24\sum_{d_0|m_0} d_0,$$

showing that

$$r_4(m) = 8\sum_{\substack{d|m \\ 4\nmid d}} d,$$

as desired. $\square$

21. Distribution of integral points on spheres

In view of the counting formula (20.6) we know that when $n \equiv 0 \pmod 8$ the number of integral points on the sphere $\{x \in \mathbb{R}^n : \|x\|^2 = m\}$ increases to ∞ as $m \to \infty$. In this section we study the *distribution* of these integral points. More precisely, we project these points onto the unit sphere $S^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$ and then study the distribution of these projected points on S^{n-1}. For this, for any $m \in \mathbb{N}$ define

$$A_m := \left\{ \tfrac{v}{\sqrt{m}} : v \in \mathbb{Z}^n,\ \|v\|^2 = m \right\} \subset S^{n-1}. \tag{21.1}$$

We will show that $\{A_m\}_{m\in\mathbb{N}}$ become *equidistributed* on S^{n-1} as $m \to \infty$, which roughly speaking says that these point sets become more and more evenly distributed on S^{n-1}. To state the main result, we need to introduce some more definitions.

Definition 21.2. Let $(X, \mathcal{B}, \mu)$ be a probability space with X a topological space, $\mathcal{B}$ the σ-algebra of Borel sets and μ a probability Borel measure on X. Let $C_c(X)$ be the space of compactly supported continuous functions on X. Given a sequence of finite point sets $\{A_m\}_{m\in\mathbb{N}}$ of X, we say $\{A_m\}_{m\in\mathbb{N}}$ *equidistribute on* X *with respect to* μ if for *any* $f \in C_c(X)$,

$$\frac{1}{\#A_m} \sum_{x\in A_m} f(x) \longrightarrow \int_X f\, d\mu \quad \text{as } m \to \infty. \tag{21.3}$$

Remark 21.4. Note that if $f = \chi_B$ is the indicator function of some Borel set $B \subset X$ with finite measure, we have

$$\sum_{x\in A_m} f(x) = \#(A_m \cap B).$$

Using an approximation argument one can replace the above continuous test functions by indicator functions to show that $\{A_m\}_{m\in\mathbb{N}}$ equidistribute on X with respect to μ if and only if for any Borel set $B \subset X$ with boundary of measure zero

$$\frac{\#(A_m \cap B)}{\#A_m} \longrightarrow \mu(B) \quad \text{as } m \to \infty.$$

So intuitively equidistribution means that asymptotically the number of points of A_m inside *any* fixed "nice" set is proportional to the measure (or mass) of this set. In practice in order to prove equidistribution one usually uses the following equivalent statement which also follows from an approximation argument: The sets $\{A_m\}_{m\in\mathbb{N}}$ equidistribute on X with respect to μ if and only if (21.3) holds for any f from a dense subset of $C_c(X)$.

As can be seen from the above definition, in order to talk about equidistribution one needs a measure on S^{n-1} to begin with. There is a natural probability measure (called *spherical measure*) on S^{n-1} which is characterized by the property of beign rotation-invariant.

Definition 21.5. Let $\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a smooth, compactly supported non-negative function satisfying $\int_{\mathbb{R}^n} \eta(\|x\|^2)\, dx = 1$. Let σ_n be the measure on S^{n-1} defined such that for any $f \in C(S^{n-1})$,

$$\int_{S^{n-1}} f(x)\, d\sigma_n(x) := \int_{\mathbb{R}^n} \eta(\|x\|^2) f\left(\tfrac{x}{\|x\|}\right)\, dx.$$

Lemma 21.1. *The measure σ_n is a rotation-invariant probability measure on S^{n-1}.*

Proof. The fact that σ_n is a probability measure comes from the assumption that $\int_{\mathbb{R}^n} \eta(\|x\|^2)\ dx = 1$. We thus only need to show it is rotation-invariant. Note that rotations in $\mathbb{R}^n$ (with respect to the origin) are parameterized by the special orthogonal group

$$\mathrm{SO}_n(\mathbb{R}) := \left\{g \in \mathrm{SL}_n(\mathbb{R}) : g^t g = I_n\right\}.$$

