Discontinuous Galerkin Methods

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Introduction

- Discontinuous Galerkin (DG) methods are nowadays one of the main finite element methods to solve partial differential equations.
- The key feature of DG methods is the use of discontinuous test and trial spaces. This results in a local element wise discretization and a discontinuous approximation at element faces or edges.

Introduction

Benefits of discontinuous Galerkin methods:

- DG methods provide higher order accurate, element-wise conservative finite element discretizations of partial differential equations with excellent stability and convergence properties.
- The local, element based discretization in DG methods provides great flexibility to design:
  - solution adaptive numerical discretizations using local mesh refinement (*h*-adaptation) and/or local adjustment of the polynomial order (*p*-refinement).
  - efficient parallel finite element discretizations due to the minimal element connectivity. In general, only nearest neighboring elements are connected at element faces or edges.
  - DG discretizations of time-dependent problems generally result in a block-diagonal mass matrix, which is very beneficial when using an explicit time integration method.
- DG methods have a well established mathematical theory to analyse the convergence, stability and accuracy of the finite element discretization.

Introduction

Benefits of discontinuous Galerkin methods:

- Discontinuous Galerkin discretizations generally are more complicated than standard conforming finite element discretizations and also their mathematical analysis is more involved.

Introduction

To main goals of these lectures are:

- To discuss the basic mathematical techniques necessary to understand the mathematical properties of DG discretizations.

- To use these tools to study convergence, stability and accuracy of discontinuous Galerkin discretizations of hyperbolic and elliptic model problems.

Main references

These lectures are mainly based on:

- D.A. Di Pietro, A. Ern, Mathematical aspects of discontinuous Galerkin methods, Springer, 2012, ISBN 978-3-642-22979-4.
- A. Ern, J.-L. Guermond, Theory and practice of finite elements, Springer, 2004, ISBN 0-387-20574-8.
- S.C. Brenner, L.R. Scott, The mathematical theory of finite element methods, 3rd edition, Springer, 2008, ISBN 978-0-387-75933-3.

Discrete setting

- The domain Ω is a bounded, connected subset of $\mathbb{R}^d$, $d \geq 1$, with Lipschitz continuous boundary $\partial\Omega$ that has a unit outward normal vector n.

- For simplicity we will also assume that Ω is a polyhedron.

- Polyhedron:

  - P is a polyhedron in $\mathbb{R}^d$ if P is an open connected, bounded subset of $\mathbb{R}^d$ s.t. its boundary ∂P is a finite union of parts of hyperplanes.

  - Moreover, each point in the interior of P is assumed to lie only on one side of the hyperplane boundary.

  - Each polyhedron can be subdivided into a finite number of simplicial elements.

Discrete setting

- A simplex is defined as:
  - Given a family $\{a_0, \cdots, a_d\}$ of $d+1$ points in $\mathbb{R}^d$ s.t. the vectors $\{a_1 - a_0, \cdots, a_d - a_0\}$ are linearly independent.
  - The interior of the convex hull of $\{a_0, \cdots, a_d\}$ is called a non-degenerate simplex in $\mathbb{R}^d$.
- The points $\{a_0, \cdots, a_d\}$ are the vertices of the simplex.
- In $\mathbb{R}^d$, for $d = 1, 2, 3$ simplices are, respectively, a line segment, a triangle, and a tetrahedron.
- Unit or reference simplex

$$S_d := \{(x_1, \cdots, x_d) \in \mathbb{R}^d \mid x_i > 0 \ \forall i \in \{1, \cdots, d\} \text{ and } x_1 + \cdots + x_d < 1\}$$

Discrete setting

Simplex faces and mesh:

- Let S be a non-degenerate simplex with vertices $\{a_0, \cdots, a_d\}$.

  For each $i \in \{0, \cdots, d\}$ the convex hull of $\{a_0, \cdots, a_d\} \backslash \{a_i\}$ is a face of simplex S.

- A simplicial mesh $\mathcal{T}$ of the (polyhedral) domain Ω is a finite collection of disjoint non-degenerate simplices $\mathcal{T} = \{T\}$ forming a partition of Ω,

$$\overline{\Omega} = \cup_{T \in \mathcal{T}} \overline{T},$$

  with each $T \in \mathcal{T}$ a mesh element.

  The outward unit normal vector at ∂T is denoted n_T.

Discrete setting

General mesh:

- A general mesh $\mathcal{T}$ of a domain Ω is a finite collection of disjoint polyhedra $\mathcal{T} = \{T\}$ forming a partition of Ω.
- Note, a general mesh allows hanging nodes.
- Let $\mathcal{T}$ be a (general) mesh of Ω. For all $T \in \mathcal{T}$, h_T denotes the diameter of T and the mesh size is defined as

$$h := \max_{T \in \mathcal{T}} h_T.$$

- $\mathcal{T}_h$ is a mesh $\mathcal{T}$ with mesh size h.

Discrete setting

Mesh faces:

- Let $\mathcal{T}_h$ be a mesh of Ω. A (closed) subset F of $\overline{\Omega}$ is a mesh face if F has a positive $(d-1)$-dimensional Hausdorff measure and if either one of the two conditions is satisfied:
  - there are distinct mesh elements T_1 and T_2 s.t. $F = \partial T_1 \cap \partial T_2$, then F is called an interface;
  - there is a $T \in \mathcal{T}_h$ s.t. $F = \partial T \cap \partial\Omega$, then F is called a boundary face.
- Interfaces are collected in the set $\mathcal{F}_h^i$, boundary faces in the set $\mathcal{F}_h^b$, hence

$$\mathcal{F}_h := \mathcal{F}_h^i \cup \mathcal{F}_h^b.$$

Discrete setting

- The set $\mathcal{F}_T := \{F \in \mathcal{F}_h \mid F \subset \partial T\}$ collects the mesh faces composing the boundary of element T.
- The maximum number of mesh faces composing the boundary of mesh elements is

$$N_\partial := \max_{T \in \mathcal{T}_h} \operatorname{card}(\mathcal{F}_T).$$

- For any mesh face $F \in \mathcal{F}_h$ define the set

$$\mathcal{T}_F := \{T \in \mathcal{T}_h \mid F \subset \partial T\}.$$

- Note $\mathcal{T}_F$ consists of two mesh elements if $F \in \mathcal{F}_h^i$ and one mesh element if $F \in \mathcal{F}_h^b$.

Jumps and Averages

- Let v be a scalar valued function on Ω, sufficiently smooth to admit $\forall F \in \mathcal{F}_h^i$ a possibly two-valued trace.
- Denote with $v|_T$ for all $T \in \mathcal{T}_h$ the restriction of v to T with trace at ∂T.
- For all $F \in \mathcal{F}_h^i$ and a.e. $x \in F$ the average of v is defined as

$$\{\!\{v\}\!\}_F(x) := \frac{1}{2}\left(v|_{T_1}(x) + v|_{T_2}(x)\right),$$

  and the jump of v as

$$[\![v]\!]_F(x) := v|_{T_1}(x) - v|_{T_2}(x).$$

- If v is a vector then the average and jump operators act component wise on v.

Normal vectors

- For all $F \in \mathcal{F}_h$ and a.e. $x \in F$ the unit normal n_F to F at x is defined as
  - $n_F = n_{T_1}$, the normal vector to F at x pointing from element T_1 to element T_2 if $F \in \mathcal{F}_h^i$ with $F = \partial T_1 \cap \partial T_2$.

    At $F \in \mathcal{F}_h^i$ we have $n_{T_2} = -n_{T_1}$.

    The orientation of $n_F = n_{T_1}$ is arbitrary, depending on the choice of T_1 and T_2, but this orientation must be kept fixed.
  - n, the outward normal to Ω at x if $F \in \mathcal{F}_h^b$.

Lebesgue spaces

- Consider functions $v : \Omega \subset \mathbb{R}^d \to \mathbb{R}$, $d \geq 1$, that are Lebesgue measurable.

