Discontinuous Galerkin Methods

Jaap van der Vegt

Numerical Analysis and Computational Science Group
Department of Applied Mathematics
Universiteit Twente

Enschede, The Netherlands

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Steady-advection-reaction equation

- The steady advection-reaction equation with homogeneous inflow boundary condition is

$$\beta \cdot \nabla u + \mu u = f \qquad \text{in } \Omega,$$

$$u = 0 \qquad \text{on } \partial\Omega^-,$$

with $\beta \in \mathbb{R}^d$ the advection velocity, μ the reaction coefficient, and f the source term.

The inflow boundary $\partial\Omega^-$ is defined as

$$\partial\Omega^- := \{x \in \partial\Omega \mid \beta(x) \cdot n(x) < 0\}.$$

- The steady advection-reaction equation can also be written in the conservative form

$$\nabla \cdot (\beta u) + \widetilde{\mu} u = 0,$$

with $\widetilde{\mu} := \mu - \nabla \cdot \beta$.

Steady-advection-reaction equation

- The data μ and β are assumed to be in the function spaces,

$$\mu \in L^{\infty}(\Omega), \qquad \beta \in [Lip(\Omega)]^{d},$$

with $Lip(\Omega)$ the space spanned by Lipschitz continuous functions: $v \in Lip(\Omega)$ if there exists a Lipschitz constant L_v s.t. $\forall x, y \in \Omega$,

$$|v(x) - v(y)| \leq L_v |x - y|,$$

with $|x - y|$ the Euclidian norm of $x - y$ in $\mathbb{R}^d$.

- In addition, we assume that Ω is a polyhedron in $\mathbb{R}^d$ and that there exists a $\mu_0 > 0$ s.t.

$$f \in L^2(\Omega) \quad \text{and} \quad \Lambda := \mu - \frac{1}{2}\nabla \cdot \beta \geq \mu_0 \quad \text{a.e. in } \Omega.$$

Steady-advection-reaction equation

- Since $\beta \in [Lip(\Omega)]^d$ there holds $\beta \in [W^{1,\infty}(\Omega)]^d$ with $\|\nabla\beta_i\|_{[L^\infty(\Omega)]^d} \leq L_{\beta_i}$, $\forall i \in \{1, \cdots, d\}$, with $(\beta_1, \cdots, \beta_d) = \beta$ (Brenner and Scott, Math. Theory. FEM, 2008).

  We also define $L_\beta := \max_{1 \leq i \leq d} L_{\beta_i}$.

- The regularity assumption on β can be weakened to a bound on $\|\beta\|_{[L^\infty(\Omega)]^d}$ and $\|\nabla \cdot \beta\|_{L^\infty(\Omega)}$.

- Define the parameters

$$\tau_c := \{\max(\|\mu\|_{L^\infty(\Omega)}, L_\beta)\}^{-1}, \qquad \beta_c := \|\beta\|_{[L^\infty(\Omega)]d},$$

  which can be considered as a reference time and velocity.

- Note, τ_c is finite since if $\|\mu\|_{L^\infty(\Omega)} = L_\beta = 0$ this implies $\Lambda = 0$, which contradicts $\Lambda \geq \mu_0 > 0$ a.e. in Ω.

Graph space

Next we need to consider the function spaces in which the solution of the advection-reaction equation must be sought.

- Let $C_0^\infty(\Omega)$ be the space of infinitely differentiable functions with compact support, which is dense in $L^2(\Omega)$.

  For a function $v \in L^2(\Omega)$, the statement $\beta \cdot \nabla v \in L^2(\Omega)$ means that the linear form

  $$C_0^\infty(\Omega) \ni \phi \mapsto -\int_\Omega v\nabla \cdot (\beta\phi) \in \mathbb{R},$$

  is bounded in $L^2(\Omega)$. That is there exists C_v s.t.

  $$\forall \phi \in C_0^\infty(\Omega) \qquad \int_\Omega v\nabla \cdot (\beta\phi) \leq C_v \|\phi\|_{L^2(\Omega)}.$$

  Using the Riesz representation theorem, the function $\beta \cdot \nabla v$ is thus defined as the function representing the linear form $-\int_\Omega v\nabla \cdot (\beta\phi)$ in $L^2(\Omega)$.

Graph space

- Proof.

$$\begin{aligned}
-\int_\Omega v\nabla\cdot(\beta\phi) &= -\int_{\partial\Omega} v(n\cdot\beta)\phi + \int_\Omega(\beta\cdot\nabla v)\phi \\
&= \int_\Omega(\beta\cdot\nabla v)\phi \qquad \text{(since } \phi\in C_0^\infty(\Omega)) \\
&\leq \|\beta\cdot\nabla v\|_{L^2(\Omega)}\|\phi\|_{L^2(\Omega)} \\
&\leq C_v\|\phi\|_{L^2(\Omega)} \ \ \text{(with } C_v = \|\beta\cdot\nabla v\|_{L^2(\Omega)} < \infty \text{ since } \beta\cdot\nabla v\in L^2(\Omega))
\end{aligned}$$

Graph space

- (Graph space) The graph space is defined as

$$V := \{v \in L^2(\Omega) \mid \beta \cdot \nabla v \in L^2(\Omega)\},$$

and is equipped with the scalar product: For all $v, w \in V$,

$$(v, w)_V := (v, w)_{L^2(\Omega)} + (\beta \cdot \nabla v, \beta \cdot \nabla w)_{L^2(\Omega)}$$

and graph norm $\|v\|_V := (v, v)_V^{\frac{1}{2}}$.

Graph space

- Lemma. (Hilbertian structure of graph space). The graph space V together with the scalar product $(v, w)_V$ is a Hilbert space.

  Proof. Let $(v_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in V, then $(v_n)_{n\in\mathbb{N}}$ and $(\beta\cdot\nabla v_n)_{n\in\mathbb{N}}$ are Cauchy sequences in $L^2(\Omega)$.

  Let v and w be the limits of $(v_n)_{n\in\mathbb{N}}$ and $(\beta\cdot\nabla v_n)_{n\in\mathbb{N}}$ in $L^2(\Omega)$ as $n\to\infty$.

  Let $\phi\in C_0^\infty(\Omega)$, then $\forall n\in\mathbb{N}$,

$$\int_\Omega v_n\nabla\cdot(\beta\phi) = -\int_\Omega(\beta\cdot\nabla v_n)\phi$$

  so that

$$\int_\Omega v\nabla\cdot(\beta\phi) \leftarrow \int_\Omega v_n\nabla\cdot(\beta\phi) = -\int_\Omega(\beta\cdot\nabla v_n)\phi \to -\int_\Omega w\phi$$

  This means that $v\in V$ with $\beta\cdot\nabla v = w$.

Traces in graph space V

- Define the space

$$L^2(|\beta \cdot n|; \partial\Omega) := \{v \text{ is measurable on } \partial\Omega \mid \int_{\partial\Omega} |\beta \cdot n| v^2 < \infty\},$$

and the outflow boundary

$$\partial\Omega^+ := \{x \in \partial\Omega \mid \beta(x) \cdot n(x) > 0\}.$$

Assume that the inflow and outflow boundaries are well separated

$$\mathrm{dist}(\partial\Omega^-, \partial\Omega^+) := \min_{(x,y) \in \partial\Omega^- \times \partial\Omega^+} |x - y| > 0$$

Note, this means that $\partial\Omega^-$ and $\partial\Omega^+$ must be separated by a part of $\partial\Omega$ with $|\beta \cdot n| = 0$.

Traces in graph space V

- Lemma. (Traces and integration by parts rule). The trace operator

$$\gamma : C^0(\overline{\Omega}) \ni v \mapsto \gamma(v) := v|_{\partial\Omega} \in L^2(|\beta \cdot n|; \partial\Omega)$$

extends continuously to V, meaning that there is a C_γ s.t. $\forall v \in V$,

$$\|\gamma(v)\|_{L^2(|\beta\cdot n|;\partial\Omega)} \leq C_\gamma \|v\|_V.$$

Moreover, the following integration by parts formula holds. For all $v, w \in V$,

$$\int_\Omega \big((\beta \cdot \nabla v)w + (\beta \cdot \nabla w)v + (\nabla \cdot \beta)vw\big) = \int_{\partial\Omega} (\beta \cdot n)vw.$$

Traces in graph space V

Proof. Assume that the inflow and outflow boundaries are separated.

Then we can define then the functions $\psi^-, \psi^+ \in C^\infty(\overline{\Omega})$ such that

$$\psi^- + \psi^+ = 1 \quad \text{in } \overline{\Omega},$$

and

$$\phi^-|_{\partial\Omega^+} = 0 \ \text{ and } \ \phi^+|_{\partial\Omega^-} = 0.$$

Traces in graph space V

Let $v \in C^\infty(\overline{\Omega})$, then

$$\int_{\partial\Omega} v^2|\beta\cdot n| = \int_{\partial\Omega} v^2(\psi^+ + \psi^-)|\beta\cdot n| = \int_{\partial\Omega^-} v^2\psi^-|\beta\cdot n| + \int_{\partial\Omega^+} v^2\psi^+|\beta\cdot n|$$

(since at $\partial\Omega\backslash(\partial\Omega^- \cup \partial\Omega^+)$ we have $|\beta\cdot n| = 0$)

$$= \int_{\partial\Omega} v^2(\psi^+ - \psi^-)(\beta\cdot n) = \int_\Omega \nabla\cdot(v^2(\psi^+ - \psi^-)\beta) \quad \text{(use Gauss' theorem)}$$

$$= \int_\Omega \nabla\cdot((\psi^+ - \psi^-)\beta)v^2 + 2(\psi^+ - \psi^-)(\beta\cdot\nabla v)v$$

$$\leq \|\nabla\cdot((\psi^+ - \psi^-)\beta)\|_{L^\infty(\Omega)}\|v\|^2_{L^2(\Omega)} + \|\psi^+ - \psi^-\|_{L^\infty(\Omega)}\left(\|v\|^2_{L^2(\Omega)} + \|\beta\cdot\nabla v\|^2_{L^2(\Omega)}\right)$$

$$\leq C_\gamma\|v\|^2_V, \qquad \text{with } C_\gamma = \|\nabla\cdot((\psi^+ - \psi^-)\beta)\|_{L^\infty(\Omega)} + \|\psi^+ - \psi^-\|_{L^\infty(\Omega)}.$$

Traces in graph space V

Hence for all $v \in C^\infty(\overline{\Omega})$ we have

$$\|v\|_{L^2(|\beta\cdot n|;\partial\Omega)} \le C_\gamma \|v\|_V^2.$$

Next, consider $v \in V$. Since $C^\infty(\overline{\Omega})$ is dense in V there is a sequence $(v_n)_{n\in\mathbb{N}}$ in $C^\infty(\overline{\Omega})$ converging to $v \in V$.

The inequality

$$\|\gamma(v_n)\|_{L^2(|\beta\cdot n|;\partial\Omega)} \le C_\gamma \|v_n\|_V$$

implies that $\gamma(v_n)$ is a Cauchy sequence in $L^2(|\beta \cdot n|; \partial\Omega)$, with limit $\gamma(v)$ as $n \to \infty$.

The integration by parts formula can also be proven using a density argument.

Traces in graph space V

- Counter example for inflow-outflow separation.

  The separation of inflow and outflow boundaries is necessary if one wants traces in $L^2(|\beta \cdot n|; \partial\Omega)$.

  Consider the triangular domain

$$\Omega := \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_2 < 1 \text{ s.t. } |x_1| < x_2\},$$

  and set $\beta = (1, 0)^t$.

  Then the function $u(x_1, x_2) = x_2^\alpha$ is in V provided $\alpha > -1$, but $\gamma(u)$ is only in $L^2(|\beta \cdot n|; \partial\Omega)$ if $\alpha > -\frac{1}{2}$.

Traces in graph space V

Proof.

$$\int_\Omega u^2 dx_1 dx_2 = \int_0^1 x_2^{2\alpha}\Big(\int_{-x_2}^{x_2} dx_1\Big) dx_2 = \int_0^1 2\alpha^{2\alpha+1} dx_2,$$

which is finite is $\alpha > -1$, then $u \in L^2(\Omega)$.

Since $\beta \cdot \nabla u = (1,0)^t \cdot (0, \alpha x_2^{\alpha-1}) = 0$ we have $u \in V$.