We thus need to show

$$\int_{S^{n-1}} f(gx)\ d\sigma_n(x) = \int_{S^{n-1}} f(x)\ d\sigma_n(x)$$

for any $f \in C(S^{n-1})$ and any $g \in \mathrm{SO}_n(\mathbb{R})$. For this by definition

$$\begin{aligned}\int_{S^{n-1}} f(gx)\ d\sigma_n(x) &= \int_{\mathbb{R}^n} \eta(\|x\|^2) f\left((\tfrac{gx}{\|gx\|}\right)\ dx\\ &= \int_{\mathbb{R}^n} \eta(\|x\|^2) f\left(\tfrac{x}{\|x\|}\right)\ dx\\ &= \int_{S^{n-1}} f(x)\ d\sigma_n(x),\end{aligned}$$

as desired. Here for the second equality we made a change of variable $gx \mapsto x$ and used the facts that $g \in \mathrm{SO}_n(\mathbb{R})$ preserves $\|\cdot\|$ and the Lebesgue measure. $\square$

Remark 21.6. Although our definition of σ_n depends on the choice of the function η, the defining properties of σ_n that it being rotation-invariant and a probability measure uniquely determine σ_n. Indeed, we have the following more intrinsic description of σ_n that for any Borel set $A \subset S^{n-1}$,

$$\sigma_n(A) = \frac{\mathrm{Leb}(\tilde{A})}{\mathrm{Leb}(B_1)},$$

where Leb is the usual Lebesgue measure on $\mathbb{R}^n$, $B_1 \subset \mathbb{R}^n$ is the unit ball centered at origin and $\tilde{A} := \{tx : 0 \le x \le 1,\ x \in A\}$. Note also that σ_n naturally gives an inner product on the function space $C(S^{n-1})$, namely, for any $f, g \in C(S^{n-1})$ we can define $\langle f, g\rangle := \int_{S^{n-1}} f\overline{g}\ d\sigma_n$.

We can now state the main result of this section.

Theorem 21.2. *Assume $n \equiv 0 \pmod 8$. The point sets $\{A_m\}_{m\in\mathbb{N}}$ with $A_m \subset S^{n-1}$ given as in (21.1) become equidistributed on S^{n-1} with respect to σ_n as $m \to \infty$.*

The remaining of this section is devoted to proving this theorem.

21.1. **Harmonic analysis on spheres.** In this section we collect some results from spherical harmonic analysis which are necessary for our proof to Theorem 21.2. The main reference is the online note by Garrett [Gar14]. Let

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

be the usual Laplace operator on $\mathbb{R}^n$. A function f on $\mathbb{R}^n$ is called *harmonic* if it is twice continuously differentiable in all variables and is annihilated by the Laplace operator, that is, $\Delta f = 0$. Let $\mathbb{C}[x]$ be the polynomial ring in variables $x =$

$(x_1, \ldots, x_n)$, and for any $d \geq 0$ let $C^{(d)}[x] := \{f \in \mathbb{C}[x] : f(\lambda x) = \lambda^d f(x) \ \forall\ \lambda \in \mathbb{C}\}$ be the subspace of homogeneous polynomials of degree d. Let

$$H_d := \left\{ f \in \mathbb{C}^{(d)}[x] : \Delta f = 0 \right\}$$

be the subspace of harmonic homogeneous polynomials of degree d and let

$$\mathcal{H}_d := \{ f|_{S^{n-1}} : f \in H_d \}$$

be the restriction of functions in H_d to S^{n-1}. Elements in $\mathcal{H}_d$ are usually called *spherical harmonics*. The main goal of this section is to prove the following theorem regarding $\mathcal{H}_d$.

Theorem 21.3. *The subspace $\oplus_{d\geq 0}\mathcal{H}_d$ is dense in $C(S^{n-1})$. Moreover, $\langle \mathcal{H}_d, \mathcal{H}_{d'} \rangle = 0$ whenever $d \neq d'$.*

We first prove the first statement of this theorem. Define a pairing

$$\mathbb{C}[x] \times \mathbb{C}[x] \to \mathbb{C}, \quad (P, Q) \mapsto \left(\overline{Q}(\partial) P(x)\right)\big|_{x=0},$$

where $Q(\partial)$ is $Q(x)$ but with $x = (x_1, \cdots, x_n)$ replaced by $(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})$ and $\overline{Q}(\partial)$ is its complex conjugate. For instance, if $Q(x) = x_1^2 + 2x_1x_2 - 3x_3^2$, then $Q(\partial) = \frac{\partial^2}{\partial x_1^2} + 2\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} - 3\frac{\partial^2}{\partial x_3^2}$. We list a few properties of this pairing.