  Let $1 \leq p \leq \infty$ be a real number and define the norms

$$\|v\|_{L^p(\Omega)} := \left(\int_\Omega |v|^p \right)^{\frac{1}{p}} \qquad 1 \leq p < \infty,$$

$$\begin{aligned} \|v\|_{L^\infty(\Omega)} &:= \sup \mathrm{ess}\{|v(x)| \mid \text{for almost every } x \in \Omega\} \\ &= \inf\{M > 0 \mid |v(x)| \leq M \text{ for almost every } x \in \Omega\}. \end{aligned}$$

Lebesgue spaces

- The Lebesgue space is defined as

$$L^p(\Omega) := \{v \text{ is Lebesgue measurable} \mid \|v\|_{L^p(\Omega)} < \infty\}$$

- The Lebesgue space with norm $\|v\|_{L^p(\Omega)} < \infty$ is a Banach space for all $1 \leq p \leq \infty$.
- For all $1 \leq p < \infty$ the space $C_0^\infty(\Omega)$ of infinitely differentiable functions with compact support is dense in $L^p(\Omega)$.
- For $p = 2$, the space $L^2(\Omega)$ is a Hilbert space, equipped with the scalar product

$$(v, w)_{L^2(\Omega)} := \int_\Omega vw.$$

- Holder's inequality.

  For all $1 \le p, q \le \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ all $v \in L^p(\Omega)$ and all $w \in L^q(\Omega)$ there holds $vw \in L^1(\Omega)$ and

$$\int_\Omega vw \le \|v\|_{L^p(\Omega)} \|w\|_{L^q(\Omega)}.$$

- For $p = q = 2$ Holder's inequality becomes the Cauchy-Schwarz inequality.

  For all $v, w \in L^2(\Omega)$, $vw \in L^1(\Omega)$ and

$$(v, w)_{L^2(\Omega)} \le \|v\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)}.$$

Sobolev spaces

- Given a Cartesian basis in $\mathbb{R}^d$ with coordinates $(x_1, \cdots, x_d)$, then ∂_i with $i \in \{1, \cdots, d\}$ denotes the distributional partial derivative with respect to x_i.

- For $\alpha \in \mathbb{N}^d$, then $\partial^\alpha v$ denotes the distributional or weak derivative $\partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} v$ of v with $\partial^{(0,\cdots,0)} v = v$.

  A function $f \in L^1_{loc}(\Omega)$ has a distributional or weak derivative $\partial^\alpha f$ provided there exists a function $g \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega g(x)\phi(x)dx = (-1)^{|\alpha|} \int_\Omega f(x)\phi^{(\alpha)}dx \quad \forall \phi \in C_0^\infty(\Omega).$$

  If such a g exists, then we define $\partial^\alpha f = g$.

Sobolev spaces

Example. Take $d = 1$, $\Omega = [-1, 1]$ and $f(x) = 1 - |x|$.

Then for $\phi \in C_0^\infty(\Omega)$, arbitrary, we have,

$$\int_{-1}^{1} f(x)\partial_1^1\phi(x)dx = \int_{-1}^{0} f(x)\partial_1^1\phi(x)dx + \int_{0}^{1} f(x)\partial_1^1\phi(x)dx$$

$$= -\int_{-1}^{0}(+1)\phi(x)dx + f\phi\big|_{-1}^{0} - \int_{0}^{1}(-1)\phi(x)dx + f\phi\big|_{0}^{1} \quad \text{(integration by parts)}$$

$$= -\Big(\int_{-1}^{0}(+1)\phi(x)dx + \int_{0}^{1}(-1)\phi(x)dx\Big) + (f\phi)(0-) - (f\phi)(0+)$$

$$\text{(since } \phi(-1) = \phi(1) = 0\text{)}$$

$$= -\int_{-1}^{1} g(x)\phi(x)dx \quad \text{(since } f \text{ is continuous at } x = 0\text{)},$$

with

$$g(x) = \begin{cases} 1 & x < 0, \\ -1 & x > 0. \end{cases}$$

The weak derivative $\partial_1^1 f(x)$ is then given by $\partial_1^1 f(x) = g(x)$.

Sobolev spaces

- For $1 \leq p \leq \infty$, $p \in \mathbb{R}$, define for all $\xi \in \mathbb{R}^d$, with $\xi = (\xi_1, \cdots, \xi_d)$ in the Cartesian basis of $\mathbb{R}^d$, the norm

$$|\xi|_{\ell^p} := \left(\sum_{i=1}^{d} |\xi_i|^p \right)^{\frac{1}{p}}, \qquad 1 \leq p < \infty,$$

$$|\xi|_{\ell^\infty} := \max_{1 \leq i \leq d} |\xi_i|.$$

- Sobolev spaces

Let $m \geq 0$, $1 \leq p \leq \infty$. The Sobolev space $W^{m,p}(\Omega)$ is defined as

$$W^{m,p}(\Omega) := \{ v \in L^p(\Omega) \mid \forall \alpha \in A_d^m,\ \partial^\alpha v \in L^p(\Omega) \},$$

where $A_p^m := \{ \alpha \in \mathbb{N}^d \mid |\alpha|_{\ell^1} \leq m \}$.

Note, $W^{0,p}(\Omega) = L^p(\Omega)$.

Sobolev spaces

- The Sobolev spaces $W^{m,p}(\Omega)$ are a Banach space when equipped with the norm

$$\|v\|_{W^{m,p}(\Omega)} := \left(\sum_{\alpha \in A_d^m} \|\partial^\alpha v\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}, \qquad 1 \le p < \infty,$$

$$\|v\|_{W^{m,\infty}(\Omega)} := \max_{\alpha \in A_d^m} \|\partial^\alpha v\|_{L^\infty(\Omega)}.$$

- The semi-norm $|\cdot|_{W^{m,p}(\Omega)}$ is obtained by keeping only the derivatives of global order m, hence $\overline{A}_d^m := \{\alpha \in \mathbb{N}^d \mid |\alpha|_{\ell^1} = m\}$.

Hilbert spaces

- For $p = 2$ we use the notation $H^m(\Omega) := W^{m,2}(\Omega)$, hence

$$H^m(\Omega) = \{v \in L^2(\Omega) \mid \forall \alpha \in A_d^m,\ \partial^\alpha v \in L^2(\Omega)\}$$

- $H^m(\Omega)$ is a Hilbert space when equipped with the scalar product

$$(v, w)_{H^m(\Omega)} := \sum_{\alpha \in A_d^m} (\partial^\alpha v, \partial^\alpha w)_{L^2(\Omega)},$$

resulting in the norm and semi-norm

$$\|v\|_{H^m(\Omega)} := \Big(\sum_{\alpha \in A_d^m} \|\partial^\alpha v\|_{L^2(\Omega)}^2 \Big)^{\frac{1}{2}}, \qquad |v|_{H^m(\Omega)} := \Big(\sum_{\alpha \in \bar{A}_d^m} \|\partial^\alpha v\|_{L^2(\Omega)}^2 \Big)^{\frac{1}{2}}.$$

- For $m = 1$ we can consider the gradient $\nabla v = (\partial_1 v, \cdots, \partial_d v)^T \in \mathbb{R}^d$. The norm on $W^{1,p}(\Omega)$ then is equal to

$$\|v\|_{W^{1,p}(\Omega)} = \left(\|v\|^p_{L^p(\Omega)} + \|\nabla v\|^p_{[L^p(\Omega)]^d}\right)^{\frac{1}{p}}, \qquad 1 \leq p < \infty,$$

with

$$\|\nabla v\|^p_{[L^p(\Omega)]^d} := \left(\int_\Omega |\nabla v|^p_{\ell^p}\right)^{\frac{1}{p}} = \left(\int_\Omega \sum_{i=1}^d |\partial_i v|^p\right)^{\frac{1}{p}}.$$

- For $p = 2$ we have

$$(v, w)_{H^1(\Omega)} = (v, w)_{L^2(\Omega)} + (\nabla v, \nabla w)_{[L^2(\Omega)]^d}.$$

Traces

- Boundary values of functions in the Sobolev space $W^{1,p}(\Omega)$ have a meaning as traces in $L^p(\partial\Omega)$.