At $\partial\Omega^- = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = -x_1, x_1 \in (-1,0)\}$ we have $n = (-\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2})$, hence

$$\int_{\partial\Omega^-} |\beta \cdot n| u^2 ds = \int_{-1}^0 \frac{1}{2}\sqrt{2}\, t^{2\alpha}\sqrt{2} dt,$$

which integral is finite if $2\alpha > -1$, which implies $\alpha > -\frac{1}{2}$.

For $-1 < \alpha \leq -\frac{1}{2}$ we thus have $u = x_2^\alpha \in V$, but $\gamma(u) \notin L^2(|\beta \cdot n|; \partial\Omega)$.

Weak formulation and well-posedness

- For a real number x define its positive and negative parts as

$$x^{\oplus} := \frac{1}{2}(|x| + x), \qquad x^{\ominus} := \frac{1}{2}(|x| - x).$$

  Note, $x^{\oplus}$ and $x^{\ominus}$ are both non-negative.

- Define the bilinear form $a(v, w) : V \times V \to \mathbb{R}$,

$$a(v, w) := \int_{\Omega} \mu v w + \int_{\Omega} (\beta \cdot \nabla v) w + \int_{\partial\Omega} (\beta \cdot n)^{\ominus} v w.$$

Weak formulation and well-posedness

- The bilinear form $a(v, w)$ is bounded on $V \times V$,

$$|a(v,w)| \leq (1 + \|\mu\|^2_{L^\infty(\Omega)})^{\frac{1}{2}} \|v\|_V \|w\|_{L^2(\Omega)} + C_\gamma^2 \|v\|_V \|w\|_V.$$

Proof.

$$|a(v,w)| \leq \|\mu\|_{L^\infty(\Omega)} \|v\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)} + \|\beta \cdot \nabla v\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)} + C_\gamma^2 \|v\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)}$$

since

$$\begin{aligned}
\int_{\partial\Omega} (\beta \cdot n)^{\ominus} vw &= \int_{\partial\Omega} \sqrt{(\beta \cdot n)^{\ominus}}\, v \sqrt{(\beta \cdot n)^{\ominus}}\, w \\
&\leq \Big(\int_{\partial\Omega} (\beta \cdot n)^{\ominus} v^2 \Big)^{\frac{1}{2}} \Big(\int_{\partial\Omega} (\beta \cdot n)^{\ominus} w^2 \Big)^{\frac{1}{2}} \\
&\leq \|\gamma(v)\|_{L^2(|\beta \cdot n|; \partial\Omega)} \|\gamma(w)\|_{L^2(|\beta \cdot n|; \partial\Omega)} \\
&\leq C_\gamma^2 \|v\|_V \|w\|_V.
\end{aligned}$$

Weak formulation and well-posedness

- The steady advection-reaction equation can be represented in the weak form

$$\text{Find } u \in V, \text{ s.t.} \quad a(u, w) = \int_\Omega fw \qquad \forall w \in V. \tag{1}$$

  with bilinear form $a(v, w) : V \times V \to \mathbb{R}$,

$$a(v, w) := \int_\Omega \mu vw + \int_\Omega (\beta \cdot \nabla v) w + \int_{\partial\Omega} (\beta \cdot n)^{\ominus} vw.$$

- (Characterization of the solution of (1).) Assume that $u \in V$ solves (1). Then

$$\begin{aligned} \beta \cdot \nabla u + \mu u &= f \qquad \text{a.e. in } \Omega, \\ u &= 0 \qquad \text{a.e. in } \partial\Omega^-. \end{aligned} \tag{2}$$

Weak formulation and well-posedness

- Proof. Taking $w = \phi \in C_0^\infty(\Omega)$ in the weak formulation (1) gives

$$\int_\Omega (\mu u + \beta \cdot \nabla u - f)\phi = 0,$$

hence (2) follows since ϕ is dense in $L^2(\Omega)$. Using (2) into (1) then implies for all $w \in V$ that

$$\int_{\partial\Omega} (\beta \cdot n)^\ominus uw = 0.$$

Choosing $w = u$ then gives

$$\int_{\partial\Omega} (\beta \cdot n)^\ominus u^2 = 0.$$

This implies $u = 0$ at $\partial\Omega^-$ since $(\beta \cdot n)^\ominus > 0$ at $\partial\Omega^-$.

Weak formulation and well-posedness

- Lemma. (L^2-coercivity of $a(u,v)$). The bilinear form $a(u,v)$ is $L^2(\Omega)$-coercive on V, namely,

$$\forall v \in V, \qquad a(v,v) \geq \mu_0 \|v\|^2_{L^2(\Omega)} + \int_{\partial\Omega} \frac{1}{2} |\beta \cdot | v^2$$

Proof.

$$a(v,v) = \int_\Omega \mu v^2 + \int_\Omega (\beta \cdot \nabla v) v + \int_{\partial\Omega} (\beta \cdot n)^{\ominus} v^2$$

Using the integration by parts rule with $w = v$ gives

$$\int_\Omega (\beta \cdot \nabla v) v + (\beta \cdot \nabla v) v + (\nabla \cdot \beta) v^2 = \int_{\partial\Omega} (\beta \cdot n) v^2,$$

thus

$$\int_\Omega (\beta \cdot \nabla v) v = -\int_\Omega \frac{1}{2} (\nabla \cdot \beta) v^2 + \frac{1}{2} \int_{\partial\Omega} (\beta \cdot n) v^2.$$

Weak formulation and well-posedness

Hence

$$
\begin{aligned}
a(v, v) &= \int_\Omega \left(\mu v^2 - \frac{1}{2}(\nabla \cdot \beta) v^2\right) + \int_{\partial\Omega} \frac{1}{2}(\beta \cdot n) v^2 + \int_{\partial\Omega} (\beta \cdot n)^{\ominus} v^2 \\
&= \int_\Omega \Lambda v^2 + \int_{\partial\Omega} \frac{1}{2}|\beta \cdot n| v^2 \\
&\geq \mu_0 \|v\|^2_{L^2(\Omega)} + \int_{\partial\Omega} \frac{1}{2}|\beta \cdot n| v^2.
\end{aligned}
$$

- A consequence of the coercivity of the bilinear form $a(u, v)$ is that the weak formulation (1) has at most one solution.

Well-posedness

- The steady advection-reaction equation stated in the weak form

$$\text{Find } u \in V, \text{ s.t.} \quad a(u, w) = \int_\Omega fw \qquad \forall w \in V. \tag{3}$$

  is well-posed.

- Proof. First, consider (3) with boundary condition $u = 0$ at $\partial\Omega^-$ strongly enforced.

  Define $V_0 := \{v \in V \mid v|_{\partial\Omega^-} = 0\}$ and consider

$$\text{Find } u \in V_0 \text{ s.t. } a_0(u, w) = \int_\Omega fw \qquad \forall w \in L^2(\Omega),$$

  with

$$a_0(v, w) := \int_\Omega \mu v w + \int_\Omega (\beta \cdot \nabla v) w.$$

Well-posedness

For well-posedness we need to use the BNB-theorem

- Theorem (BNB theorem). Let X be a Banach space and let Y be a reflexive space. Let $a \in \mathcal{L}(X \times Y, \mathbb{R})$ and let $f \in Y'$. Then the problem:

$$\text{Find } u \in X \text{ s.t. } a(u, w) = \langle f, w \rangle_{Y', Y} \;\; \forall w \in Y$$

is well posed if and only if

i) There is a $C_{sta} > 0$ s.t.

$$\forall v \in X, \quad C_{sta} \|v\|_X \leq \sup_{w \in Y \setminus \{0\}} \frac{a(v, w)}{\|w\|_Y}.$$

ii) For all $w \in Y$, $a(v, w) = 0$ implies that $w = 0$, $\forall v \in X$.

Use the spaces $X = V_0$ and $Y = L^2(\Omega)$. Since V_0 is a closed subspace of the Hilbert space V, V_0 is also a Hilbert space.

We also have that $L^2(\Omega)$ is reflexive (actually $(L^2(\Omega))' = L^2(\Omega)$).

Well-posedness

Check the conditions of the BNB-theorem

$$\text{Let } v \in V_0 \text{ and set} \qquad \overline{S} = \sup_{w \in L^2(\Omega)\setminus\{0\}} \frac{a_0(v,w)}{\|w\|_{L^2(\Omega)}}.$$

Using the coercivity of $a(u,w)$, we have

$$\begin{aligned}
a_0(v,v) &= \int_\Omega (\mu - \frac{1}{2}\nabla\cdot\beta)v^2 + \int_{\partial\Omega} \frac{1}{2}(\beta\cdot n)v^2 \\
&\geq \mu_0 \|v\|^2_{L^2(\Omega)} + \int_{\partial\Omega^+} \frac{1}{2}(\beta\cdot n)v^2 \qquad \text{(since } v = 0 \text{ at } \partial\Omega^-\text{)} \\
&\geq \mu_0 \|v\|^2_{L^2(\Omega)} \qquad \text{(since } \beta\cdot n > 0 \text{ at } \partial\Omega^+\text{).}
\end{aligned}$$

Well-posedness

Hence, for $v \neq 0$, we have

$$\|v\|_{L^2(\Omega)}^2 \leq \frac{1}{\mu_0} a_0(v,v) \leq \frac{1}{\mu_0} \frac{a_0(v,v)}{\|v\|_{L^2(\Omega)}} \|v\|_{L^2(\Omega)}$$

$$\leq \frac{1}{\mu_0} \sup_{w \in L^2(\Omega) \setminus \{0\}} \frac{a_0(v,w)}{\|w\|_{L^2(\Omega)}} \|v\|_{L^2(\Omega)} = \frac{1}{\mu_0} \overline{S} \|v\|_{L^2(\Omega)}.$$

Thus $\|v\|_{L^2(\Omega)} \leq \frac{1}{\mu_0} \overline{S}$ for all $v \in V_0$.

Moreover

$$\|\beta \cdot \nabla v\|_{L^2(\Omega)} = \sup_{w \in L^2(\Omega) \setminus \{0\}} \frac{\int_\Omega (\beta \cdot \nabla v) w}{\|w\|_{L^2(\Omega)}} = \sup_{w \in L^2(\Omega) \setminus \{0\}} \frac{a_0(v,w) - \int_\Omega \mu v w}{\|w\|_{L^2(\Omega)}}$$

$$\leq \overline{S} + \|\mu\|_{L^\infty(\Omega)} \|v\|_{L^2(\Omega)}$$

$$\leq (1 + \mu_0^{-1} \|\mu\|_{L^\infty(\Omega)}) \overline{S}.$$

Well-posedness

Collecting all terms gives

$$\|v\|_V^2 = \|v\|_{L^2(\Omega)}^2 + \|\beta \cdot \nabla v\|_{L^2(\Omega)}^2 \leq \left(\mu_0^{-2} + (1 + \mu_0^{-1}\|\mu\|_{L^\infty(\Omega)})^2\right)\overline{S}^2,$$

hence

$$\|v\|_V \leq \left(\mu_0^{-2} + (1 + \mu_0^{-1}\|\mu\|_{L^\infty(\Omega)})^2\right)^{\frac{1}{2}} \sup_{w \in L^2(\Omega)\setminus\{0\}} \frac{a_0(v, w)}{\|w\|_{L^2(\Omega)}},$$

which is condition i) in the BNB-theorem.

Well-posedness

- Proof of condition ii) in the BNB-theorem.

  Let $w \in L^2(\Omega)$ be such that $a_0(v, w) = 0$ for all $v \in V_0$.

  Since $C_0^\infty(\Omega) \subset V_0$ (and dense) we obtain for $v \in C_0^\infty(\Omega)$ that $a_0(v, w) = 0$ implies there is a distribution

$$0 = a_0(v, w) = \int_\Omega \mu v w + \int_\Omega (\beta \cdot \nabla v) w = \int_\Omega (\mu w - \nabla \cdot (\beta w)) v \quad \text{(since } v \in C_0^\infty(\Omega)\text{)},$$

  which implies

$$\mu w - \nabla \cdot (\beta w) = 0 \qquad \text{in } \Omega,$$

  hence $\beta \cdot \nabla w = (\mu - \nabla \cdot \beta) w \in L^2(\Omega)$, thus $w \in V$.