Proposition 21.4. *Let $(\cdot,\cdot)$ be the paring as above.*

(1) *$(\cdot,\cdot)$ is a Hermitian form, that is for any $\lambda_1, \lambda_2 \in \mathbb{C}$, $P_1, P_2, Q_1, Q_2 \in \mathbb{C}[x]$,*
$$(\lambda_1 P_1 + \lambda_2 P_2, Q) = \lambda_1 (P_1, Q) + \lambda_2 (P_2, Q)$$
and
$$(P, \lambda_1 Q_1 + \lambda_2 Q_2) = \overline{\lambda}_1 (P, Q_1) + \overline{\lambda}_2 (P, Q_2).$$
(2) *For any $P, Q \in \mathbb{C}[x]$*
$$(\Delta P, Q) = (P, \|x\|^2 Q).$$
(3) *$(P, Q) = 0$ whenever P, Q are two homogeneous polynomials of different degrees.*
(4) *For any $(a_1, \cdots, a_n), (b_1, \cdots, b_n) \in \mathbb{Z}_{\geq 0}$ with $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$ we have*
$$(x_1^{a_1} \cdots x_n^{a_n}, x_1^{b_1} \cdots x_n^{b_n}) = \begin{cases} a_1! \cdots a_n! & \text{if } (a_1, \ldots, a_n) = (b_1, \ldots, b_n), \\ 0 & \text{otherwise.} \end{cases}$$
In particular, $(\cdot,\cdot)$ is positive definite on the subspace $C^{(d)}[x]$.

Proof. (1) can be checked directly. (2) is true since by definition

$$\begin{aligned}(P, \|x\|^2 Q) &= \overline{\|x\|^2 Q(\partial)} P(x)\big|_{x=0} = \Delta \overline{Q(\partial)} P(x)\big|_{x=0} \\ &= \overline{Q(\partial)} \Delta P(x)\big|_{x=0} = (\Delta P, Q).\end{aligned}$$

For (3), suppose $\deg(P) > \deg(Q)$, then $\overline{Q}(\partial)P(x)$ is a homogeneous polynomial of degree $\deg(P) - \deg(Q)$ and thus its evaluation at 0 vanishes. On the other hand if $\deg(P) < \deg(Q)$, then $\overline{Q}(\partial)P(x) = 0$. In particular its evaluation at 0 also vanishes. For (4) the case when $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$ follows from direct computation. If $(a_1, \ldots, a_n) \neq (b_1, \ldots, b_n)$, since $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$, there exists some $1 \leq i \leq n$ such that $b_i > a_i$. Then by direct computation $\frac{\partial^{b_i}}{\partial x_i^{b_i}}(x_1^{a_1} \cdots x_n^{a_n}) = 0$ implying that $(x_1^{a_1} \cdots x_n^{a_n}, x_1^{b_1} \cdots x_n^{b_n}) = 0$. The in particular part follows since the above assertion implies that $\{x_1^{a_1}\cdots x_n^{a_n} : a_i \geq 0,\ \sum_i a_i = d\}$ is an orthogonal basis with respect to $(\cdot,\cdot)$ and $(v,v) > 0$ for any element v from this basis. $\square$

Corollary 21.5. *For any $d \geq 2$, the map $\Delta : \mathbb{C}^{(d)}[x] \to \mathbb{C}^{(d-2)}[x]$ is surjective. Moreover, $(f, \|x\|^2 h) = 0$ for any $f \in H_d$ and $h \in C^{(d-2)}[x]$.*

Proof. For the first half, since $(\cdot,\cdot)$ is positive definite on $\mathbb{C}^{(d)}[x]$, it suffices to prove the statement that for any $h \in \mathbb{C}^{(d-2)}[x]$, $(\Delta f, h) = 0$ for any $f \in \mathbb{C}^{(d)}[x]$ implies that $h = 0$. This is true since by (2) of Proposition 21.4 we have

$$(f, \|x\|^2 h) = (\Delta f, h) = 0, \quad \forall\ f \in \mathbb{C}^{(d)}[x].$$