- Trace inequalities:

  For all $1 \leq p \leq \infty$ there is a constant C s.t.

$$\|v\|_{L^p(\partial\Omega)} \leq C\|v\|_{L^p(\Omega)}^{1-\frac{1}{p}}\|v\|_{W^{1,p}(\Omega)}^{\frac{1}{p}}, \qquad \forall v \in W^{1,p}(\Omega).$$

  For $p = 2$ this gives

$$\|v\|_{L^2(\partial\Omega)} \leq C\|v\|_{L^2(\Omega)}^{\frac{1}{2}}\|v\|_{H^1(\Omega)}^{\frac{1}{2}}, \qquad \forall v \in H^1(\Omega).$$

- We will also consider Hilbert-Sobolev spaces $H^s(\Omega)$, $s \in \mathbb{R}, s > 0$, e.g. functions in $H^{\frac{1}{2}+\epsilon}(\Omega)$, $\epsilon > 0$ have a trace in $L^2(\Omega)$.

Polynomial spaces

- The space of polynomials $\mathcal{P}_d^k$ of total degree at most k, with $k \geq 0$, integer, is defined as

$$\mathcal{P}_d^k := \{p : \mathbb{R}^d \ni x \mapsto p(x) \in \mathbb{R} \mid \exists (\gamma_\alpha)_{\alpha \in A_d^k} \in \mathbb{R}^{\mathrm{card}(A_d^k)} \text{ s.t. } p(x) = \sum_{\alpha \in A_d^k} \gamma_\alpha x^\alpha\},$$

with for $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, $x^\alpha := \Pi_{i=1}^d x_i^{\alpha_i}$ and

$$A_d^k = \{\alpha \in \mathbb{N}^d \mid |\alpha|_{\ell^1} \leq k\}$$

- The dimension of $\mathbb{P}_d^k$ is

$$\dim\left(\mathbb{P}_d^k\right) = \mathrm{card}(A_d^k) = \begin{pmatrix} k+d \\ k \end{pmatrix} = \frac{(k+d)!}{k!d!}.$$

Broken polynomial spaces

- The broken polynomial space $\mathbb{P}_d^k(\mathcal{T}_h)$ is defined as

$$\mathbb{P}_d^k(\mathcal{T}_h) := \{v \in L^2(\Omega) \mid \forall T \in \mathcal{T}_h,\ v|_T \in \mathbb{P}_d^k(T)\},$$

  with $\mathbb{P}_d^k(T)$ spanned by the restriction to T of polynomials in $\mathbb{P}_d^k$.

- The dimension of $\mathbb{P}_d^k(\mathcal{T}_h)$ is

$$\dim(\mathbb{P}_d^k(\mathcal{T}_h)) = \mathrm{card}(\mathcal{T}_h) \times \dim(\mathbb{P}_d^k).$$

Broken Sobolev spaces

- Let $\mathcal{T}_h$ be a mesh in Ω. The broken Sobolev spaces are defined as

$$H^m(\mathcal{T}_h) := \{v \in L^2(\Omega) \mid \forall T \in \mathcal{T}_h, \;\; v|_T \in H^m(T)\},$$

$$W^{m,p}(\mathcal{T}_h) := \{v \in L^p(\Omega) \mid \forall T \in \mathcal{T}_h, \;\; v|_T \in W^{m,p}(T)\},$$

with $m \geq 0$, integer, $1 \leq p \leq \infty$ a real number.

Broken Sobolev spaces

- The continuous trace inequality gives $\forall v \in W^{1,p}(\mathcal{T}_h)$ and $\forall T \in \mathcal{T}_h$,

$$\|v\|_{L^p(\partial T)} \leq C\|v\|_{L^p(T)}^{1-\frac{1}{p}}\|v\|_{W^{1,p}(T)}^{\frac{1}{p}},$$

with for $p = 2$, $\forall v \in H^m(\mathcal{T}_h)$ and $\forall T \in \mathcal{T}_h$,

$$\|v\|_{L^2(\partial T)} \leq C\|v\|_{L^p(\Omega)}^{\frac{1}{2}}\|v\|_{H^1(T)}^{\frac{1}{2}}.$$

- The broken gradient $\nabla_h : W^{1,p}(\mathcal{T}_h) \to [L^p(\Omega)]^d$ is defined as

$$\forall T \in \mathcal{T}_h, \qquad (\nabla_h v)|_T := \nabla(v|_T), \qquad \forall v \in W^{1,p}(\mathcal{T}_h).$$

Note, the subscript h will not be used if ∇_h is used inside an integral over a fixed mesh element $T \in \mathcal{T}_h$.

Broken Sobolev spaces

- Lemma 1. (Broken gradient on usual Sobolev spaces). Let $m \geq 0$, $1 \leq p \leq \infty$. There holds $W^{m,p}(\Omega) \subset W^{m,p}(\mathcal{T}_h)$.

  Moreover, $\forall v \in W^{1,p}(\Omega)$, $\nabla_h v = \nabla v$ in $[L^p(\Omega)]^d$.

  Proof. Take $m = 1$. Let $v \in W^{1,p}(\Omega)$. For all $\Phi \in [C_0^\infty(T)]^d$ we can since $\Phi = 0$ at ∂T define the extension of Φ by zero as $E\Phi \in [C_0^\infty(\Omega)]^d$. Then

$$\int_T \nabla(v|_T) \cdot \Phi = -\int_T v(\nabla \cdot \Phi) = -\int_\Omega v(\nabla \cdot (E\Phi))$$

$$= \int_\Omega \nabla v \cdot E\Phi = \int_T (\nabla v)|_T \cdot \Phi.$$

  Since Φ is arbitrary, this implies $\nabla(v|_T) = (\nabla v)|_T$.

  Since $T \in \mathcal{T}_h$ is arbitrary, using $(\nabla_h v)|_T := \nabla(v|_T)$, we obtain that $\nabla_h v = \nabla v$. Hence $v \in W^{1,p}(\mathcal{T}_h)$.

Broken Sobolev spaces

- The reverse inclusion, namely $W^{m,p}(\mathcal{T}_h) \subset W^{m,p}(\Omega)$, is in general not true (except for $m = 0$) since functions in $W^{m,p}(\mathcal{T}_h)$ can have non-zero jumps at interfaces.

Broken Sobolev spaces

- Lemma. (Characterization of $W^{1,p}(\Omega)$). Let $1 \le p \le \infty$. A function $v \in W^{1,p}(\mathcal{T}_h)$ belongs to $W^{1,p}(\Omega)$ if and only if

$$[[v]] = 0 \qquad \forall F \in \mathcal{F}_h^i.$$

Proof.