Well-posedness

Use now the integration by parts formula

$$
\begin{aligned}
\int_{\partial\Omega}(\beta\cdot n)vw &= \int_\Omega \Big((\beta\cdot\nabla v)w + (\beta\cdot\nabla w)v + (\nabla\cdot\beta)vw\Big) \\
&= a_0(v,w) - \int_\Omega(\mu-\nabla\cdot\beta)vw + \int_\Omega(\beta\cdot\nabla w)v \\
&= a_0(v,w) \qquad (\text{since } \beta\cdot\nabla w = (\mu-\nabla\cdot\beta)w \in L^2(\Omega)) \\
&= 0.
\end{aligned}
$$

Taking $v = \psi^+ w$, with $\psi^+|_{\partial\Omega^-} = 0$, then $v \in V_0$ yields

$$
\int_{\partial\Omega}(\beta\cdot n)vw = \int_{\partial\Omega}(\beta\cdot n)\psi^+ w^2 = \int_{\partial\Omega^+}(\beta\cdot n)w^2 = \int_{\partial\Omega}(\beta\cdot n)^{\oplus} w^2 = 0,
$$

hence $w|_{\partial\Omega^+} = 0$.

Well-posedness

Since $\mu w - \nabla \cdot (\beta w) = 0$ in Ω, we have from $a_0(w, w) = 0$ and the coercivity of a_0 that

$$0 = \int_\Omega \mu w^2 - \nabla \cdot (\beta w) w = \int_\Omega (\mu - \frac{1}{2} \nabla \cdot \beta) w^2 - \int_{\partial\Omega} \frac{1}{2} (\beta \cdot n) w^2 \geq \mu_0 \|w\|^2_{L^2(\Omega)},$$

where we used $\int_{\partial\Omega} \frac{1}{2} (\beta \cdot n) w^2 \leq 0$ since $w|_{\partial\Omega^+} = 0$.

Hence $\|w\|_{L^2(\Omega)} \leq 0$, which implies that $w = 0$.

Condition ii) in the BNB-theorem is thus satisfied.

The existence of the solution results from the fact that u solves

$$\text{find } u \in V \text{ s.t. } \; a(u, w) = \int_\Omega f w \qquad \forall w \in V$$

since for $u \in V_0 \subset V$ we have for all $w \in V$ that $a(u, w) = a_0(u, w)$.

Finally, uniqueness of the solution follows from the $L^2(\Omega)$ coercivity of a_0.

Nonhomogeneous boundary condition

- Consider the nonhomogeneous boundary condition

$$u = g \quad \text{on } \partial\Omega^- .$$

Extend the data g to $\partial\Omega$ by setting $g = 0$ at $\partial\Omega \backslash \partial\Omega^-$ and assume that

$$g \in L^2(|\beta \cdot n|; \partial\Omega).$$

The steady advection-reaction equation weak form is

$$\text{Find } u \in V \text{ s.t. } \; a(u, w) = \int_\Omega fw + \int_{\partial\Omega} (\beta \cdot n)^\ominus gw \qquad \forall w \in V.$$

Surjectivity of traces

We first need to consider the surjectivity of the trace operator γ.

Consider the trace operator $\gamma : V \to L^2(|\beta \cdot n|; \partial\Omega)$ with $\gamma(v) := v|_{\partial\Omega}$.

- (Surjectivity of traces). For all $g \in L^2(|\beta \cdot n|; \partial\Omega)$, there is a $u_g \in V$ s.t. $u_g = g$ a.e. in $\partial\Omega^- \cap \partial\Omega^+$.

  Moreover, there is a C, only depending on Ω and β s.t.

$$\|u_g\|_V \leq C\|g\|_{L^2(|\beta \cdot n|;\partial\Omega)}.$$

Proof. Let $g \in L^2(|\beta \cdot n|; \partial\Omega)$ and $\psi_g : V \to \mathbb{R}$ be the linear map s.t. $\forall w \in V$,

$$\psi_g(w) = \int_{\partial\Omega} (\beta \cdot n) g w.$$

Surjectivity of traces

From the trace theorem we have

$$\|w\|_{L^2(|\beta\cdot n|;\partial\Omega)} \leq C_\gamma \|w\|_V$$

hence using the Cauchy-Schwarz inequality

$$\begin{aligned}|\psi_g(w)| &\leq \|g\|_{L^2(|\beta\cdot n);\partial\Omega)}\|w\|_{L^2(|\beta\cdot n);\partial\Omega)}\\ &\leq C_\gamma \|g\|_{L^2(|\beta\cdot n);\partial\Omega)}\|w\|_V,\end{aligned}$$

thus $\psi_g \in V'$ and $\|\psi_g\|_{V'} = \sup_{w\in V\setminus\{0\}} \frac{\int_{\partial\Omega}(\beta\cdot n)gw}{\|w\|_V} \leq C_\gamma \|g\|_{L^2(|\beta\cdot n|;\partial\Omega)}$

From the Riesz representation theorem we obtain that there exists a $z \in V$ s.t. $\forall w \in V$,

$$(z,w)_V = \int_\Omega zw + \int_\Omega (\beta\cdot\nabla z)(\beta\cdot\nabla w) = \psi_g(w) = \int_{\partial\Omega}(\beta\cdot n)gw. \tag{4}$$

Surjectivity of traces

Set $u_g := \beta \cdot \nabla z \in L^2(\Omega)$. Next, we check if $u_g \in V$. Taking $w = \phi \in C_0^\infty(\Omega)$ in (4) gives

$$\int_\Omega u_g(\beta \cdot \nabla \phi) = -\int_\Omega z\phi \qquad \text{(since } \phi = 0 \text{ at } \partial\Omega\text{)}.$$

Use the relation

$$\int_\Omega u_g(\beta \cdot \nabla \phi) = \int_\Omega u_g \nabla \cdot (\beta \phi) - u_g(\nabla \cdot \beta)\phi,$$

then we obtain

$$\begin{aligned} \int_\Omega u_g \nabla \cdot (\beta\phi) &= \int_\Omega u_g(\beta \cdot \nabla\phi) + u_g(\nabla \cdot \beta)\phi \\ &= -\int_\Omega z\phi + \int_\Omega (\nabla \cdot \beta) u_g \phi. \end{aligned}$$

Surjectivity of traces

We also have the relation

$$\int_\Omega u_g \nabla \cdot (\beta\phi) = \int_\Omega \nabla \cdot (u_g \beta \phi) - \nabla u_g \cdot \beta\phi$$

$$= -\int_\Omega (\beta \cdot \nabla u_g)\phi \qquad \text{(using integration by parts and } \phi = C_0^\infty(\Omega)\text{)}.$$

Hence from

$$\int_\Omega u_g \nabla \cdot (\beta\phi) = -\int_\Omega z\phi + \int_\Omega (\nabla \cdot \beta) u_g \phi$$

we obtain

$$-\int_\Omega (\beta \cdot \nabla u_g)\phi + \int_\Omega z\phi - \int_\Omega (\nabla \cdot \beta) u_g \phi = 0 \qquad \forall \phi \in C_0^\infty,$$

which implies $\beta \cdot \nabla u_g = z - (\nabla \cdot \beta) u_g \in L^2(\Omega)$. Thus $u_g \in V$.

Surjectivity of traces

The relations $u_g = \beta \cdot \nabla z$ and $\beta \cdot \nabla u_g = z - (\nabla \cdot \beta) u_g$ can be used to obtain the estimate

$$\begin{aligned}
\|u_g\|_V^2 &= \|u_g\|_{L^2(\Omega)}^2 + \|\beta \cdot \nabla u_g\|_{L^2(\Omega)}^2 \\
&= \|\beta \cdot \nabla z\|_{L^2(\Omega)}^2 + \|z - (\nabla \cdot \beta) u_g\|_{L^2(\Omega)}^2 \\
&\leq \|\beta \cdot \nabla z\|_{L^2(\Omega)}^2 + 2\|z\|_{L^2(\Omega)}^2 + 2|\nabla \cdot \beta|^2 \|u_g\|_{L^2(\Omega)}^2 \\
&\leq \|\beta \cdot \nabla z\|_{L^2(\Omega)}^2 + 2\|z\|_{L^2(\Omega)}^2 + 2|\nabla \cdot \beta|^2 \|\beta \cdot \nabla z\|_{L^2(\Omega)}^2 \\
&= (1 + 2|\nabla \cdot \beta|^2)\|\beta \cdot \nabla z\|_{L^2(\Omega)}^2 + 2\|z\|_{L^2(\Omega)}^2.
\end{aligned}$$

Hence

$$\|u_g\|_V \leq C\|z\|_V.$$

Surjectivity of traces

Next, since

$$\|z\|_V^2 = \int_{\partial\Omega} (\beta \cdot n) g z = \psi_g(z) \leq C_\gamma \|g\|_{L^2(|\beta \cdot n|;\partial\Omega)} \|z\|_V,$$

we have

$$\|z\|_V \leq C' \|\psi_g\|_{V'} \leq C \|g\|_{L^2(|\beta \cdot n|;\partial\Omega)},$$

hence

$$\|u_g\|_V \leq C' \|z\|_V \leq C \|g\|_{L^2(|\beta \cdot n|;\partial\Omega)},$$

where C, C' only depend on Ω and β.

Surjectivity of traces

Using the integration by parts formula we obtain for all $w \in V$,

$$
\begin{aligned}
\int_{\partial\Omega} (\beta \cdot n) u_g w &= \int_\Omega (\beta \cdot \nabla w) u_g + \int_\Omega (\beta \cdot \nabla u_g) w + \int_\Omega (\nabla \cdot \beta) u_g w \\
&= \int_\Omega (\beta \cdot \nabla w)(\beta \cdot \nabla z) + \int_\Omega z w \\
&\qquad\qquad\qquad\qquad (\text{using } u_g = \beta \cdot \nabla z \text{ and } \beta \cdot \nabla u_g = z - (\nabla \cdot \beta) u_g) \\
&= \int_{\partial\Omega} (\beta \cdot n) g w,
\end{aligned}
$$

where in the last step we used the Riesz representation theorem, namely that there exists a $z \in V$ s.t. $\forall w \in V$ s.t.

$$
(z, w)_V = \int_\Omega z w + \int_\Omega (\beta \cdot \nabla z)(\beta \cdot \nabla w) = \psi_g(w) = \int_{\partial\Omega} (\beta \cdot n) g w.
$$

Surjectivity of traces

Since

$$\int_{\partial\Omega} (\beta \cdot n)(u_g - g)w = 0 \qquad \forall w \in V,$$

and using the density of $C_0^\infty(\overline{\Omega})$ in V we have

$$u_g = g \quad \text{a.e. in } \Omega^+ \cap \Omega^-.$$

Well-posedness

- Theorem. (Well-posedness) The weak formulation

$$\text{Find } u \in V \text{ s.t. } \; a(u,w) = \int_\Omega fw + \int_{\partial\Omega} (\beta \cdot n)^\ominus g w \qquad \forall w \in V$$

is well-posed. Moreover, its unique solution $u \in V$ satisfies

$$\begin{aligned} \beta \cdot \nabla u + \mu u &= f && \text{a.e. in } \Omega \\ u &= g && \text{a.e. in } \partial\Omega^-. \end{aligned}$$

Well-posedness

Proof. Let $u_g \in V$ with $u_g = g$ a.e. in $\partial\Omega^- \cap \partial\Omega^+$ be given. Consider

$$\text{Find } v \in V \text{ s.t. } \; a(v, w) = \int_\Omega fw - a_0(u_g, w) \qquad \forall w \in V,$$

with

$$a_0(v, w) = \int_\Omega \mu vw + (\beta \cdot \nabla v)w.$$

The map $V \ni w \mapsto a_0(u_g, w) \in \mathbb{R}$ is bounded in $L^2(\Omega)$ since $\forall w \in L^2(\Omega)$,

$$\begin{aligned}
\left|a_0(u_g, w)\right| &= \Big| \int_\Omega \mu u_g w + (\beta \cdot \nabla u_g) w \Big| \\
&\leq \left(1 + \|\mu\|^2_{L^\infty(\Omega)}\right)^{\frac{1}{2}} \|u_g\|_V \|w\|_{L^2(\Omega)} \\
&\leq C \|g\|_{L^2(|\beta\cdot|;\partial\Omega)} \|w\|_{L^2(\Omega)}.
\end{aligned}$$

Well-posedness

Using Riesz' representation theorem we have

$$\int_\Omega fw - a_0(u_g, w) = \int_\Omega \widetilde{f} w \qquad \text{for some } \widetilde{f} \in L^2(\Omega),$$

hence the weak formulation is well-posed with the modified righthand side $\widetilde{f}$,

As before we can prove that $u = v + u_g$ satisfies

$$\mu u + \beta \cdot \nabla u = f \qquad \text{in } \Omega,$$

and with $v = 0$ and $u_g = g$ on $\partial\Omega^-$ we obtain,

$$u = g \qquad \text{on } \partial\Omega^-.$$

Central fluxes

- Discrete problem:

  Assume $\mu \in L^\infty(\Omega)$, $\beta \in [Lip(\Omega)]^d$. Seek a solution of the advection-reaction equation in the broken polynomial space $\mathbb{P}_d^k(\mathcal{T}_h)$.