Since $(\cdot,\cdot)$ is positive definite on $\mathbb{C}^{(d)}[x]$, the above condition implies that $\|x\|^2 h = 0$ which then implies that $h = 0$. This proves the first half of this corollary. The second half follows since for for $f \in H_d$ and $h \in \mathbb{C}^{(d-2)}[x]$, again by (2) of Proposition 21.4 $(f, \|x\|^2 h) = (\Delta f, h) = (0, h) = 0$. $\square$

Corollary 21.6. *For any $d \geq 0$, if d is even then $\mathbb{C}^{(d)}[x] = H_d \oplus \|x\|^2 H_{d-2} \oplus \cdots \oplus \|x\|^d H_0$, while if d is odd then $\mathbb{C}^{(d)}[x] = H_d \oplus \|x\|^2 H_{d-2} \oplus \cdots \oplus \|x\|^{d-1} H_1$. In particular,*

$$\mathbb{C}[x]\big|_{S^{n-1}} = \bigoplus_{d\geq 0} \mathcal{H}_d. \tag{21.7}$$

Proof. For the first half, by Corollary 21.5 we have $(H_d, \|x\|^2\mathbb{C}^{(d-2)}[x]) = 0$ and $\dim H_d + \dim \mathbb{C}^{(d-2)}[x] = \dim \mathbb{C}^{(d)}[x]$ (since $\Delta : \mathbb{C}^{(d)}[x] \to \mathbb{C}^{(d-2)}[x]$ is surjective with kernel being H_d). These two conditions together with the fact that $(\cdot,\cdot)$ is positive definite on $\mathbb{C}^{(d)}[x]$ imply that

$$\mathbb{C}^{(d)}[x] = H_d \oplus \|x\|^2 \mathbb{C}^{(d-2)}[x]. \tag{21.8}$$

Then the desired decomposition of $\mathbb{C}^{(d)}[x]$ follows by applying (21.8) repeatedly. For the in particular part we need to show for any polynomial P, there exists $f_l \in \mathcal{H}_l$ for finitely many $l \geq 0$ such that $P|_{S^{n-1}} = \sum_l f_l$. Without loss of generality we may assume f is homogeneous of degree d for some $d \geq 0$. Then by the first half we have

$$P = P_d + \|x\|^2 P_{d-2} + \cdots$$

with $P_l \in H_l$. Restricting to S^{n-1} and noting that $\|x\|^2 = 1$ for $x \in S^{n-1}$ we get

$$P|_{S^{n-1}} = f_d + f_{d-2} + \cdots,$$

where $f_l := P_l|_{S^{n-1}} \in \mathcal{H}_l$. This finishes the proof. $\square$

Remark 21.9. From the above proof we see that $\dim H_d = \dim \mathbb{C}^{(d)}[x] - \dim \mathbb{C}^{(d-2)}[x]$ where the latter two can be easily computed. Indeed one can show $\dim \mathbb{C}^{(d)}[x] = \binom{n+d-1}{n-1}$ implying that $\dim H_d = \binom{n+d-1}{n-1} - \binom{n+d-3}{n-1}$.

Remark 21.10. Finally we give a more precise description of functions in H_d. First $H_0 = \mathbb{C}$ is just the space of constant functions, and $H_1 = \mathbb{C}^{(1)}[x]$ is the whole su bspace of linear polynomials (since Δ annihilates any linear polynomials). For $d \geq 2$, we state without proof that H_d is spanned by functions of the form $P_u(x) := (u^t x)^d$ with $u \in \mathbb{C}^n$ such that $\|u\|^2 = 0$. It is easy to see that any function of this form is an element in H_d. However the other direction is nontrivial and we refer the reader to [Iwa97, Theorem 9.1].

We can now prove the first half of Theorem 21.3.

(Sketch). By the Weierstrauss approximation theorem (see e.g. [Gar14, Appendix]) $\mathbb{C}[x]|_{S^{n-1}}$ is dense in the function space $C(S^{n-1})$, but the former set is just $\oplus_{d\geq 0}\mathcal{H}_d$ by Corollary 21.6. □

Next, we prove the the second half of (21.3). For this we introduce the Laplace operator on S^{n-1}. To define this operator, it suffices to define its action on smooth functions.

Definition 21.11. The *spherical Laplace operator* on S^{n-1}, denoted by Δ^S, is defined such that for any $f \in C^\infty(S^{n-1})$,

$$\Delta^S f := (\Delta F)\big|_{S^{n-1}},$$

where $F(x) := f\left(\frac{x}{\|x\|}\right)$ for any $x \in \mathbb{R}^n \setminus \{0\}$.