We will use for all $\Phi \in [C_0^\infty(\Omega)]^d$ the relation

$$\sum_{T \in \mathcal{T}_h} \int_{\partial T} v_T(\Phi \cdot n_T) = \sum_{F \in \mathcal{F}_h} \int_F [[\Phi \cdot n_F]]\{\{v\}\} + \sum_{F \in \mathcal{F}_h^i} \int_F \{\{\Phi \cdot n_T\}\}[[v]],$$

Using the fact that Φ is continuous across interfaces,

$$[[\Phi \cdot n_F]] = 0 \quad \text{and} \quad \{\{\Phi \cdot n_F\}\} = \Phi \cdot n_F \qquad \text{for } \forall F \in \mathcal{F}_h^i,$$

we obtain

$$\sum_{T \in \mathcal{T}_h} \int_{\partial T} v_T(\Phi \cdot n_T) = \sum_{F \in \mathcal{F}_h^i} \int_F (\Phi \cdot n_T)[[v]].$$

Broken Sobolev spaces

Let $v \in W^{1,p}(\mathcal{T}_h)$. Then $\forall \Phi \in [C_0^\infty(\Omega)]^d$, we obtain by integrating by parts element-wise that

$$\int_\Omega \nabla_h v \cdot \Phi = \sum_{T \in \mathcal{T}_h} \int_T \nabla(v|_T) \cdot \Phi = -\sum_{T \in \mathcal{T}_h} \int_T v(\nabla \cdot \Phi) + \sum_{T \in \mathcal{T}_h} \int_{\partial T} v|_T(\Phi \cdot n_T)$$

$$= -\int_\Omega v(\nabla \cdot \Phi) + \sum_{F \in \mathcal{F}_h^i} \int_F (\Phi \cdot n_F)[[v]]. \qquad (1)$$

The condition $[[v]] = 0$, $\forall F \in \mathcal{F}_h^i$ then implies

$$\int_\Omega \nabla_h v \cdot \Phi = -\int_\Omega v \nabla \cdot \Phi = \int_\Omega \nabla v \cdot \Phi \qquad \forall \Phi \in [C_0^\infty(\Omega)]^d,$$

hence $\nabla v = \nabla_h v$ in $[L^p(\Omega)]^d$, thus $v \in W^{1,p}(\Omega)$.

Broken Sobolev spaces

- Conversely, if $v \in W^{1,p}(\Omega)$, then $\nabla v = \nabla_h v$ in $[L^p(\Omega)]^d$ owing to Lemma 1. Hence (1) implies

$$\sum_{F \in \mathcal{F}_h^i} \int_F (\Phi \cdot n_F)[[v]] = \int_\Omega \nabla_h v \cdot \Phi + \int_\Omega v(\nabla \cdot \Phi)$$

$$= \int_\Omega \nabla_h v \cdot \Phi - \int_\Omega \nabla v \cdot \Phi \qquad (\text{since } \Phi \in [C_0^\infty(\Omega)]^d)$$

$$= 0 \qquad (\text{since } \nabla v = \nabla_h v \text{ for } v \in W^{1,p}(\Omega)).$$

This implies $[[v]] = 0$, $\forall F \in \mathcal{F}_h^i$, by choosing the support of Φ only to contain the two elements $T_1, T_2 \in \mathcal{T}_h$ connected to $F \in \mathcal{F}_h^i$, and Φ being arbitrary.

Well-posedness for linear variational equations

For the well-posedness we consider:

- Let X and Y be two Banach spaces equipped with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$.
- Let $\mathcal{L}(X, Y)$ be the vector space spanned by linear operators from X to Y, equipped with the norm

$$\|A\|_{\mathcal{L}(X,Y)} := \sup_{v \in X \setminus \{0\}} \frac{\|Av\|_Y}{\|v\|_X} \qquad \forall A \in \mathcal{L}(X, Y).$$

- Define the linear model model problem as

$$\text{Find } u \in X \text{ s.t.} \qquad a(u, v) = \langle f, w \rangle_{Y', Y} \qquad \forall w \in Y, \tag{2}$$

where $a \in \mathcal{L}(X \times Y, \mathbb{R})$ is a bounded bilinear form, $f \in Y' := \mathcal{L}(Y, \mathbb{R})$ is a bounded linear form, and $\langle \cdot, \cdot \rangle_{Y', Y}$ denotes the duality pairing between Y' and Y.

Well-posedness of linear variational equations

- Alternatively, we can introduce the bounded linear operator $A \in \mathcal{L}(X, Y)$ s.t.

$$\langle Av, w \rangle_{Y', Y} := a(v, w) \qquad \forall (v, w) \in X \times Y,$$

  and consider

$$\text{Find } u \in X \text{ s.t.} \quad Au = f \quad \text{in } Y'. \tag{3}$$

- Problems (2) and 3) are equivalent; u solves (2) if and only if u solves (3).
- Problems (2) and 3) are well-posed if they admit one and only one solution $u \in X$.

Well-posedness of linear variational equations

- The well-posedness of (3) requires A to be an isomorphism (bijective mapping that preserves the structure).

  In Banach spaces this implies that if $A \in \mathcal{L}(X, Y')$ is an isomorphism, then A^{-1} is bounded, that is

$$\|A^{-1}\|_{\mathcal{L}(X,Y')} \leq C,$$

  which implies that

$$\|u\|_X = \|A^{-1}f\|_X \leq C\|f\|_{Y'}.$$

Banach-Nečas-Babuška theorem

- The Banach-Nečas-Babuška (BNB) theorem provides necessary and sufficient conditions for well-posedness for linear variational equations.

- Theorem (BNB theorem). Let X be a Banach space and let Y be a reflexive space.

  Let $a \in \mathcal{L}(X \times Y, \mathbb{R})$ and let $f \in Y'$. Then the problem:

$$\text{Find } u \in X \text{ s.t. } a(u, w) = \langle f, w \rangle_{Y', Y} \quad \forall w \in Y$$

  is well posed if and only if

  i) There is a $C_{sta} > 0$ s.t.

$$\forall v \in X, \quad C_{sta} \|v\|_X \leq \sup_{w \in Y \setminus \{0\}} \frac{a(v, w)}{\|w\|_Y}.$$

  ii) For all $w \in Y$, $a(v, w) = 0$ implies that $w = 0$, $\forall v \in X$.

Banach-Nečas-Babuška theorem

- Moreover, the following a priori estimate holds true

$$\|u\|_X \leq \frac{1}{C_{sta}} \|f\|_{Y'}.$$

- Note, the condition

$$\forall v \in X, \quad C_{sta}\|v\|_X \leq \sup_{w \in Y \setminus \{0\}} \frac{a(v,w)}{\|w\|_Y}$$

is equivalent to the inf-sup condition

$$C_{sta} \leq \inf_{v \in X \setminus \{0\}} \sup_{w \in Y \setminus \{0\}} \frac{a(v,w)}{\|v\|_X \|w\|_Y}.$$

- The BNB-theorem is a direct result of the Banach Closed Range theorem and the Banach Open Mapping Theorem.

Lax-Milgram lemma

- Let X be a Hilbert space, $Y = X$. Let $a \in \mathcal{L}(X \times X, \mathbb{R})$.

  The bilinear form a is coercive on X if there is $C_{sta} > 0$ s.t.

$$\forall v \in X, \quad C_{sta} \|v\|_X^2 \leq a(v, v).$$

- Equivalently, a bounded linear operator $A \in \mathcal{L}(X, X')$ defined by

$$\langle Av, w \rangle_{X',X} := a(v, w) \quad \forall (v, w) \in X \times X$$

  is coercive if $\exists C_{sta} > 0$ s.t.

$$\forall v \in X, \quad C_{sta} \|v\|_X^2 \leq \langle Av, v \rangle_{X',X}.$$

- The Lax-Milgram lemma provides sufficient conditions for well-posedness.

Lax-Milgram lemma

- Lemma (Lax-Milgram) Let X be a Hilbert space. Let $a \in \mathcal{L}(X \times X, \mathbb{R})$ and let $f \in X'$.

  Then the problem

$$\text{Find } u \in X \text{ s.t. } a(u,w) = \langle f, w \rangle_{X',X} \quad \forall w \in X$$

  is well posed if the bilinear form a is coercive on X.

  Equivalently, the problem

$$\text{Find } u \in X, \text{ s.t. } \; Au = f \text{ in } X'$$

  is well-posed if the linear operator $A \in \mathcal{L}(X, X')$ is coercive.

  Moreover, the following estimate holds true

$$\|u\|_X \le \frac{1}{C_{sta}} \|f\|_{X'}.$$

Proof.

- Let a be coercive, then for all $v \in X\backslash\{0\}$,

$$C_{sta}\|v\|_X \le \frac{a(v,v)}{\|v\|_X} \le \sup_{w \in X\backslash\{0\}} \frac{a(v,w)}{\|w\|_X},$$

  and this condition also holds for $v = 0$.