  Set $V_h := \mathbb{P}_d^k(\mathcal{T}_h)$ and consider the discrete problem

  $$\text{Find } u_h \in V_h \text{ s.t. } \; a_h(u_h, v_h) = \int_\Omega f v_h \qquad (a_h \text{ yet to be defined}).$$

  In order to prove consistency of the DG discretization by plugging in the exact solution into a_h we need slightly more regularity.

Central fluxes

- (Regularity of exact solution) Assume that there is a partition $P_\Omega = \{\Omega_i\}_{1\le i\le N_\Omega}$ of Ω into disjoint polyhedra such that the exact solution u satisfies

  $$u \in V_* := V \cap H^1(P_\Omega),$$

  and set $V_{*h} := V_* + V_h$.

  This assumption implies that $\forall T \in \mathcal{T}_h$, $u|_T$ has traces on each face $F \in \mathcal{F}_T$ and $\mathrm{trace}(u) \in L^2(\mathcal{F})$.

- Lemma. (Jumps of u across interfaces). The exact solution $u \in V_*$ is s.t. $\forall F \in \mathcal{F}_h^i$

  $$(\beta \cdot n_F)[\![u]\!](x) = 0 \quad \text{a.e. for } x \in F.$$

Central fluxes

Proof. Let $F \in \mathcal{F}_h^i$ with $F = \partial T_1 \cap \partial T_2$.

This interface can be partitioned into a finite number of subsets $\{F_j\}_{1 \leq j \leq N_F}$ s.t. each F_j is shared by at most two elements in P_Ω.

Assume F_j is shared by $\Omega_1, \Omega_2 \in P_\Omega$. Let $\phi \in C_0^\infty(\Omega)$ with support only intersecting F_j and $\Omega_1,\ \Omega_2$.

Since $\phi \in C_0^\infty(\Omega)$, $u \in V$, the integration by parts formula gives

$$\int_\Omega \Big((\nabla \cdot \beta) u \phi + (\beta \cdot \nabla u)\phi + u(\beta \cdot \nabla \phi) \Big) = 0.$$

Using that the support of ϕ is only non-zero over Ω_1, Ω_2, we have

$$0 = \int_\Omega \{\cdots\} = \int_{T_1 \cap \Omega} \{\cdots\} + \int_{T_2 \cap \Omega} \{\cdots\} = \int_{F_j} (\beta \cdot n_F) [\![u]\!] \phi$$

Since $\phi \in C_0^\infty(\Omega)$ is arbitrary we thus have $(\beta \cdot n_F)[\![u]\!](x) = 0$ for a.e. $x \in F$.

Central fluxes

- Remark. The condition $(\beta \cdot n_F)[\![u]\!](x) = 0$ does not say anything on the jumps of the exact solution when $\beta \cdot n_F = 0$.

- Remark. (Weaker regularity assumption). The assumption $u \in V \cap H^{\frac{1}{2}+\epsilon}(P_\Omega)$, $\epsilon > 0$ is also sufficient since this ensures that $\mathrm{trace}(u)$ at F is in $L^2(F)$.

Heuristic derivation DG discretization

- Starting point for the DG discretization is a discrete bilinear form $a_h^{(0)}$ obtained from a by replacing $\beta \cdot \nabla$ with $\beta \cdot \nabla_h$.

  Define $a_h^{(0)} : V_{*h} \times V_h \to \mathbb{R}$ as

$$a_h^{(0)}(v, w_h) := \int_\Omega (\mu v w_h + (\beta \cdot \nabla_h v) w_h) + \int_{\partial\Omega} (\beta \cdot n)^{\ominus} v w_h.$$

  The bilinear form $a_h^{(0)}$ is consistent since the exact solution satisfies

$$\begin{aligned} \beta \cdot \nabla u + \mu u = f \qquad & \text{a.e. in } \Omega \\ u = 0 \qquad & \text{a.e. on } \partial\Omega^-. \end{aligned}$$

Heuristic derivation DG discretization

- Coercivity of the bilinear form a is not transferred to the discrete bilinear form $a_h^{(0)}$.

  Consider $v_h \in V_h$,

$$a_h^{(0)}(v_h, v_h) = \int_\Omega \mu v_h^2 + \sum_{T \in \mathcal{T}_h} \int_T (\beta \cdot \nabla v_h) v_h + \int_{\partial\Omega} (\beta \cdot n)^\ominus v_h^2.$$

  Use $\nabla \cdot (\frac{1}{2} v_h^2 \beta) = (\beta \cdot \nabla v_h) v_h + \frac{1}{2} v_h^2 \nabla \cdot \beta$, then

$$a_h^{(0)}(v_h, v_h) = \int_\Omega (\mu v_h^2 - \frac{1}{2}(\nabla \cdot \beta) v_h^2) + \sum_{T \in \mathcal{T}_h} \int_{\partial T} \frac{1}{2} v_h^2 (\beta \cdot n_T) + \int_{\partial\Omega} (\beta \cdot n)^\ominus v_h^2$$

$$= \int_\Omega \Lambda v_h^2 + \sum_{T \in \mathcal{T}_h} \int_{\partial T} \frac{1}{2} (\beta \cdot n_T) v_h^2 + \int_{\partial\Omega} (\beta \cdot n)^\ominus v_h^2$$

  with $\Lambda = \mu - \frac{1}{2} \nabla \cdot \beta$.

Heuristic derivation DG discretization

Use the fact that $n_{T_1} = -n_{T_2}$ if the elements T_1 and T_2 are connected at a face $F = \overline{T}_1 \cap \overline{T}_2$.

This gives the relation

$$\sum_{T \in \mathcal{T}_h} \int_{\partial T} \frac{1}{2}(\beta \cdot n_T) v_h^2 = \sum_{F \in \mathcal{F}_h^i} \int_F \frac{1}{2}(\beta \cdot n_F)[\![v_h^2]\!] + \sum_{F \in \mathcal{F}_h^b} \int_F \frac{1}{2}(\beta \cdot n) v_h^2,$$

with $[\![v_h]\!] = v_h|_L - v_h|_R$.

For all $F \in \mathcal{F}_h^i$ with $F = \partial T_1 \cap \partial T_2$, $v_i|_{T_i}$, $i \in \{1, 2\}$ we have

$$\frac{1}{2}[\![v_h^2]\!] = \frac{1}{2}(v_1^2 - v_2^2) = \frac{1}{2}(v_1 - v_2)(v_1 + v_2) = \frac{1}{2}[\![v_h]\!]\{\!\{v_h\}\!\},$$

hence

$$a_h^{(0)}(v_h, v_h) = \int_\Omega \Lambda v_h^2 + \sum_{F \in \mathcal{F}_h^i} \int_F (\beta \cdot n_F)[\![v_h]\!]\{\!\{v_h\}\!\} + \sum_{F \in \mathcal{F}_h^b} \int_F \frac{1}{2}(\beta \cdot n) v_h^2 + \int_{\partial\Omega} (\beta \cdot n)^{\ominus} v_h^2.$$

Heuristic derivation DG discretization

Next, we use the relation

$$\frac{1}{2}(\beta \cdot n)v_h^2 + (\beta \cdot n)^{\ominus} v_h^2 = \frac{1}{2}(\beta \cdot n)v_h^2 + \frac{1}{2}|\beta \cdot n|v_h^2 - \frac{1}{2}(\beta \cdot n)v_h^2$$
$$= \frac{1}{2}|\beta \cdot n|v_h^2.$$

The bilinear form $a_h^{(0)}$ is thus equal to

$$a_h^{(0)}(v_h, v_h) = \int_\Omega \Lambda v_h^2 + \sum_{F \in \mathcal{F}_h^i} \int_F (\beta \cdot n_F)[\![v_h]\!]\{\!\{v_h\}\!\} + \int_{\partial\Omega} \frac{1}{2}|\beta \cdot n|v_h^2.$$

The second term on the righthand side has no sign a priori and must be removed to obtain coercivity.

Heuristic derivation DG discretization

- Define the bilinear form $a_h^{cf} : V_{*h} \times V_h \to \mathbb{R}$ as

$$a_h^{cf}(v, w_h) := \int_\Omega \Big(\mu v w + (\beta \cdot \nabla_h v) w_h \Big) + \int_{\partial\Omega} (\beta \cdot n)^{\ominus} v w_h$$
$$- \sum_{F \in \mathcal{F}_h^i} \int_F (\beta \cdot n_F) [\![v]\!] \{\!\{ w_h \}\!\}.$$

Note, since $(\beta \cdot n_F)[\![u]\!] = 0$ for all $F \in \mathcal{F}_h^i$ the bilinear form is still consistent.

The coercivity of a_h^{cf} can be expressed in the following norm on V_{*h}

$$|\!|\!| v |\!|\!|_{cf}^2 := \tau_c^{-1} \|v\|_{L^2(\Omega)}^2 + \int_{\partial\Omega} \frac{1}{2} |\beta \cdot n| v^2,$$

with $\tau_c = \{\max(\|\mu\|_{L^\infty(\Omega)}, L_\beta)\}^{-1}$ and L_β the Lipschitz constant for β.

Note, $|\!|\!| \cdot |\!|\!|_{cf}$ is a norm since $\| \cdot \|_{L^2(\Omega)}$ is a norm.

Heuristic derivation DG discretization

- Lemma. (Consistency and discrete coercivity). The discrete bilinear form a_h^{cf} satisfies the following properties

  i) Consistency, namely for the exact solution $u \in V_*$,

  $$a_h^{cf}(u, v_h) = \int_\Omega f v_h \qquad \forall v_h \in V_h.$$

  ii) Coercivity on V_h wrp. to the $|||\cdot|||_{cf}$ norm,

  $$\forall v_h \in V_h, \quad a_h^{cf}(v_h, v_h) \geq C_{sta} \, |||\, v_h|||_{cf}^2,$$

  with $C_{sta} := \min(1, \tau_c \mu_0)$.

Proof. Consistency was already verified.

Coercivity follows directly from the construction of a_h^{cf} since

$$a_h^{cf}(v_h, v_h) = \int_\Omega \Lambda v_h^2 + \int_{\partial\Omega} \frac{1}{2}|\beta \cdot n| v_h^2$$

$$\geq C_{sta} \, ||| \, v_h |||_{cf}^2,$$

with $C_{sta} := \min(1, \tau_c \mu_0)$.

Heuristic derivation DG discretization

- Lemma. (Equivalent expression for a_h^{cf}). For all $(v, w_h) \in V_{*h} \times V_h$ there holds

$$a_h^{cf}(v, w_h) = \int_\Omega \Big((\mu - \nabla \cdot \beta) v w_h - v(\beta \cdot \nabla_h w_h) \Big) + \int_{\partial\Omega} (\beta \cdot n)^{\oplus} v w_h$$
$$+ \sum_{F \in \mathcal{F}_h^i} \int_F (\beta \cdot n_F) \{\!\{ v \}\!\} [\![w_h]\!].$$

This expression is useful to identify the numerical fluxes in the DG discretization and to analyze an upwind-type DG discretization.