Recall from Remark 21.6 that we have introduced an inner product structure on $C^\infty(S^{n-1})$, the space of smooth functions on S^{n-1}, that for any $f, g \in C^\infty(S^{n-1})$, $\langle f, g\rangle = \int_{S^{n-1}} f\overline{g}\, d\sigma_n$. The following proposition shows that Δ^S is self-adjoint and non-positive with respect to this inner product.

Proposition 21.7. *For any $f, g \in C^\infty(S^{n-1})$ we have*

$$\langle \Delta^S f, g\rangle = \langle f, \Delta^S g\rangle, \tag{21.12}$$

and

$$\langle \Delta^S f, f\rangle \leq 0 \tag{21.13}$$

with the equality holds if and only if f is a constant.

Proof. Let $F(x) := f(x/\|x\|)$ and $G(x) := g(x/\|x\|)$. Abbreviate $r := \|x\|$ then by definition

$$\langle \Delta^S f, g\rangle = \int_{\mathbb{R}^n} \eta(r^2)(\Delta F)\,(x/r)\,\overline{G}\,(x/r)\; dx,$$

where η is the function as fixed in Definition 21.5. Note that $F(tx) = F(x)$ and $G(tx) = G(x)$ for any $t > 0$. Applying Δ to both sides of the equation $F(tx) = F(x)$ we get

$$t^2(\Delta F)(tx) = \Delta F(x).$$

Taking $t = 1/r$ we get $(\Delta F)\,(x/r) = r^2\Delta F(x)$, implying that

$$\langle \Delta^S f, g\rangle = \int_{\mathbb{R}^n} \eta(r^2)r^2(\Delta F)(x)\overline{G}(x)\; dx.$$

Now by integration by parts and noting that η is compactly supported we get (with $\delta(r^2) := r^2\eta(r^2)$)

$$\begin{aligned}\langle \Delta^S f, g\rangle &= -\int_{\mathbb{R}^n} \sum_{j=1}^n F_j(x)\frac{\partial}{\partial x_j}\left(\delta(r^2)\overline{G}(x)\right)\; dx\\ &= -\int_{\mathbb{R}^n} \sum_{j=1}^n F_j(x)\left(2x_j\delta'(r^2)\overline{G}(x) + \delta(r^2)\overline{G_j}(x)\right)\; dx.\end{aligned}$$

Since $F(tx) = F(x)$ for any $t > 0$ and $x \in \mathbb{R}^n \setminus \{0\}$, differentiating both sides with respect to t gives

$$\sum_{j=1}^{n} x_j F_j(tx) = 0, \quad \forall\, t > 0.$$

Plugging this identity (with $t = 1$) to the previous expression for $\langle \Delta^S f, g\rangle$ we get

$$\langle \Delta^S f, g\rangle = -\int_{\mathbb{R}^n} \sum_{j=1}^{n} F_j(x)\delta(r^2)\overline{G_j(x)}\, dx.$$

The above expression for $\langle \Delta^S f, g\rangle$ is symmetric in F and G, giving (21.12). For (21.13) taking $g = f$ (so that $F = G$) and we see from the above equation that

$$\langle \Delta^S f, f\rangle = -\int_{\mathbb{R}^n} \sum_{j=1}^{n} |F_j(x)|^2 \delta(r^2)\, dx \leq 0$$

with the equality holds if and only if $F_j = 0$ for all $1 \leq j \leq n$. The latter condition implies that F is a constant, or equivalently f is a constant. $\square$

We have the following direct corollary stating that eigenfunctions of Δ^S of distinct eigenvalues are mutually orthogonal with respect to $\langle \cdot, \cdot \rangle$.