- To prove the second statement in the BNB theorem, namely,

$$\text{For all } w \in X,\ a(v,w) = 0 \text{ implies that } w = 0,\ \forall v \in X.$$

  Let $w \in X$ be such that $a(v,w) = 0$, $\forall v \in X$. Then, choosing $v = w$ yields $\|w\|_X = 0$ due to the coercivity of $a(v,w)$. Hence $w = 0$.

Abstract nonconforming error analysis

- Let $V_h \subset L^2(\Omega)$ be a finite dimensional function space, e.g. V_h is a broken polynomial space.

  Consider the discrete problem

  $$\text{Find } u_h \in V_h \text{ s.t. } a_h(u_h, w_h) = l_h(w_h) \qquad \forall w_h \in V_h,$$

  with discrete bilinear form $a_h : V_h \times V_h \to \mathbb{R}$ and discrete linear form $l_h : V_h \to \mathbb{R}$.

  Since functions in V_h can be discontinuous across mesh elements, we have $V_h \not\subset X$ and $V_h \not\subset Y$.

  Hence we have a nonconforming finite element discretization.

Abstract nonconforming error analysis

- Alternatively, consider the discrete linear operator $A_h : V_h \to V_h$ s.t. $\forall v_h, w_h \in V_h$

$$(A_h v_h, w_h)_{L^2(\Omega)} := a_h(v_h, w_h)$$

and the discrete function $L_h \in V_h$ s.t. $\forall w_h \in V_h$,

$$(L_h, w_h)_{L^2(\Omega)} := l_h(w_h),$$

which gives the formulation

$$\text{Find } u_h \in V_h \text{ s.t. } A_h u_h = L_h \quad \text{in } V_h.$$

Abstract nonconforming error analysis

- Assume that the data $f \in L^2(\Omega)$, then $< f, w >_{Y', Y} = (f, w)_{L^2(\Omega)}$ and

$$l_h(w_h) = (L_h, w_h)_{L^2(\Omega)} = (f, w_h)_{L^2(\Omega)}, \quad \text{and} \quad L_h = \pi_h f,$$

with $\pi_h : L^2(\Omega) \to V_h$ the $L^2(\Omega)$-orthogonal projection onto V_h so that $\forall v \in L^2(\Omega)$, $\pi_h v \in V_h$ with

$$(\pi_h v, y_h)_{L^2(\Omega)} = (v, y_h)_{L^2(\Omega)} \qquad \forall y_h \in V_h.$$

- Note, $\pi_h v$ can be computed in an element T independently from other elements in $\mathcal{T}_h$, hence $\forall T \in \mathcal{T}_h$, $\pi_h v|_T \in \mathbb{P}_d^k(T)$, s.t.

$$(\pi_h v|_T, \xi)_{L^2(T)} = (v, \xi) L^2(T) \qquad \forall \xi \in \mathbb{P}_d^k(T).$$

Discrete stability

Define the norm $|||\cdot|||$ on V_h.

- (Discrete stability) A discrete bilinear form a_h has discrete stability on V_h if there is a $C_{sta} > 0$ s.t.

$$\forall v_h \in V_h, \quad C_{sta} \, |||v_h||| \leq \sup_{w_h \in V_h \setminus \{0\}} \frac{a_h(v_h, w_h)}{|||w_h|||}. \tag{4}$$

- Property (4) is called the discrete inf-sup condition and is equivalent to

$$C_{sta} \leq \inf_{v_h \in V_h \setminus \{0\}} \sup_{w_h \in V_h \setminus \{0\}} \frac{a_h(v_h, w_h)}{|||v_h|||\,|||w_h|||}.$$

- The coefficient C_{sta} can depend on the mesh size h, but for convergence analysis it is important to ensure C_{sta} is independent of h.

Discrete stability

- Lemma. (Discrete well-posedness) The discrete problem

$$\text{Find } u_h \in V_h \text{ s.t. } a_h(u_h, w_h) = l(w_h) \quad \forall w_h \in V_h,$$

is well-posed if and only if the discrete inf-sup condition (4) is satisfied.

Proof. The discrete inf-sup condition is the discrete counterpart of the inf-sup condition in the BNB theorem.

Since $V_h \not\subset V$ in a DG discretization the discrete inf-sup condition does not follow from the inf-sup condition in the space V, and must be separately proven.

Discrete stability

- A sufficient condition for discrete stability (and easier to verify) is coercivity:

$$\text{There is a } C_{sta} > 0 \text{ s.t. } \forall v_h \in V_h, \quad C_{sta} |||\, v_h |||^2 \leq a_h(v_h, v_h). \tag{5}$$

- Discrete coercivity implies the discrete inf-sup condition since $\forall v_h \in V_h \setminus \{0\}$,

$$C_{sta} |||\, v_h ||| \leq \frac{a_h(v_h, v_h)}{|||v_h|||} \leq \sup_{w_h \in V_h \setminus \{0\}} \frac{a_h(v_h, w_h)}{|||w_h|||}.$$

- Property (5) is the discrete counterpart of the Lax-Milgram lemma.

Consistency

- A (rather strong) form of consistency requires that the exact solution u of the variational equation satisfies the discrete problem

$$\text{Find } u_h \in V_h \text{ s.t. } a_h(u_h, w_h) = l(w_h) \quad \forall w_h \in V_h. \tag{6}$$

  This requires that $a_h(u, w_h)$ has a meaning, which may not be possible since a_h is only defined on $V_h \times V_h$.

- Assume that there is a subspace $X_* \subset X$ s.t. the exact solution belongs to X_* and that the bilinear form can be extended to $X_* \times V_h$.

- (Consistency) The discrete problem (6) is consistent if for the exact solution $u \in X_*$,

$$a_h(u, w_h) = l(w_h) \qquad \forall w_h \in V_h.$$

Consistency

- Consistency is equivalent to the Galerkin orthogonality property

$$a_h(u - u_h, w_h) = 0 \qquad \forall\, w_h \in V_h. \tag{7}$$

Proof. Substracting

$$a_h(u, w_h) = l_h(w_h) \qquad w_h \in V_h,$$
$$a_h(u_h, w_h) = l_h(w_h) \qquad w_h \in V_h,$$

and using the linearity of a_h gives (7).

Boundedness

- Define the vector space

$$X_{*h} := X_* + V_h$$

  with $X_* \subset X$ the space for the exact solution and V_h the discrete space.

  The approximation error then is $u - u_h \in X_{*h}$.

- Assume that the discrete norm $|||\cdot|||$ can be extended to X_{*h}.

  For many problems to prove boundedness in the space $X_{*h} \times V_h$ and we need to define also a norm $|||\cdot|||_*$ on X_{*h} s.t.

$$\forall v \in X_{*h}, \quad |||v||| \leq |||v|||_*$$

Boundedness

- (Boundedness) A discrete bilinear form a_h is bounded in $X_{*h} \times V_h$ if there is $C_{bnd} > 0$ s.t.

$$\forall (v, w) \in X_{*h} \times V_h, \quad |a_h(v, w)| \leq C_{bnd} ||| v |||_* \, ||| w ||| \, .$$

We assume that C_{bnd} is independent of h.

Error estimate

- Theorem (Abstract error estimate) Let u solve

$$\text{Find } u \in X \text{ } s.t. \text{ } a(u,w) = (f,w)_{L^2(\Omega)} \qquad \forall w \in Y$$

with $f \in L^2(\Omega)$. Let u_h solve

$$\text{Find } u_h \in V_h \text{ } s.t. \text{ } a_h(u_h,w_h) = (f,w_h)_{L^2(\Omega)} \qquad \forall w_h \in V_h.$$

Let $X_* \subset X$ and assume $u \in X_*$. Set $X_{*h} = X_* + V_h$ and assume that the bilinear form a_h can be extended to $X_{*h} \times V_h$.

Let $|||\cdot|||$ and $|||\cdot|||_*$ be two norms defined on X_{*h} s.t. $\forall v \in X_{*h}$, $|||v||| \leq |||v|||_*$.