Heuristic derivation DG discretization

Proof. We start with the original expression for a_h^{cf}

$$a_h^{cf}(v, w_h) := \int_\Omega \Big(\mu v w + (\beta \cdot \nabla_h v) w_h\Big) + \int_{\partial\Omega} (\beta \cdot n)^{\ominus} v w_h$$
$$- \sum_{F \in \mathcal{F}_h^i} \int_F (\beta \cdot n_F) [\![v]\!] \{\!\{ w_h \}\!\}.$$

Use

$$\nabla_h \cdot (\beta v w_h) = (\beta \cdot \nabla_h v) w_h + (\beta \cdot \nabla_h w_h) v + v w_h \nabla \cdot \beta,$$

then we obtain

$$a_h^{cf}(v, w_h) = \int_\Omega \Big((\mu - \nabla \cdot \beta) v w_h - v(\beta \cdot \nabla_h w_h)\Big) + \sum_{T \in \mathcal{T}_h} \int_{\partial T} (\beta \cdot n_T) v w_h$$
$$+ \int_{\partial\Omega} (\beta \cdot n)^{\ominus} v w_h - \sum_{F \in \mathcal{F}_h^i} \int_F (\beta \cdot n_F) [\![v]\!] \{\!\{ w_h \}\!\}.$$

Use the relation

$$\sum_{T \in \mathcal{T}_h} \int_{\partial T} (\beta \cdot n_T) v w_h = \sum_{F \in \mathcal{F}_h^i} \int_F (\beta \cdot n_F) [\![v w_h]\!] + \sum_{F \in \mathcal{F}_h^b} \int_F (\beta \cdot n) v w_h$$

since β has a continuous normal component at $F \in \mathcal{F}_h^i$. Next, use, with $v_i = v|_{T_i}$, $w_i = w|_{T_i}$, $i \in \{1, 2\}$,

$$\begin{aligned} [\![v w_h]\!] &= v_1 w_1 - v_2 w_2 \\ &= \frac{1}{2}(v_1 - v_2)(w_1 + w_2) + \frac{1}{2}(v_1 + v_2)(w_1 - w_2) \\ &= [\![v]\!] \{\!\{ w_h \}\!\} + \{\!\{ v \}\!\} [\![w_h]\!]. \end{aligned}$$

Heuristic derivation DG discretization

The integrals at the element faces and domain boundary then can be evaluated as

$$
\begin{aligned}
&\sum_{T\in\mathcal{T}_h}\int_{\partial T}(\beta\cdot n_T)vw_h + \int_{\partial\Omega}(\beta\cdot n)^{\ominus}vw_h - \sum_{F\in\mathcal{F}_h^i}\int_F(\beta\cdot n_F)[\![v]\!]\{\!\{w_h\}\!\} \\
&= \sum_{F\in\mathcal{F}_h^i}\int_F\Big((\beta\cdot n_F)[\![v]\!]\{\!\{w_h\}\!\} + (\beta\cdot n_F)[\![v]\!]\{\!\{w_h\}\!\}\Big) \\
&+ \sum_{F\in\mathcal{F}_h^b}\int_F(\beta\cdot n)vw_h + \int_{\partial\Omega}\frac{1}{2}(|\beta\cdot n| - (\beta\cdot n))vw_h - \sum_{F\in\mathcal{F}_h^i}\int_F(\beta\cdot n_F)[\![v]\!]\{\!\{w_h\}\!\} \\
&= \sum_{F\in\mathcal{F}_h^i}\int_F(\beta\cdot n_F)\{\!\{v\}\!\}[\![w_h]\!] + \sum_{F\in\mathcal{F}_h^b}\int_F\frac{1}{2}(|\beta\cdot n| + (\beta\cdot n))vw_h \\
&= \sum_{F\in\mathcal{F}_h^i}\int_F(\beta\cdot n_F)\{\!\{v\}\!\}[\![w_h]\!] + \sum_{F\in\mathcal{F}_h^b}\int_F(\beta\cdot n)^{\oplus}vw_h.
\end{aligned}
$$

Combining all terms gives the alternative formulation for a_h^{cf}.

Boundedness bilinear form a_h^{cf}

- Consider the discrete problem

$$\text{Find } u_h \in V_h \text{ s.t. } a_h^{cf}(u_h, v_h) = \int_\Omega f v_h \qquad \forall v_h \in V_h.$$

  The problem is well-posed due to the coercivity of a_h^{cf} on V_h.

  Define on V_{*h} the norm

$$|||v|||_{cf,*}^2 = |||v\,|||_{cf}^2 + \sum_{T\in\mathcal{T}_h} \tau_c \|\beta \cdot \nabla v\|_{L^2(T)}^2 + \sum_{T\in\mathcal{T}_h} \tau_c \beta_c^2 h_T^{-1} \|v\|_{L^2(\partial T)}^2,$$

  with time scale τ_c and reference velocity β.

- Lemma. (Boundedness) There holds

$$\forall (v, w_h) \in V_{*h} \times V_h, \quad a_h^{cf}(v, w_h) \leq C_{bnd}\, |||\, v\, |||_{cf,*}\, |||w_h|||_{cf},$$

  where C_{bnd} is independent of h and the data μ, β.

Boundedness bilinear form a_h^{cf}

Proof. Let $(v, w_h) \in V_{*h} \times V_h$ and use the Cauchy-Schwarz inequality, then

$$
\begin{aligned}
&\int_\Omega \Big(\mu v w_h + (\beta \cdot \nabla_h v) w_h\Big) + \int_{\partial\Omega} (\beta \cdot n)^{\ominus} v w_h \\
&\le \|\mu\|_{L^\infty(\Omega)} \|v\|_{L^2(\Omega)} \|w_h\|_{L^2(\Omega)} + \sum_{T \in \mathcal{T}_h} \|\beta \cdot \nabla v\|_{L^2(T)} \|w_h\|_{L^2(T)} \\
&+ \|v\|_{L^2(|\beta \cdot n|;\partial\Omega)} \|w_h\|_{L^2(|\beta \cdot n|;\partial\Omega)} \\
&\le \max(\|\mu\|_{L^\infty(\Omega)}, 1) \Big(\|v\|_{L^2(\Omega)}^2 + \sum_{T \in \mathcal{T}_h} \|\beta \cdot \nabla v\|_{L^2(T)}^2 + \|v\|_{L^2(|\beta \cdot n|;\partial\Omega)}^2\Big) \\
&\qquad \Big(2\|w_h\|_{L^2(\Omega)}^2 + \|w_h^2\|_{L^2(|\beta \cdot n|;\partial\Omega)}\Big) \\
&\le 2 \,|||\, v \,|||_{cf,*} \,|||\, w_h \,|||_{cf} \, .
\end{aligned}
$$

Boundedness bilinear form

To bound the integral over the interior faces, we use the Cauchy-Schwarz inequality

$$\sum_{F\in\mathcal{F}_h^i}\int_F(\beta\cdot n_F)[\![v]\!]\{\!\{w_h\}\!\} \leq \Big(\sum_{F\in\mathcal{F}_h^i}\frac{1}{2}\tau_c\beta_c^2\{\!\{h\}\!\}^{-1}\|[\![v]\!]\|_{L^2(F)}^2\Big)^{\frac{1}{2}}$$

$$\times\Big(\sum_{F\in\mathcal{F}_h^i}2\tau_c^{-1}\{\!\{h\}\!\}\|\{\!\{w_h\}\!\}\|_{L^2(F)}^2\Big)^{\frac{1}{2}},$$

where for all $F\in\mathcal{F}_h^i$, with $F=\partial T_1\cap\partial T_2$, we have $\{\!\{h\}\!\}=\frac{1}{2}(h_{T_1}+h_{T_2})$.

Set $v_i=v|_{T_i}$, $w_i=w_h|_{T_i}$, $i\in\{1,2\}$, then we have the relations

$$\frac{1}{2}[\![v]\!]^2\leq v_1^2+v_2^2,$$

$$2\{\!\{w_h\}\!\}^2\leq w_1^2+w_2^2.$$

Boundedness bilinear form

The computational mesh satisfies the relation

$$C_\rho^{-1} \max(h_{T_1}, h_{T_2}) \leq \{\!\{h\}\!\} \leq C_\rho \min(h_{T_1}, h_{T_2}),$$

where C_ρ only depends on the mesh regularity. Hence,

$$\sum_{F \in \mathcal{F}_h^i} \int_F (\beta \cdot n_F)[\![v]\!]\{\!\{w_h\}\!\} \leq C_\rho \, |\!|\!| \, v \, |\!|\!|_{cf,*} \left(\sum_{T \in \mathcal{T}_h} \tau_c^{-1} h_T \|w_h\|^2_{L^2(\partial T)} \right)^{\frac{1}{2}}.$$

Boundedness bilinear form

Finally, use the trace inequality

$$h_T^{\frac{1}{2}} \|w_h\|_{L^2(\partial T)} \leq C_{tr} N_\partial^{\frac{1}{2}} \|w_h\|_{L^2(T)},$$

then

$$\sum_{F \in \mathcal{F}_h^i} \int_F (\beta \cdot n_F) [\![v]\!] \{\!\{ w_h \}\!\} \leq C_\rho \; |||\, v \,|||_{cf,*} \left(\sum_{T \in \mathcal{T}_h} \tau_c^{-1} C_{tr}^2 N_\partial \|w_h\|^2_{L^2(T)} \right)^{\frac{1}{2}}$$

$$= \tau_c^{-\frac{1}{2}} C_\rho C_{tr} N_\partial^{\frac{1}{2}} \; |||\, v \,|||_{cf,*} \; \|w_h\|_{L^2(\Omega)}.$$

Combining all terms then gives the boundedness of the bilinear form $a_h^{cf}(v, w_h)$,

$$\forall (v, w_h) \in V_{*h} \times V_h, \quad a_h^{cf}(v, w_h) \leq C_{bnd} \; |||\, v \,|||_{cf,*} \; |||w_h|||_{cf},$$

Error estimate

From the discrete stability, consistency and boundedness we immediately obtain the error estimate.

- Theorem 1. (Error estimate). Let u and u_h, respectively, solve

$$\text{Find } u \in V, \text{ s.t. } \ a(u, w) = \int_\Omega fw, \qquad \forall w \in V,$$

$$\text{Find } u_h \in V_h, \text{ s.t. } \ a_h^{cf}(u_h, w_h) = \int_\Omega fw_h, \qquad \forall w_h \in V_h,$$

with

$$a_h^{cf}(v, w_h) = \int_\Omega \left(\mu v w_h + (\beta \cdot \nabla_h v) w_h\right) + \int_{\partial\Omega} (\beta \cdot n)^{\ominus} v w_h - \sum_{F \in \mathcal{F}_h^i} \int_F (\beta \cdot n_F) [\![v]\!] \{\!\{ w_h \}\!\},$$

and $V_h = \mathbb{P}_d^k(\mathcal{T}_h)$, with $k \geq 1$ and $\mathcal{T}_h$ belonging to an admissible mesh sequence. Then, there holds

$$|||u - u_h|||_{cf} \leq C \inf_{y_h \in V_h} |||u - y_h|||_{cf,*}$$

with C independent of h and only depending on the data through $\{\min(1, \tau_c \mu_0)\}^{-1}$.

Error estimate

To obtain the convergence rate, assume u is smooth enough.

Take $y_h = \pi_h u$, with π_h the L^2-orthogonal projection of u onto V_h and use the interpolation error estimates.

- Corollary. (Convergence rate for smooth solution). Besides the assumptions of Theorem 1, assume $u \in H^{k+1}(\Omega)$. Then there holds

$$|||u - u_h|||_{cf} \leq Ch^k \|u\|_{H^{k+1}(\Omega)},$$

where C is independent of h and only depending on the data through $\{\min(1, \tau_c \mu_0)\}^{-1}$.

Note, this convergence rate is suboptimal.

An optimal convergence rate for the error in the $L^2(\Omega)$ norm should be order $k+1$ and for the boundary contribution order $k + \frac{1}{2}$ if the solution is smooth enough.

A better convergence rate can be obtained using an upwind DG discretization.

Numerical fluxes

Since the DG discretization uses broken polynomial spaces the DG discretization can also be considered on an individual element $T \in \mathcal{T}_h$.

Consider an arbitrary polynomial $\xi \in \mathbb{P}_d^k(T)$. For a set $S \subset \Omega$ denote the characteristic function χ_S as

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Insert the test function $v_h = \xi \chi_T$ into the DG discretization and use

$$[\![\xi \chi_T]\!] = \epsilon_{T,F} \xi \quad \text{with } \epsilon_{T,F} := n_T \cdot n_F$$

since

$$[\![\xi \chi_T]\!] = \begin{cases} \xi|_{T_1} & \text{if } T = T_1, \\ -\xi|_{T_2} & \text{if } T = T_2. \end{cases}$$

and $n_T \cdot n_F$=1 if $T = T_1$, $n_T \cdot n_F$=-1 if $T = T_2$, assuming $n_{T_1} = n_F$ and using $n_{T_2} = -n_{T_1}$.