Corollary 21.8. (1) *For any $f \in C^\infty(S^{n-1})$, suppose $\Delta^S f = \lambda f$ for some $\lambda \in \mathbb{C}$, then $\lambda \leq 0$.*
(2) *For any $f_1, f_2 \in C^\infty(S^{n-1})$, suppose $\Delta^S f_i = \lambda_i f_i$ with $\lambda_1 \neq \lambda_2$, then $\langle f_1, f_2\rangle = 0$.*

Proof. For (1) we have by (21.13),

$$\lambda\langle f, f\rangle = \langle \Delta^S f, f\rangle \leq 0.$$

If $\langle f, f\rangle > 0$, we then have $\lambda < 0$. If $\langle f, f\rangle = 0$, then $f = 0$ implying that $\Delta^S f = 0$, i.e. $\lambda = 0$. In both cases we have $\lambda \leq 0$. This proves (1). For (2) by (21.12) we have

$$\lambda_1\langle f_1, f_2\rangle = \langle \Delta^S f_1, f_2\rangle = \langle f_1, \Delta^S f_2\rangle = \lambda_2\langle f_1, f_2\rangle,$$

where for the last equality we used that $\lambda_2 \leq 0$ is real. This implies that $(\lambda_1 - \lambda_2)\langle f_1, f_2\rangle = 0$. Since $\lambda_1 \neq \lambda_2$, we must have $\langle f_1, f_2\rangle = 0$ as desired. $\square$

We now give the proof of the second half of Theorem 21.3.

Proof. In view of (2) of Corollary 21.8 it suffices to prove the following claim that

$$\Delta^S f = -d(d+n-2)f, \quad \forall\, f \in \mathcal{H}_d. \tag{21.14}$$

Take any $f \in \mathcal{H}_d$, by definition $f = P|_{S^{n-1}}$ for some $P \in H_d$, and

$$\Delta^S f = (\Delta F)|_{S^{n-1}}$$

with $F(x) = f(\frac{x}{\|x\|})$. Since f agrees with P on S^{n-1} and P is homogeneous of degree d we have $F(x) = P(\frac{x}{\|x\|}) = \|x\|^{-d}P(x)$. By direct computation we have for any $1 \leq j \leq n$,

$$\frac{\partial}{\partial x_j}F(x) = -d\|x\|^{-d-2}x_j P(x) + \|x\|^{-d}P_j(x),$$

and

$$\frac{\partial^2}{\partial x_j^2}F(x) = \frac{\partial}{\partial x_j}\left(-d\|x\|^{-d-2}x_jP(x) + \|x\|^{-d}P_j(x)\right)$$
$$= \left(d(d+2)\|x\|^{-d-4}x_j^2 - d\|x\|^{-d-2}\right)P(x) - 2d\|x\|^{-d-2}x_jP_j(x) + \|x\|^{-d}P_{jj}(x),$$

where $P_j := \frac{\partial}{\partial x_j}P$ and $P_{jj} := \frac{\partial^2}{\partial x_j^2}P$. Thus we have

$$\Delta F(x) = (d(d+2) - nd)\|x\|^{-d-2}P(x) - 2d\|x\|^{-d-2}\sum_{j=1}^{n} x_jP_j(x) + \|x\|^{-d}\Delta P(x).$$

Since $P \in H_d$, $\Delta P = 0$. Moreover note that for any homogeneous polynomial Q of degree d,

$$\sum_{j=1}^{n} x_jQ_j = dQ. \tag{21.15}$$

This identity can be directly checked for monomials and then extended to a general homogenous polynomial by linearity. Now applying (21.15) to the above expression for ΔF we get

$$\Delta F(x) = (d(d+2) - nd)\|x\|^{-d-2}P(x) - 2d^2\|x\|^{-d-2}P(x)$$
$$= -d(d+n-2)\|x\|^{-d-2}P(x).$$

Finally restricting to S^{n-1} gives the desired formula for $\Delta^S f$ (noting that $(\Delta F)|_{S^{n-1}} = \Delta^S f$ and $\|x\|^{-d-2}P|_{S^{n-1}} = f$). $\square$

21.2. **Theta series revisited.** Another main ingredient for our proof to Theorem 21.2 is to realize certain sum that appears naturally when proving equidistribution as Fourier coefficients of a cusp form. For this we need to study generalized theta series associated to harmonic polynomials that we now introduce.