Assume discrete stability, consistency and boundedness. Then the following error estimate holds true

$$|||u - u_h||| \leq \inf_{y_h \in V_h} |||u - y_h|||_*,$$

with $C = 1 + C_{sta}^{-1} C_{bnd}$.

Error estimate

- Proof. Let $y_h \in V_h$. Use the discrete stability and consistence, then

$$
\begin{aligned}
|||u_h - y_h||| &\le C_{sta}^{-1} \sup_{w_h \in V_h \setminus \{0\}} \frac{a_h(u_h - y_h, w_h)}{|||w_h|||} && \text{(discrete stability)} \\
&\le C_{sta}^{-1} \sup_{w_h \in V_h \setminus \{0\}} \frac{a_h(u - y_h, w_h)}{|||w_h|||} && \text{(orthogonality)} \\
&\le C_{sta}^{-1} C_{bnd} \sup_{w_h \in V_h \setminus \{0\}} \frac{|||u - y_h|||_* \, |||w_h|||}{|||w_h|||} && \text{boundedness)} \\
&= C_{sta}^{-1} C_{bnd} \, |||\, u - y_h \,|||_* \, .
\end{aligned}
$$

Error estimate

- Next, use the triangle inequality and the fact that

$$|||u - y_h||| \leq |||u - y_h|||_*$$

Then

$$\begin{aligned} |||u - u_h||| &\leq |||u - y_h\ ||| + |||\ u_h - y_h||| \\ &\leq |||u - y_h\ |||_* + C_{sta}^{-1} C_{bnd}\ |||\ u - y_h|||_* \\ &\leq (1 + C_{sta}^{-1} C_{bnd}) \inf_{y_h \in V_h} |||u - y_h|||_*, \end{aligned}$$

since $y_h \in V_h$ is arbitrary.

Admissible meshes

- Consider a mesh sequence $\mathcal{T}_{\mathcal{H}} := (\mathcal{T}_h)_{h \in \mathcal{H}}$.

  $\mathcal{H}$ denotes a countable subset of $\mathbb{R}_{>0} := \{x \in \mathbb{R} \mid x > 0\}$ having 0 as only accumulation point.

- (Matching simplicial mesh) A mesh $\mathcal{T}_h$ is a matching simplicial mesh if it is a simplicial mesh and if for any element $T \in \mathcal{T}_h$ with vertices $\{a_0, \cdots, a_d\}$, the set $\partial T \cap \partial T'$ for any $T' \in \mathcal{T}_h$, $T' \neq T$ is the convex hull of a (possibly empty) subset of $\{a_0, \cdots, a_d\}$.

  For instance in 2D, the set $\partial T \cap \partial T'$ for two distinct elements of a matching simplicial mesh is either a common vertex or common edge.

Simplicial submesh

- (Matching simplicial submesh) Let $\mathcal{T}_h$ be a general polyhedral mesh. Then $\mathscr{C}_h$ is a matching simplicial submesh of $\mathcal{T}_h$ if

  i) $\mathscr{C}_h$ is a matching simplicial mesh.

  ii) For all $T' \in \mathscr{C}_h$, there is only one $T \in \mathcal{T}_h$ s.t. $T' \subset T$.

  iii) For all $F' \in \mathscr{F}_h$, which is the set collecting the mesh faces of $\mathscr{C}_h$, there is at most one $F \in \mathcal{F}_h$ s.t. $F' \subset F$.

  The simplices in $\mathscr{C}_h$ are called subelements. The mesh faces in $\mathscr{F}_h$ are called subfaces.

  Define for all $T \in \mathcal{T}_h$ the sets

$$\mathscr{C}_T := \{T' \in \mathscr{C}_h \mid T' \subset T\},$$

$$\mathscr{F}_T := \{F' \in \mathscr{F}_h \mid F' \subset \partial T\},$$

$$\forall F \in \mathcal{F}_h: \quad \mathscr{F}_F := \{F' \in \mathscr{F}_h \mid F' \subset F\}.$$

Shape and contact regularity

- A mesh sequence $\mathcal{T}_{\mathcal{H}}$ is shape and contact regular if $\forall h \in \mathcal{H}$, $\mathcal{T}_h$ admits a matching simplicial submesh $\mathscr{C}_h$ s.t.

  i) The mesh sequence $\mathscr{C}_{\mathcal{H}}$ is shape regular, namely $\exists \rho_1 > 0$, independent of h, s.t. $\forall T' \in \mathscr{C}_h$

  $$\rho_1 h_{T'} \leq r_{T'},$$

  where $h_{T'}$ is the diameter of T' and $r_{T'}$ the radius of the largest ball inscribed in T'.

  ii) $\exists \rho_2 > 0$, independent of h, s.t. $\forall T \in \mathcal{T}_h$ and $\forall T' \in \mathscr{C}_h$

  $$\rho_2 h_T \leq h_{T'}.$$

The parameters ρ_1 and ρ_2 are called mesh regularity parameters and are denoted as ρ.

If $\mathcal{T}_h$ itself is matching and simplicial, then $\mathscr{C}_h = \mathcal{T}_h$ and the only requirement is shape regularity, $\rho_1 > 0$, independent of h.

Geometric properties of the mesh

- Lemma (Bound on $\operatorname{card}(\mathscr{C}_T)$). Let $\mathcal{T}_{\mathcal{H}}$ be a shape- and contact -regular mesh sequence. Then, for all $h \in \mathcal{H}$ and all $T \in \mathcal{T}_h$, $\operatorname{card}(\mathscr{C}_T)$ is bounded uniformly in h.

Proof. Let $|\cdot|_d$ denote the d-dimensional Haussdorff measure and let B_d be the unit ball in $\mathbb{R}^d$. Then,

$$
\begin{aligned}
h_T^d \geq |T|_d = \sum_{T' \in \mathscr{C}_T} |T'|_d &\geq \sum_{T' \in \mathscr{C}_T} |B_d|_d \, r_{T'}^d \geq \sum_{T' \in \mathscr{C}_T} |B_d|_d \, \rho_1^d h_{T'}^d \\
&\geq \sum_{T' \in \mathscr{C}_T} |B_d|_d \, \rho_1^d \rho_2^d h_T^d \\
&\geq |B_d|_d \, \rho_1^d \rho_2^d \operatorname{card}(\mathscr{C}_T) h_T^d,
\end{aligned}
$$

hence

$$
\operatorname{card}(\mathscr{C}_T) \leq \frac{1}{|B_d|_d \, \rho_1^d \rho_2^d}.
$$

Geometric properties of the mesh

- Lemma. (Bound on $\mathrm{card}(\mathcal{F}_T)$, $\mathrm{card}(\mathscr{F}_T)$, N_∂ and $\mathrm{card}(\mathscr{F}_F)$) Let $\mathcal{T}_\mathcal{H}$ be a shape- and contact-regular mesh sequence with parameter ρ.

  Then, for all $h \in \mathcal{H}$ and $\forall\, T \in \mathcal{T}_h$, $\mathrm{card}(\mathcal{F}_T)$, $\mathrm{card}(\mathscr{F}_T)$, and N_∂ are bounded uniformly in h.

  In addition, for all $F \in \mathcal{F}_h$, $\mathrm{card}(\mathscr{F}_F)$ is bounded uniformly in h.

  Proof. Observe that

$$\mathrm{card}(\mathcal{F}_T) \leq \mathrm{card}(\mathscr{F}_T) \leq (d+1)\mathrm{card}\,(\mathscr{C}_T),$$

  where in the last inequality we used the fact that a simplicial element has $d+1$ faces.