Numerical fluxes

Recall the alternative expression for a_h^{cf}

$$a_h^{cf}(v, w_h) = \int_\Omega \Big((\mu - \nabla \cdot \beta) v w_h - v(\beta \cdot \nabla_h w_h) \Big) + \int_{\partial\Omega} (\beta \cdot n)^\oplus v w_h$$
$$+ \sum_{F \in \mathcal{F}_h^i} \int_F (\beta \cdot n_F) \{\!\{ v \}\!\} [\![w_h]\!].$$

The bilinear form $a_h^{cf}(u_h, \xi\chi_T)$ on an individual element $T \in \mathcal{T}_h$ then becomes

$$\int_T \Big((\mu - \nabla \cdot \beta) u_h \xi - u_h(\beta \cdot \nabla\xi) \Big) + \sum_{F \in \mathcal{F}_T} \epsilon_{T,F} \int_F \phi_F(u_h) \xi = \int_T f\xi,$$

with the numerical fluxes

$$\phi_F(u_h) := \begin{cases} (\beta \cdot n_F) \{\!\{ u_h \}\!\} & \text{if } F \in \mathcal{F}_h^i, \\ (\beta \cdot n)^\oplus u_h & \text{if } F \in \mathcal{F}_h^b. \end{cases}$$

Numerical fluxes

The numerical fluxes $\phi_F(u_h)$

$$\phi_F(u_h) := \begin{cases} (\beta \cdot n_F)\{\!\{u_h\}\!\} & \text{if } F \in \mathcal{F}_h^i, \\ (\beta \cdot n)^{\oplus} u_h & \text{if } F \in \mathcal{F}_h^b, \end{cases}$$

are called centered fluxes, because the average value of u_h is used on each face $F \in \mathcal{F}_h^i$.

Since the numerical fluxes are single valued at each face $F \in \mathcal{F}_h^i$ the DG discretization is element-wise conservative. Taking, $\xi = 1$ gives

$$\int_T (\mu - \nabla \cdot \beta) u_h + \sum_{F \in \mathcal{F}_T} \epsilon_{T,F} \int_F \phi_F(u_h) = \int_T f,$$

which is the balance equation used in finite volume discretizations.

Upwinding

- In order to improve the stability of the DG discretization and it's convergence rate upwinding is introduced.

  Consider the upwind bilinear form

  $$a_h^{upw}(v_h, w_h) := a_h^{cf}(v_h, w_h) + s_h(v_h, w_h),$$

  with stabilization bilinear form

  $$s_h(v_h, w_h) = \sum_{F \in \mathcal{F}_h^i} \int_F \frac{\eta}{2} |\beta \cdot n_F| [\![v_h]\!] [\![w_h]\!],$$

  with $\eta > 0$ a user-dependent stability parameter.

Upwinding

The bilinear form a_h^{upw} then is equal to

$$a_h^{upw}(v_h, w_h) := \int_\Omega \Big(\mu v_h w_h + (\beta \cdot \nabla_h v_h) w_h\Big) + \int_{\partial\Omega} (\beta \cdot n)^\ominus v_h w_h$$

$$- \sum_{F \in \mathcal{F}_h^i} \int_F (\beta \cdot n_F) [\![v_h]\!] \{\!\{ w_h \}\!\} + \sum_{F \in \mathcal{F}_h^i} \int_F \frac{\eta}{2} |\beta \cdot n_F| [\![v_h]\!] [\![w_h]\!],$$

or equivalently

$$a_h^{upw}(v_h, w_h) = \int_\Omega \Big((\mu - \nabla \cdot \beta) v_h w_h - v_h (\beta \cdot \nabla_h w_h)\Big) + \int_{\partial\Omega} (\beta \cdot n)^\oplus v_h w_h$$

$$+ \sum_{F \in \mathcal{F}_h^i} \int_F (\beta \cdot n_F) \{\!\{ v_h \}\!\} [\![w_h]\!] + \sum_{F \in \mathcal{F}_h^i} \int_F \frac{\eta}{2} |\beta \cdot n_F| [\![v_h]\!] [\![w_h]\!].$$

Note, a_h^{upw} and a_h^{cf} use the same stencil.

Upwinding

- Define on V_{*h} the norm

$$
\begin{aligned}
|||v|||_{upb}^2 &:= |||v\,|||_{cf}^2 + \sum_{F\in\mathcal{F}_h^i} \int_F \frac{\eta}{2}|\beta\cdot n_F|[\![v]\!]^2 \\
&= \tau_c^{-1}\|v\|_{L^2(\Omega)}^2 + \int_{\partial\Omega}\frac{1}{2}|\beta\cdot n|v^2 + \sum_{F\in\mathcal{F}_h^i}\int_F \frac{\eta}{2}|\beta\cdot n_F|[\![v]\!]^2.
\end{aligned}
$$

- Lemma. (Consistency and discrete coercivity). The upwind DG bilinear form a_h^{upw} satisfies

  i) Consistency, namely for the exact solution $u \in V_*$,

$$
a_h^{upw}(u, v_h) = \int_\Omega f v_h \qquad \forall v_h \in V_h.
$$

  ii) Coercivity on V_h w.r.p to the $|||\cdot|||_{upb}$-norm,

$$
\forall v_h \in V_h, \quad a_h^{upw}(v_h, v_h) \geq C_{sta}\,|||\,v_h|||_{upb}^2,
$$

  with $C_{sta} = \min(1, \tau_c\mu_0)$.

Upwinding

Proof.

i) Consistency follows from the consistency of $a_h^{cf}(u, v_h)$ and the fact that $(\beta \cdot n_F)[\![u]\!] = 0$ at faces $F \in \mathcal{F}_h^i$ for the exact solution $u \in V_*$.

ii) Coercivity of a_h^{upw} follows from the coercivity of a_h^{cf} and the fact that

$$s_h(v_h, v_h) = \sum_{F \in \mathcal{F}_h^i} \int_F \frac{\eta}{2} |\beta \cdot n_F| [\![v_h]\!]^2 \geq 0.$$

The coercivity of a_h^{upw} on V_h implies well-posedness of the upwind DG-discretization.

For accuracy it is important not to use large values of the penalty parameter η. Optimal values of η can be derived.

Error estimates based on coercivity

- The norm $|||\cdot|||_{upb}$ is not strong enough to establish boundedness of the upwind DG bilinear form a_h^{upw}.

  This requires either a stronger norm or we can restrict the functions in the first argument of a_h^{upw} to those functions in V_{*h} that are L^2-orthogonal to V_h.

  Thus to functions of the form $v - \pi_h v$ for $v \in V_*$, with π_h the L^2-projection, which are called orthogonal subscales.

- Definition. (Boundedness on orthogonal subscales). Boundedness on orthogonal subscales holds true for a_h^{upw} (uniformly in h, μ, β) if there exists $C_{bnd} > 0$, independent of h, μ, β, s.t. $\forall (v, w_h) \in V_* \times V_h$,

$$|a_h^{upw}(v - \pi_h v, w_h)| \leq C_{bnd} \, ||| \, v - \pi_h v \, |||_{uwb,*} \, |||w_h|||_{uwb},$$

  for a norm $|||\cdot|||_{uwb,*}$ defined on V_{*h} s.t. $\forall v \in V_{*h}$, $|||v|||_{uwb} \leq |||v|||_{uwb,*}$.

Error estimates based on coercivity

- Lemma. (Boundedness on orthogonal subscales). Boundedness on orthogonal subscales holds true for the upwind DG bilinear form a_h^{upw} when defining on V_{*h} the norm

$$|||v|||^2_{uwb,*} := |||v\,|||^2_{uwb} + \sum_{T\in\mathcal{T}_h} \beta_c \|v\|^2_{L^2(\partial T)}.$$

Proof. Let $(v, w_h) \in V_* \times V_h$ and set $y = v - \pi_h v$.

We also have

$$|||v|||^2_{uwb,*} := \tau_c^{-1} \|v\|^2_{L^2(\Omega)} + \int_{\partial\Omega} \frac{1}{2} |\beta\cdot n| v^2 + \sum_{F\in\mathcal{F}_h^i} \int_F \frac{\eta}{2} |\beta\cdot n_F| [\![v]\!]^2 + \sum_{T\in\mathcal{T}_h} \beta_c \|v\|^2_{L^2(\partial T)}.$$

Error estimates based on coercivity

From the alternative formulation of a_h^{cf}

$$a_h^{upw}(y_h, w_h) = \int_\Omega \Big((\mu - \nabla \cdot \beta) y_h w_h - y_h(\beta \cdot \nabla_h w_h) \Big) + \int_{\partial\Omega} (\beta \cdot n)^{\oplus} y_h w_h$$
$$+ \sum_{F \in \mathcal{F}_h^i} \int_F (\beta \cdot n_F) \{\!\{ y_h \}\!\} [\![w_h]\!] + \sum_{F \in \mathcal{F}_h^i} \int_F \frac{\eta}{2} |\beta \cdot n_F| [\![y_h]\!] [\![w_h]\!].$$

we have

$$\int_\Omega (\mu - \nabla \cdot \beta) y w_h + \int_{\partial\Omega} (\beta \cdot n)^{\oplus} y w_h$$
$$\leq \tau_c^{-1} \|y\|_{L^2(\Omega)} \|w_h\|_{L^2(\Omega)} + \Big(\int_{\partial\Omega} |\beta \cdot n|^{\oplus} y^2 \Big)^{\frac{1}{2}} \Big(\int_{\partial\Omega} |\beta \cdot n|^{\oplus} w_h^2 \Big)^{\frac{1}{2}}$$
$$\leq C_1 \, ||| \, y \, |||_{upb} \, ||| w_h \, |||_{uwb} \, .$$

with C_1 independent of h, μ and β.

Error estimates based on coercivity

Denote with $\langle\beta\rangle_T$ the mean value of β on each $T \in \mathcal{T}_h$. Then we have

$$\|\beta - \langle\beta\rangle_T\|_{L^\infty(T)} \leq L_\beta h_T \leq \frac{1}{\tau_c} h_T \quad \text{(since } \beta \text{ is Lipschitz continuous).}$$

Note, from the mean value theorem for integrals there exists a $\overline{x} \in T$ s.t. $\langle\beta\rangle_T = \beta(\overline{x})$. Then

$$\begin{aligned}
\|\beta - \langle\beta\rangle_T\|_{L^\infty(\Omega)} &= \operatorname*{ess\,sup}_{x\in T} |\beta(x) - \beta(\overline{x})| \\
&\leq L_\beta |x - \overline{x}| \leq L_\beta h_T.
\end{aligned}$$

We also have $\forall w_h \in V_h$ that $\langle\beta\rangle_T \cdot \nabla w_h \in \mathbb{P}_d^{k-1}(T) \subset \mathbb{P}_d^k(T)$, hence

$$\begin{aligned}
\forall T \in \mathcal{T}_h, \quad \int_T y\langle\beta\rangle_T \cdot \nabla w_h &= \int_T (v - \pi_h v)\langle\beta\rangle_T \cdot \nabla w_h \\
&= 0
\end{aligned}$$

since π_h is the L^2-orthogonal projection.

Error estimates based on coercivity

Use now the inverse inequality

$$\|\nabla v_h\|_{[L^2(\Omega)]^d} \leq C_{inv} h_T^{-1} \|v_h\|_{L^2(T)},$$

to obtain

$$\begin{aligned}
\int_\Omega y\beta \cdot \nabla_h w_h &= \sum_{T\in\mathcal{T}_h} \int_T y\beta \cdot \nabla w_h = \sum_{T\in\mathcal{T}_h} \int_T y(\beta - \langle\beta\rangle_T) \cdot \nabla w_h \\
&\leq \sum_{T\in\mathcal{T}_h} \|y\|_{L^2(T)} \tau_c^{-1} h_T \|\nabla w_h\|_{L^2(\Omega)]^d} \quad \text{(use } \|\beta - \langle\beta\rangle_T\|_{L^\infty(\Omega)} \leq \tau_c^{-1} h_T\text{)} \\
&\leq \sum_{T\in\mathcal{T}_h} \|y\|_{L^2(T)} \tau_c^{-1} C_{inv} \|w_h\|_{L^2(T)} \\
&\leq C_{inv} \, |||\, y \,|||_{upb} \, |||\, w_h \,|||_{upb} \,.
\end{aligned}$$

In addition, the Cauchy-Schwarz inequality yields

$$\sum_{F\in\mathcal{F}_h^i}\int_F(\beta\cdot n_F)\{\!\{y\}\!\}[\![w_h]\!] \le \Big(\sum_{F\in\mathcal{F}_h^i}\int_F 2\eta^{-1}|\beta\cdot n_F|\{\!\{y\}\!\}^2\Big)^{\frac12}\Big(\sum_{F\in\mathcal{F}_h^i}\int_F \frac{\eta}{2}|\beta\cdot n_F|[\![w_h]\!]^2\Big)^{\frac12}$$

$$\le \Big(\eta^{-1}\sum_{T\in\mathcal{T}_h}\beta_c\|y\|^2_{L^2(\partial T)}\Big)^{\frac12}|||\,w_h|||_{upb}$$

$$\le C_2\,|||\,y\,|||_{upb,*}\,|||w_h\,|||_{upb}\,.$$

Collecting all terms gives

$$a_h^{cf}(y,w_h)\le C_2\,|||\,y\,|||_{upb,*}\,|||w_h|||_{upb},$$

with C_2 independent of h, μ and β. Finally, the bound on $|a_h^{upw}(y_h,w_h)|$ is obtained using

$$\sum_{F\in\mathcal{F}_h^i}\int_F\frac{\eta}{2}|\beta\cdot n_F|[\![y]\!][\![w_h]\!]\le|||y\,|||_{upb}\,|||w_h\,|||_{upb}\,.$$

Error estimates based on coercivity

- Theorem 2. (Error estimate). Let u and u_h, respectively, solve

$$\text{Find } u \in V, \text{ s.t. } a(u,w) = \int_\Omega fw, \qquad \forall w \in V,$$

$$\text{Find } u_h \in V_h, \text{ s.t. } a_h^{upw}(u_h,w_h) = \int_\Omega fw_h, \qquad \forall w_h \in V_h,$$

and $V_h = \mathbb{P}_d^k(\mathcal{T}_h)$, with $k \geq 0$ and T belonging to an admissible mesh sequence.