Definition 21.16. Let d be a non-negative integer and let $P \in H_d$. The *theta series associated to the quadratic form Q_n (cf. (20.1)) and P* is defined by

$$\Theta_n(z;P) := \sum_{v\in\mathbb{Z}^n} P(v)e^{\pi i\|v\|^2 z}, \quad z \in \mathbb{H}.$$

Note that $\Theta_n(z;1)$ agrees with the theta series defined in (19.2) (with $Q = Q_n$). The connection between $\Theta_n(z;P)$ and the equidistribution problem is from the following simple identity which generalizes (19.2):

$$\Theta(z;P) = P(0) + \sum_{v\in\mathbb{Z}^n\setminus\{0\}} P(v)e^{\pi i\|v\|^2 z} = \delta_{d0} + \sum_{m=1}^{\infty}\sum_{\substack{v\in\mathbb{Z}^n\\ \|v\|^2=m}} P(v)e^{\pi imz}. \tag{21.17}$$

Here for the second equality we used that $P(0) = 0$ if $d \geq 1$. The main goal of this section is to prove the following theorem stating that when $n \equiv 0 \pmod 8$ and $d > 0$ the theta series $\Theta_n(z;P)$ is a cusp form of weight $\frac{n}{2} + d$.

Theorem 21.9. *Assume $n \equiv 0 \pmod 8$ and $d > 0$. Then for any $P \in H_d$, $\Theta_n(z;P) \in \mathcal{S}_{k+d}(\Gamma_\theta)$ with $k = \frac{n}{2}$ and $\Gamma_\theta = \langle T^2, S\rangle$ as before.*

The main tool for our proof to Theorem 21.9 is an inversion formula for $\Theta_n(z;P)$ which generalizes Theorem 19.4.

Theorem 21.10. *Let $n \in \mathbb{N}$, $d \geq 0$ and $k = \frac{n}{2}$. For any $x \in \mathbb{R}^n$, $P \in H_d$ and for any $z \in \mathbb{H}$ we have*

$$\sum_{v \in \mathbb{Z}^n} P(v+x) e^{\pi i \|v+x\|^2 z} = i^k z^{-k-d} \sum_{v \in \mathbb{Z}^n} P(v) e^{\pi i \left(-\|v\|^2/z + 2v^t x\right)}. \tag{21.18}$$

In particular, taking $x = 0$ we get

$$\Theta_n(z; P) = i^k z^{-k-d} \Theta_n(-1/z; P). \tag{21.19}$$

Proof. First note that when $d = 0$, $H_0 = \mathbb{C}$ is the space of constant functions and we may assume $f = 1$ in which case (21.18) reads as

$$\sum_{v \in \mathbb{Z}^n} e^{\pi i \|v+x\|^2 z} = i^k z^{-k} \sum_{v \in \mathbb{Z}^n} e^{\pi i \left(-\|v\|^2/z + 2v^t x\right)}. \tag{21.20}$$

Note that Jacobi's inversion formula (19.6) (with $L = \mathbb{Z}^n$) is the special case of this identity when $x = 0$. Using the same arguments as in the proof of Theorem 19.4 but with the generalzied Poisson summation formula (19.11) in place of (19.8) (with $L = \mathbb{Z}^n$) we can prove (21.19), i.e. the $d = 0$ case of (21.18). Next for $d \geq 1$, in view of Remark 21.10 we may assume $P(x) = (u^t x)^d$ where $u \in \mathbb{C}^n$ and satisfies $\|u\|^2 = 0$ if $d \geq 2$. Define a differential operator

$$\mathcal{D} := \sum_{j=1}^n u_j \frac{\partial}{\partial x_j}.$$

Applying $\mathcal{D}$ to the left hand side of (21.20) and by direct computation we get

$$\mathcal{D}(\mathrm{LHS}) = (2\pi i z) \sum_{v \in \mathbb{Z}^n} u^t (v+x) e^{\pi i \|v+x\|^2 z}.$$

More generally applying $\mathcal{D}$ d times to the left hand side of (21.20) we get

$$\begin{aligned} \mathcal{D}^d(\mathrm{LHS}) &= (2\pi i z)^d \sum_{v \in \mathbb{Z}^n} (u^t(v+x))^d e^{\pi i \|v+x\|^2 z} \\ &= (2\pi i z)^d \sum_{v \in \mathbb{Z}^n} P(v+x) e^{\pi i \|v+x\|^2 z}. \end{aligned}$$

Here when $d \geq 2$ we need to use the assumption that $\|u\|^2 = 0$. Similarly, applying $\mathcal{D}$ d times to the right hand side of (21.20) we get

$$\mathcal{D}^d(\mathrm{RHS}) = (2\pi i)^d i^k z^{-k} \sum_{v \in \mathbb{Z}^n} P(v) e^{\pi i \left(-\|v\|^2/z + 2v^t x\right)}.$$