  Since $\mathrm{card}(\mathscr{C}_T)$ is uniformly bounded in h, then also $\mathrm{card}(\mathcal{F}_T)$ and $\mathrm{card}(\mathscr{F}_T)$ are uniformly bounded in h. Hence,

$$N_\partial = \max_{T \in \mathcal{T}_h} \mathrm{card}(\mathcal{F}_T)$$

  is also bounded in h. Finally, take $T \in \mathcal{T}_h$ s.t. $F \in \mathcal{F}_T$, use $\mathrm{card}(\mathscr{F}_F) \leq \mathrm{card}(\mathscr{C}_T)$.

- Lemma. (Lower bound on face diameters). Let $\mathcal{T}_{\mathcal{H}}$ be a shape- and contact-regular mesh sequence with parameter ρ.

  Then for all $h \in \mathcal{H}$, all $T \in \mathcal{T}_h$ and all $F \in \mathcal{F}_T$,

  $$\delta_F \geq \rho_1 \rho_2 h_T,$$

  where δ_F is the diameter of F.

  Proof. Let $T \in \mathcal{T}_h$, $F \in \mathcal{F}_T$. Then, take an $F' \in \mathscr{F}_F$ and denote by $T' \in \mathscr{C}_T$ the simplex to which the subface F' belongs. Then

  $$\delta_F \geq \delta_{F'} \geq r_{T'} \geq \rho_1 h_{T'} \geq \rho_1 \rho_2 h_T.$$

Inverse and trace inequalities

- Lemma. (Inverse inequality) Let $\mathcal{T}_{\mathcal{H}}$ be a shape- and contact regular mesh sequence with parameter ρ.

  Then, for all $h \in \mathcal{H}$ and all $v_h \in \mathbb{P}_d^k(\mathcal{T}_h)$ and all $T \in \mathcal{T}_h$,

  $$\|\nabla v_h\|_{[L^2(T)]^d} \leq C_{inv} h_T^{-1} \|v_h\|_{L^2(T))},$$

  where C_{inv} only depends on ρ, d and k.

  Proof. Let $v_h \in \mathbb{P}_d^k(\mathcal{T}_h)$, $T \in \mathcal{T}_h$. For all $T' \in \mathscr{C}_T$, the restriction $v_h|_{T'} \in \mathbb{P}_d^k(T')$.

  Use the inverse inequality on simplices, see e.g. Brenner & Scott, Math. Theory. of FEM or Ern & Guermond, Theory and Practice FEM,

  $$\|\nabla v_h\|_{[L^2(T')]^d} \leq C_{inv,s} h_{T'}^{-1} \|v_h\|_{L^2(T')},$$

  where $C_{inv,s}$ only depends on ρ_1, d and k.

Using the shape- and contact regularity of the mesh, namely

$$\exists \rho_2 > 0 \text{ s.t. } \rho_2 h_T \leq h_{T'}, \quad \text{hence} \quad \frac{1}{h_{T'}} \leq \frac{1}{\rho_2 h_T},$$

gives

$$\|\nabla v_h\|_{[L^2(T')]^d} \leq C_{inv,s} \rho_2^{-1} h_T^{-1} \|v_h\|_{[L^2(T')]^d}$$

Squaring the inequality and summing over all $T' \in \mathscr{C}_T$ proves the result.

Inverse and trace inequalities

- Lemma. (Discrete trace inequality) Let $\mathcal{T}_{\mathcal{H}}$ be a shape- and contact regular mesh sequence with parameter ρ.

  Then, for all $h \in \mathcal{H}$, all $v_h \in \mathbb{P}_d^k(\mathcal{T}_h)$, all $T \in \mathcal{T}_h$ and all $F \in \mathcal{F}_T$,

  $$h_T^{\frac{1}{2}} \|v_h\|_{L^2(F)} \leq C_{tr} \|v_h\|_{L^2(T)},$$

  where C_{tr} only depends on ρ, d, and k.

Inverse and trace inequalities

- Proof. Let $v_h \in \mathbb{P}_d^k(\mathcal{T}_h)$, let $T \in \mathcal{T}_h$, $F \in \mathcal{F}_T$. First assume that $\mathcal{T}_h$ is a matching simplicial mesh.

  Let $\widehat{T}$ be the unit simplex in $\mathbb{R}^d$, and let F_T be the bijective map such that $F_T(\widehat{T}) = T$.

  Let $\widehat{F}$ be any face of $\widehat{T}$. Since the unit sphere in $\mathbb{P}_d^k(\widehat{T})$ for the $L^2(\widehat{T})$-norm is a compact set ($\mathbb{P}_d^k$ is finite dimensional), there is a $\widehat{C}_{d,k}(\widehat{F})$, only depending on d, k and $\widehat{F}$ s.t. $\forall \widehat{v} \in \mathbb{P}_d^k(\widehat{T})$

$$\|\widehat{v}\|_{L^2(\widehat{F})} \leq \widehat{C}_{d,k}(\widehat{F}) \|\widehat{v}\|_{L^2(\widehat{T})}. \tag{8}$$

  Applying inequality (8) now to the function $\widehat{v} = v_h|_T \circ F_T^{-1}$, which is in $\mathbb{P}_d^k(\widehat{T})$, gives

$$|F|_{d-1}^{-\frac{1}{2}} \|v_h\|_{L^2(F)} \leq \widehat{C}_{d,k} |T|_d^{-\frac{1}{2}} \|v_h\|_{L^2(T)}.$$

Note,

$$\frac{|T|_d}{|F|_{d-1}} = \frac{\mathrm{Vol}(T)}{\mathrm{Area}(F)} = \frac{h_{T,F}}{d} \geq \frac{1}{d} r_T \geq \frac{1}{d} \rho_1 h_T, \tag{9}$$

where $h_{T,F}$ denotes the distance of the vertex opposite to F to that face, and r_T is the radius of the largest ball inscribed in T. Hence,

$$|F|_{d-1}^{-\frac{1}{2}} \|v_h\|_{L^2(F)} \leq \widehat{C}_{d,k} |T|_d^{-\frac{1}{2}} \|v_h\|_{L^2(T)}$$

is equal to

$$\left(\frac{|T|_d}{|F|_{d-1}} \right)^{\frac{1}{2}} \|v_h\|_{L^2(F)} \leq \widehat{C}_{d,k} \|v_h\|_{L^2(T)}$$

and finally using (9) we obtain

$$h_T^{\frac{1}{2}} \|v_h\|_{L^2(F)} \leq C_{tr,s} \|v_h\|_{L^2(T)},$$

with $C_{tr,s} = d^{\frac{1}{2}} \rho_1^{-\frac{1}{2}} \widehat{C}_{d,k}$ only depending on ρ, d and k.

- General mesh.

  For each $F' \in \mathscr{F}_F$, let T' denote the simplex in $\mathscr{C}_T$ of which F' is a face.

  Since the restriction $v_h|_{T'} \in \mathbb{P}_d^k(T')$, the discrete trace inequality yields

$$h_{T'}^{\frac{1}{2}} \|v_h\|_{L^2(F')} \leq C_{tr,s} \|v_h\|_{L^2(T')} \leq C_{tr,s} \|v_h\|_{L^2(T)}.$$

  This gives

$$\Big(\sum_{F' \in \mathscr{F}_F} h_{T'} \|v_h\|^2_{L^2(F')} \Big)^{\frac{1}{2}} \leq C_{tr,s} (\mathrm{card}(\mathscr{F}_F))^{\frac{1}{2}} \|v_h\|_{L^2(T)}$$

  since $h_{T'} \geq \rho_2 h_T$ and $\mathrm{card}(\mathscr{F}_F) \leq (d+1)\mathrm{card}(\mathscr{C}_T)$ is uniformly bounded.

L^2-norm transformation rules for tetrahedron

- A. Consider a tetrahedron $K \subset \mathbb{R}^3$ with vertices $\{x_0, \cdots, x_3\}$.