Then there holds

$$|||u - u_h|||_{upb} \leq C\, |||\, u - \pi_h u|||_{upb,*},$$

with C independent of h and only depending on the data through the factor $\{\min(1, \tau_c\mu_0)\}^{-1}$.

Error estimates based on coercivity

Proof. First assume $\pi_h u \neq u_h$. Then due to the discrete coercivity and consistency of a_h^{upw} we have

$$\begin{aligned} |||u_h - \pi_h u|||_{upb} &\leq C_{sta}^{-1} \frac{a_h^{upw}(u_h - \pi_h u, u_h - \pi_h u)}{|||u_h - \pi_h u|||_{upb}} \quad \text{(using coercivity)} \\ &\leq C_{sta}^{-1} \frac{a_h^{upw}(u - \pi_h u, u_h - \pi_h u)}{|||u_h - \pi_h u|||_{upb}} \quad \text{(using consistency)}, \end{aligned}$$

with $C_{sta} = \min(1, \tau_c \mu_0)$. Hence

$$|||\, u_h - \pi_h u|||_{upb} \leq C_{sta}^{-1} C_{bnd} \, |||\, u - \pi_h u|||_{upb,*}$$

(using boundedness on orthogonal scales of a_h^{upw}).

Error estimates based on coercivity

Finally, using the triangle inequality we obtain

$$
\begin{aligned}
|||u - u_h|||_{upb} &\leq |||u - \pi_h u\,|||_{upb} + |||\,u_h - \pi_h u|||_{upb} \\
&\leq |||u - \pi_h u\,|||_{upb} + C_{sta}^{-1} C_{bnd}\,|||\,u - \pi_h u|||_{upb,*} \\
&\leq |||u - \pi_h u\,|||_{upb,*} \qquad (\text{since } |||u - \pi_h u|||_{upb} \leq |||u - \pi_h u|||_{upb,*}).
\end{aligned}
$$

- Corollary. (Convergence rate for smooth solutions). In addition to the assumptions of Theorem 2, assume $u \in H^{k+1}(\Omega)$.

  Then there holds

  $$|||u - u_h|||_{upb} \leq C h^{k+\frac{1}{2}} \|u\|_{H^{k+1}(\Omega)},$$

  with C independent of h and only depending on the data through the factor $\{\min(1, \tau_c \mu_0)\}^{-1}$.

Error estimated based on inf-sup stability

Introduce the stronger norm

$$|||v|||_{uw\sharp}^2 := |||v\,|||_{upb}^2 + \sum_{T\in\mathcal{T}_h} \beta_c^{-1} h_T \|\beta\cdot\nabla v\|_{L^2(T)}^2.$$

Note, this norm also gives control over ∇v.

To simplify arguments we assume that

$$h \leq \beta_c \tau_c.$$

Using $\tau_c = \{\max(\|\mu\|_{L^\infty(\Omega)}, L_\beta)\}^{-1}$ and $\beta_c = \|\beta\|_{[L^\infty(\Omega)]^d}$ we obtain that this assumption is equal to

$$\max(h\|\mu\|_{L^\infty(\Omega)}\|\beta\|_{[L^\infty(\Omega)]^d}^{-1}, hL_\beta\|\beta\|_{[L^\infty(\Omega)]^d}^{-1}) \leq 1.$$

The quantity $h\|\mu\|_{L^\infty(\Omega)}\|\beta\|_{[L^\infty(\Omega)]^d}^{-1}$ is the local Damköhler number (ratio chemical reaction time scale to transport time scale).

If $hL_\beta\|\beta\|_{[L^\infty(\Omega)]^d}^{-1} \leq 1$ then the mesh resolves the spatial variations of the advective velocity.

Error estimated based on inf-sup stability

- Lemma. (Discrete inf-sup condition). There is a $C'_{sta} > 0$, independent of h, μ, β s.t.

$$\forall v_h \in V_h \quad C'_{sta} C_{sta} \, |||\, v_h |||_{uw\sharp} \leq \sup_{w_h \in V_h \setminus \{0\}} \frac{a_h^{upw}(v_h, w_h)}{||| w_h |||_{uw\sharp}},$$

where $C_{sta} = \min(1, \tau_c \mu_0)$.

Error estimated based on inf-sup stability

Proof.

1.) Let $v_h \in V_h \backslash \{0\}$ and set $\overline{S} = \sup_{w_h \in V_h \backslash \{0\}} \frac{a_h^{upw}(v_h, w_h)}{|||w_h|||_{uw\sharp}}$.

From the coercivity of a_h^{upw} we obtain

$$C_{sta} \, ||| \, v_h |||_{uwb}^2 \leq a_h^{upw}(v_h, v_h) = \frac{a_h^{upw}(v_h, v_h)}{|||v_h|||_{uw\sharp}} \, ||| \, v_h |||_{uw\sharp} \leq \overline{S} \, ||| \, v_h \, |||_{uw\sharp} \, .$$

2.) Bound the advective derivative in the norm $|||v_h|||_{uw\sharp}$.

i.) Choose $w_h \in V_h$ s.t. $\forall T \in \mathcal{T}_h$, $w_h|_T = \beta_c^{-1} h_T \langle \beta \rangle_T \cdot \nabla v_h$, with $\langle \beta \rangle_T$ the average of β over T.

Bound the DG-norm $|||w_h|||_{uw\sharp}$ in terms of $|||v_h|||_{uw\sharp}$.

We abbreviate $a \leq Cb$ as $a \lesssim b$, with C independent of h, μ and β.

Error estimated based on inf-sup stability

ii.) Consider

$$\sum_{T\in\mathcal{T}_h} \beta_c h_T^{-1} \|w_h\|^2_{L^2(T)} = \sum_{T\in\mathcal{T}_h} \beta_c^{-1} h_T \|\langle\beta\rangle_T \cdot \nabla v_h\|^2_{L^2(T)}$$

(using $w_h|_T = \beta_c^{-1} h_T \langle\beta\rangle_T \cdot \nabla v_h$)

$$\leq 2 \sum_{T\in\mathcal{T}_h} \beta_c^{-1} \left(h_T \|\beta \cdot \nabla v_h\|^2_{L^2(T)} + h_T \|(\beta - \langle\beta\rangle_T) \cdot \nabla v_h\|^2_{L^2(T)} \right)$$

$$\lesssim \sum_{T\in\mathcal{T}_h} \beta_c^{-1} h_T \|\beta \cdot \nabla v_h\|^2_{L^2(T)} + \sum_{T\in\mathcal{T}_h} \tau_c^{-2} \beta_c^{-1} h_T \|v_h\|^2_{L^2(T)}$$

(using $\|\beta - \langle\beta\rangle_T\|_{[L^\infty(T)]^d} \leq \tau_c^{-1} h_T$ and

inverse inequality $\|\nabla v_h\|_{[L^2(T)]^d} \leq C_{inv} h_T^{-1} \|v_h\|_{L^2(T)}$)

$$\leq \sum_{T\in\mathcal{T}_h} \beta_c^{-1} h_T \|\beta \cdot \nabla v_h\|^2_{L^2(T)} + \sum_{T\in\mathcal{T}_h} \tau_c^{-1} \|v_h\|_{L^2(T)} \quad \text{(using } h \leq \beta_c \tau_c\text{)}$$

$$\leq |||v_h|||^2_{uw\sharp} .$$

iii.) Using the discrete trace inequality,

$$h_T^{\frac{1}{2}} \|v_h\|_{L^2(F)} \leq C_{tr} \|v_h\|_{L^2(T)},$$

and the previous result obtained in 2.ii) we obtain the estimate

$$\int_{\partial\Omega} \frac{1}{2} |\beta \cdot n| w_h^2 + \sum_{F \in \mathcal{F}_h^i} \int_F \frac{\eta}{2} |\beta \cdot n_F| [\![w_h]\!]^2 \lesssim \sum_{T \in \mathcal{T}_h} \beta_c h_T^{-1} \|w_h\|_{L^2(T)}^2$$

$$\lesssim |||v_h|||_{uw\sharp}^2 .$$

Error estimated based on inf-sup stability

iv.) Using the inverse inequality

$$\|\nabla v_h\|_{[L^2(\Omega)]^d} \leq C_{inv} h_T^{-1} \|v_h\|_{L^2(T)},$$

we obtain

$$\|w_h\|_{L^2(\Omega)} = \Big(\sum_{T \in \mathcal{T}_h^i} \beta_c^{-2} h_T^2 \, \|\langle \beta \rangle_T \cdot \nabla v_h\|_{L^2(T)}^2 \Big)^{\frac{1}{2}} \lesssim \|v_h\|_{L^2(\Omega)}$$

and

$$\sum_{T \in \mathcal{T}_h} \beta_c^{-1} h_T \|\beta \cdot \nabla w_h\|_{L^2(T)}^2 \lesssim \sum_{T \in \mathcal{T}_h} \beta_c h_T^{-1} \|w_h\|_{L^2(T)}^2 \lesssim |||v_h|||_{uw\sharp}^2$$

Error estimated based on inf-sup stability

v.) Collecting all terms gives

$$
\begin{aligned}
|||w_h|||^2_{uw\sharp} &= |||w_h\,|||^2_{uwb} + \sum_{T\in\mathcal{T}_h} \beta_c^{-1} h_T \|\beta\cdot\nabla w_h\|^2_{L^2(T)} \\
&= |||w_h\,|||^2_{cf} + \sum_{F\in\mathcal{F}_h^i} \int_F \frac{\eta}{2}|\beta\cdot n_F|[\![w_h]\!]^2 + \sum_{T\in\mathcal{T}_h} \beta_c^{-1} h_T \|\beta\cdot\nabla w_h\|^2_{L^2(T)} \\
&= \tau_c^{-1}\|w_h\|^2_{L^2(\Omega)} + \int_{\partial\Omega} \frac{1}{2}|\beta\cdot n| w_h^2 + \sum_{F\in\mathcal{F}_h^i} \int_F \frac{\eta}{2}|\beta\cdot n_F|[\![w_h]\!]^2 \\
&+ \sum_{T\in\mathcal{T}_h} \beta_c^{-1} h_T \|\beta\cdot\nabla w_h\|^2_{L^2(T)} \\
&\lesssim |||v_h\,|||^2_{uw\sharp}\,.
\end{aligned}
$$

Error estimated based on inf-sup stability

3.) Using the relation

$$\int_\Omega (\beta \cdot \nabla_h v_h) w_h = \sum_{T \in \mathcal{T}_h} \beta_c^{-1} h_T \int_T (\beta \cdot \nabla_h v_h) \langle \beta \rangle_T \cdot \nabla v_h$$

$$= \sum_{T \in \mathcal{T}_h} \beta_c^{-1} h_T \int_T (\beta \cdot \nabla_h v_h)(\beta - \langle \beta \rangle_T) \cdot \nabla v_h)$$

$$- \sum_{T \in \mathcal{T}_h} \beta_c^{-1} h_T \int_T (\beta \cdot \nabla_h v_h) \beta \cdot \nabla v_h)$$

for the advective term in a_h^{upw} we obtain

Error estimated based on inf-sup stability

$$
\begin{aligned}
\sum_{T\in\mathcal{T}_h} \beta_c^{-1} h_T \|\beta\cdot\nabla v_h\|_{L^2(T)}^2 &= a_h^{upw}(v_h, w_h) - \int_\Omega \mu v_h w_h \\
&\quad + \sum_{T\in\mathcal{T}_h} \beta_c^{-1} h_T \int_T (\beta\cdot\nabla v_h)(\beta - \langle\beta\rangle_T)\cdot\nabla v_h - \int_{\partial\Omega} (\beta\cdot n)^{\ominus} v_h w_h \\
&\quad + \sum_{F\in\mathcal{F}_h^i} \int_F (\beta\cdot n_F)[\![v_h]\!]\{\!\{w_h\}\!\} - \sum_{F\in\mathcal{F}_h^i} \int_F \frac{\eta}{2}|\beta\cdot n_F|[\![v_h]\!][\![w_h]\!] \\
&= \mathscr{T}_1 + \cdots + \mathscr{T}_6.
\end{aligned}
$$

Note, the term $\sum_{T\in\mathcal{T}_h} \beta_c^{-1} h_T \|\beta\cdot\nabla v_h\|_{L^2(T)}^2$ is the missing term in the $|||v|||_{uw\sharp}$-norm, with

$$
|||v|||_{uw\sharp}^2 = |||v\;|||_{uwb}^2 + \sum_{T\in\mathcal{T}_h} \beta_c^{-1} h_T \|\beta\cdot\nabla v_h\|_{L^2(T)}^2,
$$

in the lower bound of the stability estimate in the BNB-theorem. The coercivity already gives a lower bound in the $|||v|||_{uwb}$-norm.