Equating $\mathcal{D}^d(\mathrm{LHS})$ with $\mathcal{D}^d(\mathrm{RHS})$ gives (21.18). $\square$

With this inversion formula we can now give the

Proof of Theorem 21.9. Using similar arguments as before we see $\Theta_n(z; P)$ is a holomorphic function on $\mathbb{H}$ and vanishes at ∞ in view of the second equality in (21.17). Thus it suffices to show $F[T^2]_{k+d} = F$ and $F[S]_{k+d} = F$ with $F(z) := \Theta_n(z; P)$. The first equation is clear in view of the second equality in (21.17) while the second equation is just (21.19). This finishes the proof. $\square$

21.3. **Proof of main result.** In this section we collect all the results from the previous sections to give the proof of Theorem 21.2.

Proof. Since $\oplus_{d\geq 0}\mathcal{H}_d$ is dense in $C(S^{n-1})$ (see Theorem 21.3) and in view of the alternative criterion for equidistribution described in Remark 21.6, it suffices to show

$$\lim_{m\to\infty}\frac{1}{\#A_m}\sum_{x\in A_m}f(x)=\int_{S^{n-1}}f\,d\sigma_n \tag{21.21}$$

for any $d\geq 0$ and any $f\in\mathcal{H}_d$. If $d=0$, then f is a constant function and the above limit equation holds trivially. We thus assume $d>0$ and note that in this case by the second half of Theorem 21.3 the right hand side of (21.21) vanishes (since $\int_{S^{n-1}}f\,d\sigma_n=\langle f,1\rangle$). Thus it suffices to show

$$\lim_{m\to\infty}\frac{1}{\#A_m}\sum_{x\in A_m}f(x)=0. \tag{21.22}$$

Now by definition, there exists some $P\in H_d$ such that $f=P|_{S^{n-1}}$. Using the definition (21.1) of A_m we have

$$\sum_{x\in A_m}f(x)=\sum_{\substack{v\in\mathbb{Z}^n\\ \|v\|^2=m}}f\left(\frac{v}{\sqrt{m}}\right)=m^{-\frac{d}{2}}\sum_{\substack{v\in\mathbb{Z}^n\\ \|v\|^2=m}}P(v).$$

Now by (21.17) the sum $\sum_{\substack{v\in\mathbb{Z}^n\\ \|v\|^2=m}}P(v)$ is the m-th Fourier coefficient of the theta series $\Theta_n(z;P)$ which by Theorem 21.9 is contained in $\mathcal{S}_{k+d}(\Gamma_\theta)$ with $k=\frac{n}{2}$. Then by Hecke's bound (8.18) we have

$$\left|\sum_{\substack{v\in\mathbb{Z}^n\\ \|v\|^2=m}}P(v)\right|\ll m^{\frac{k+d}{2}}.$$

On the other hand $\#A_m=r_n(m)$ which by the asymptotic formula (20.6) can be seen to satisfy the grwoth condition that $\#A_m\asymp m^{k-1}$. Combining all these estimates we get

$$\left|\frac{1}{\#A_m}\sum_{x\in A_m}f(x)\right|\ll m^{1-k-\frac{d}{2}+\frac{k+d}{2}}=m^{1-\frac{k}{2}}.$$

With this estimate we can take $m\to\infty$ to prove (21.22) (since $k=\frac{n}{2}\geq 4$). This finishes the proof. □

Remark 21.23. Similar arguments also hold for other n's. Indeed with more involved analysis we can use Hecke's bound to establish the same equidistribution result for any $n\geq 5$, while for $n=4$ Deligne's bound on Ramanujan's conjecture (for integral weight modular forms) is sufficient. However, for ternary form Q_3 Deligne's bound just falls short for equidistribution. Indeed we know in this case by Legendre's theorem not every positive integer m is representable as a sum of three squares, i.e. A_m could be empty for m along an unbounded subsequence of $\mathbb{N}$. Nevertheless, using ergodic arguments Linnik [Lin68] proved equidistribution of $A_m\subset S^2$ along a subsequence admissible to Legendre's theorem and a certain splitting condition called *Linnik's condition*. This splitting condition was finally removed by Duke

[Duk88] using analytic methods which ultimately rely on Iwaniec's improvements to Hecke's bound for half integral modular forms; see [Iwa87].

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