Error estimated based on inf-sup stability

4.) Estimate now each term

i.)

$$|\mathscr{T}_1| = |a_h^{upw}(v_h, w_h)| = \frac{|a_h^{upw}(v_h, w_h)|}{|||w_h|||_{uw\sharp}} \, |||\, w_h|||_{uw\sharp} \leq \overline{S} \, |||\, w_h|||_{uw\sharp} \leq \overline{S} \, |||\, v_h \, |||_{uw\sharp} \, .$$

ii.) For $|\mathscr{T}_2|$, $|\mathscr{T}_4|$, $|\mathscr{T}_6|$, we obtain using the Cauchy-Schwarz inequality and the definition of $|||\cdot|||_{uwb}$,

$$|\mathscr{T}_2| + |\mathscr{T}_4| + |\mathscr{T}_6| \lesssim |||v_h\,|||_{uwb} \, |||w_h|||_{uwb} \lesssim |||v_h\,|||_{uwb} \, |||v_h\,|||_{uwb\sharp} \, .$$

Error estimated based on inf-sup stability

iii.) Using the Cauchy-Schwarz inequality together with the discrete race inequality gives

$$
\begin{aligned}
|\mathscr{T}_5| &= \sum_{F\in\mathcal{F}_h^i} \int_F (\beta\cdot n_F)[\![v_h]\!]\{\!\{w_h\}\!\} \\
&\leq \Big(\sum_{F\in\mathcal{F}_h^i} \int_F \frac{\eta}{2}|\beta\cdot n_F|[\![v_h]\!]^2 \Big)^{\frac{1}{2}} \Big(\sum_{F\in\mathcal{F}_h^i} \int_F \frac{2}{\eta}|\beta\cdot n_F|\{\!\{w_h\}\!\}^2 \Big)^{\frac{1}{2}} \\
&\lesssim |||v_h|||_{uwb} \Big(\sum_{T\in\mathcal{T}_h} \beta_c h_T^{-1} \|w_h\|^2_{L^2(T)} \Big)^{\frac{1}{2}} \quad \text{(using discrete trace inequality)} \\
&\lesssim |||v_h|||_{uwb}\, |||v_h|||_{uw\sharp}\,.
\end{aligned}
$$

iv.) Finally,

$$
\begin{aligned}
|\mathscr{T}_3| &= \sum_{T\in\mathcal{T}_h} \beta_c^{-1} h_T \int_T (\beta\cdot\nabla v_h)(\beta - \langle\beta\rangle_T)\cdot\nabla v_h \\
&\leq \Big(\sum_{T\in\mathcal{T}_h} \beta_c^{-1} h_T \|\beta\cdot\nabla v_h\|^2_{L^2(T)}\Big)^{\frac{1}{2}} \Big(\sum_{T\in\mathcal{T}_h} \beta_c^{-1} h_T \|(\beta - \langle\beta\rangle_T)\cdot\nabla v_h\|^2_{L^2(T)}\Big)^{\frac{1}{2}} \\
&\qquad \text{(use } \|\beta - \langle\beta\rangle_T\|_{[L^\infty(T)]^d} \leq \tau_c^{-1} h_T \text{ and } \|\nabla v_h\|_{[L^2(T)]^d} \leq C_{inv} h_T^{-1} \|v_h\|_{L^2(T)}\text{)} \\
&\leq \Big(\sum_{T\in\mathcal{T}_h} \beta_c^{-1} h_T \|\beta\cdot\nabla v_h\|^2_{L^2(T)}\Big)^{\frac{1}{2}} \Big(\sum_{T\in\mathcal{T}_h} \beta_c^{-1} \tau_c^{-2} C_{inv}^2 \|v_h\|^2_{L^2(T)}\Big)^{\frac{1}{2}} \\
&\lesssim \Big(\sum_{T\in\mathcal{T}_h} \beta_c^{-1} h_T \|\beta\cdot\nabla v_h\|^2_{L^2(T)}\Big)^{\frac{1}{2}} \; |||\, v_h \,|||_{uwb} \,.
\end{aligned}
$$

Error estimated based on inf-sup stability

v.) Use Young's inequality in the form $ab \leq \gamma a^2 + (4\gamma)^{-1} b^2$ with $\gamma > 0$,

$$|\mathscr{T}_3| - \frac{1}{2}\Big(\sum_{T \in \mathcal{T}_h} \beta_c^{-1} h_T \|\beta \cdot \nabla v_h\|_{L^2(T)}^2 \Big) \lesssim ||| v_h |||_{uwb}^2 .$$

vi.) Collecting all terms gives

$$\sum_{T \in \mathcal{T}_h} \beta_c^{-1} h_T \|\beta \cdot \nabla v_h\|_{L^2(T)}^2 \lesssim \overline{S} \, ||| \, v_h \, |||_{uw\sharp} + ||| \, v_h \, |||_{uwb} \, ||| v_h \, |||_{uw\sharp} + ||| \, v_h \, |||_{uwb}^2 . \qquad (5)$$

6.) Combining (5) with the coercivity bound $C_{sta} ||| \, v_h |||_{uwb}^2 \leq \overline{S} \, ||| \, v_h |||_{uw\sharp}$ gives

$$C_{sta} \, ||| \, v_h \, |||_{uwb}^2 + C_{sta} \sum_{T \in \mathcal{T}_h} \beta_c^{-1} h_T \|\beta \cdot \nabla v_h\|_{L^2(T)}^2 \leq (1 + C_{sta}) \overline{S} \, ||| \, v_h |||_{uw\sharp}$$

$$+ \, C_{sta} \, ||| \, v_h \, |||_{uwb} \, ||| v_h \, |||_{uw\sharp} + \overline{S} \, ||| \, v_h \, |||_{uw\sharp} \, .$$

Error estimated based on inf-sup stability

which is, using Young's inequality, the definition of $|||\cdot|||_{uw\sharp}$ and the coercivity bound again, equivalent to

$$C_{sta} \, ||| \, v_h |||^2_{uw\sharp} \lesssim \overline{S} \, ||| \, v_h \, |||_{uw\sharp} + C_{sta} \, ||| \, v_h |||^2_{uwb} \lesssim \overline{S} \, ||| \, v_h |||_{uw\sharp}$$

which gives the discrete inf-sup condition

$$C'_{sta} C_{sta} \, ||| \, v_h |||_{uw\sharp} \leq \sup_{w_h \in V_h \setminus \{0\}} \frac{a_h^{upw}(v_h, w_h)}{|||w_h|||_{uw\sharp}}.$$

Error estimated based on inf-sup stability

To prove boundedness of a_h^{upw} we need the norm

$$|||v|||_{uw\sharp,*} := |||v\,|||^2_{uw\sharp} + \sum_{T\in\mathcal{T}_h} \beta_c(h_T^{-1}\|v\|_{L^2(T)} + \|v\|^2_{L^2(\partial T)})$$

- Lemma. (Boundedness) There holds

$$\forall(v, w_h) \in V_{*h} \times V_h, \quad |a_h^{upw}(v, w_h)| \leq C_{bnd} \,|||\, v \,|||_{uw\sharp,*} \,|||v\,|||_{uw\sharp} \,.$$

  with C_{bnd} independent of h, μ, and β.

  Proof. Same as proof for boundedness on orthogonal scales, except for the term

$$\int_\Omega v(\beta \cdot \nabla_h w_h) \leq |||v\,|||_{uw\sharp,*} \,|||w_h\,|||_{uw\sharp} \,.$$

Error estimated based on inf-sup stability

From the discrete stability (discrete inf-sup condition), consistency and boundedness we immediately obtain an error estimate.

Error estimated based on inf-sup stability

- Theorem 3. (Error estimate). Let u and u_h, respectively, solve

$$\text{Find } u \in V, \text{ s.t. } a(u,w) = \int_\Omega fw, \qquad \forall w \in V,$$

$$\text{Find } u_h \in V_h, \text{ s.t. } a_h^{upw}(u_h, w_h) = \int_\Omega fw_h, \qquad \forall w_h \in V_h,$$

with

$$a_h^{upw}(v_h, w_h) = \int_\Omega \Big(\mu v_h w_h + (\beta\cdot\nabla_h v_h) w_h\Big) + \int_{\partial\Omega} (\beta\cdot n)^{\ominus} v_h w_h$$
$$- \sum_{F\in\mathcal{F}_h^i} \int_F (\beta\cdot n_F) [\![v_h]\!] \{\!\{ w_h \}\!\} + \sum_{F\in\mathcal{F}_h^i} \frac{\eta}{2} |\beta\cdot n_F| [\![v_h]\!] [\![w_h]\!]$$

and $V_h = \mathbb{P}_d^k(\mathcal{T}_h)$, with $k \geq 0$ and $\mathcal{T}_h$ belonging to an admissible mesh sequence. Then, there holds

$$|\!|\!| u - u_h |\!|\!|_{uw\sharp} \leq C \inf_{y_h\in V_h} |\!|\!| u - y_h |\!|\!|_{uw\sharp,*} \tag{6}$$

with C independent of h and only depending on the data through $\{\min(1, \tau_c\mu_0)\}^{-1}$.

Error estimated based on inf-sup stability

- Corollary. (Convergence rate for smooth solutions). Besides the assumptions of Theorem 3, assume that $u \in H^{k+1}(\Omega)$.

  Then there holds

$$|||u - u_h|||_{uw\sharp} \leq Ch^{k+\frac{1}{2}} \|u\|_{H^{k+1}(\Omega)},$$

  with C independent of h and only depending on the data through $\{\min(1, \tau_c \mu_0)\}^{-1}$.

  Note, this error estimate is better than the estimate based on coercivity using boundedness on orthogonal scales since it also provides a bound on the scaled advective derivative.

Numerical fluxes

- By locallizing the test functions to an individual element the local upwind DG discretization is obtained.

  For all $T \in \mathcal{T}_h$ and all $\xi \in \mathbb{P}_d^k(T)$,

$$\int_T \Big((\mu - \nabla \cdot \beta) u_h \xi - u_h (\beta \cdot \nabla \xi) \Big) + \sum_{F \in \mathcal{F}_T} \epsilon_{T,F} \int_F \phi_F(u_h) \xi = \int_T f \xi,$$

  with the numerical fluxes

$$\phi_F(u_h) = \begin{cases} (\beta \cdot n_F) \{\!\{ u_h \}\!\} + \frac{1}{2} \eta |\beta \cdot n_F| [\![u_h]\!] & \text{if } F \in \mathcal{F}_h^i, \\ (\beta \cdot n)^{\oplus} u_h & \text{if } F \in \mathcal{F}_h^b. \end{cases}$$