  Define the mapping $F_T(\widehat{T}) = T$, with $\widehat{T}$ the reference tetrahedron with vertices $\{(0,0,0),(1,0,0),(0,1,0),(0,0,1)\}$, and for $x \in T$, $\widehat{x} \in \widehat{T}$, we have the relation

$$x = B_T \widehat{x} + x_0,$$

  with Jacobian matrix

$$B_T = \frac{\partial x}{\partial \widehat{x}} = \begin{pmatrix} x_1 - x_0 & x_2 - x_0 & x_3 - x_0 \\ y_1 - y_0 & y_2 - y_0 & y_3 - y_0 \\ z_1 - z_0 & z_2 - z_0 & z_3 - z_0 \end{pmatrix}$$

  and

$$\det B_T = \frac{\mathrm{Vol}(T)}{\mathrm{Vol}(\widehat{T})}.$$

L^2-norm transformation rules for tetrahedron

- Consider the $L^2(T)$-norm:

$$\begin{aligned}
\|v\|_{L^2(T)} &= \left(\int_T |v(x)|^2 d^3x \right)^{\frac{1}{2}} \\
&= \left(\int_{\widehat{T}} |\widehat{v}(\widehat{x})|^2 \, |\det B_T| d^3\widehat{x} \right)^{\frac{1}{2}} && \text{(with } \widehat{v}(\widehat{x}) = v(F_T(\widehat{x}))\text{)} \\
&= \left(\frac{\mathrm{Vol}(T)}{\mathrm{Vol}(\widehat{T})} \right)^{\frac{1}{2}} \left(\int_{\widehat{T}} |\widehat{v}(\widehat{x})|^2 d^3\widehat{x} \right)^{\frac{1}{2}} && \text{since } \det B_T \text{ is constant} \\
&= \left(\frac{\mathrm{Vol}(T)}{\mathrm{Vol}(\widehat{T})} \right)^{\frac{1}{2}} \|\widehat{v}\|_{L^2(\widehat{T})},
\end{aligned}$$

or equivalently

$$\|\widehat{v}\|_{L^2(\widehat{T})} = \left(\frac{\mathrm{Vol}(T)}{\mathrm{Vol}(\widehat{T})} \right)^{-\frac{1}{2}} \|v\|_{L^2(T)}.$$

L^2-norm transformation rules for tetrahedron

- B. Given a tetrahedron $T \subset \mathbb{R}^3$. Consider a face $F \subset \partial T$ with vertices $\{x_0, x_1, x_2\}$.

  Define the mapping $F_F(\widehat{F}) = F$, with $\widehat{F}$ the reference triangle with vertices $\{(0,0),(1,0),(0,1)\}$, and for $x \in F$, $\widehat{x} \in \widehat{F}$, we have the relation

$$x = (x_1 - x_0)\widehat{x} + (x_2 - x_0)\widehat{y} + x_0.$$

L^2-norm transformation rules for tetrahedron

- Consider the $L^2(F)$-norm:

$$
\begin{aligned}
\|v\|_{L^2(F)} &= \Big(\int_F |v(x)|^2 dS \Big)^{\frac{1}{2}} \\
&= \Big(\int_{\widehat{F}} |v(F_F(\widehat{x}))|^2 \left| \frac{\partial F_F}{\partial \widehat{x}} \times \frac{\partial F_F}{\partial \widehat{y}} \right| d\widehat{x} d\widehat{y} \Big)^{\frac{1}{2}} \\
&= \left| (x_1 - x_0) \times (x_2 - x_0) \right|^{\frac{1}{2}} \Big(\int_{\widehat{F}} |\widehat{v}(\widehat{x})|^2 \, d\widehat{x} d\widehat{y} \Big)^{\frac{1}{2}} \\
&= \left(\frac{\mathrm{Area(F)}}{\mathrm{Area}(\widehat{F})} \right)^{\frac{1}{2}} \|\widehat{v}\|_{L^2(\widehat{F})}
\end{aligned}
$$

since

$$
\left| (x_1 - x_0) \times (x_2 - x_0) \right| = \frac{\mathrm{Area(F)}}{\mathrm{Area}(\widehat{F})}.
$$

L^2-norm transformation rules for tetrahedron

- The estimate

$$\|\widehat{v}_h\|_{L^2(\widehat{F})} \leq \widehat{C}'_{d,k} \|\widehat{v}_h\|_{L^2(\widehat{T})}$$

is thus equal to

$$\left(\frac{\mathrm{Area}(F)}{\mathrm{Area}(\widehat{F})}\right)^{-\frac{1}{2}} \|v_h\|_{L^2(F)} \leq \widehat{C}'_{d,k}(\widehat{F}) \left(\frac{\mathrm{Vol}(T)}{\mathrm{Vol}(\widehat{T})}\right)^{-\frac{1}{2}} \|v_h\|_{L^2(T)}$$

or equivalently,

$$|F|_{d-1}^{-\frac{1}{2}} \|v_h\|_{L^2(F)} \leq \widehat{C}_{d,k} |T|_d^{-\frac{1}{2}} \|v_h\|_{L^2(T)}.$$

Continuous trace inequality

- Lemma. (Continuous trace inequality) Let $\mathcal{T}_{\mathcal{H}}$ be a shape- and contact-regular mesh sequence.

  Then for all $h \in \mathcal{H}$, all $v \in H^1(\mathcal{T}_h)$ and all $T \in \mathcal{T}_h$, and all $F \in \mathcal{F}_T$,

$$\|v\|^2_{L^2(F)} \leq C_{cti}\big(2\|\nabla v\|_{[L^2(T)]^d} + dh_T^{-1}\|v\|_{L^2(T)}\big)\|v\|_{L^2(T)},$$

  with $C_{cti} = \rho^{-1}$ if $\mathcal{T}_h$ is a matching simplicial mesh and $C_{cti} = (1+d)(\rho_1\rho_2)^{-1}$ otherwise.

Continuous trace inequality

- Proof. Let $v \in H^1(\mathcal{T}_h)$ and $F \in \mathcal{F}_T$. First, assume T is a simplex with vertices $\{x_0, \cdots, x_3\}$ and consider the $\mathbb{R}^d$-valued function

$$\sigma_F = \frac{|F|_{d-1}}{d|T|_d}(x - a_F),$$

where a_F is the vertex x_0 of T opposite to face F, which has vertices $\{x_1, x_2, x_3\}$.

Note, σ_T is proportional to the lowest order Raviart-Thomas-Nédélec shape function in T.

At face F we have the normal vector

$$n_T = \frac{(x_1 - x_3) \times (x_2 - x_3)}{|(x_1 - x_3) \times (x_2 - x_3)|},$$

which gives

$$n_T = \frac{\mathrm{Area}(\widehat{F})}{\mathrm{Area}(F)}(x_1 - x_3) \times (x_2 - x_3) = \frac{|\widehat{F}|_{d-1}}{|F|_{d-1}}(x_1 - x_3) \times (x_2 - x_3).$$

With $a_F = x_0$ we have that for $x - x_0 \in F$,

$$x - a_F = x - x_0 = (x_1 - x_3)\widehat{x} + (x_2 - x_3)\widehat{y} + x_3 - x_0.$$

Then

$$\begin{aligned}
n_T \cdot \sigma_F &= \frac{|\widehat{F}|_{d-1}}{|F|_{d-1}} \frac{|F|_{d-1}}{d|T|_d} \big((x_1 - x_3) \times (x_2 - x_3)\big) \cdot \big((x_1 - x_3)\widehat{y} + (x_2 - x_3)\widehat{y} + x_3 - x_0\big) \\
&= \frac{|\widehat{F}|_{d-1}}{d|T|_d} \big((x_3 - x_1) \times (x_3 - x_2)\big) \cdot (x_3 - x_0) \\
&= \frac{|\widehat{F}|_{d-1}}{d|T|_d} \det|x_3 - x_0, x_3 - x_1, x_3 - x_2| \\
&= \frac{|\widehat{F}|_{d-1}}{d|T|_d} \frac{|T|_d}{|\widehat{T}|_d} \\
&= 1 \qquad \text{since } |\widehat{F}|_{d-1} = \frac{1}{2}, |\widehat{T}|_d = \frac{1}{6} \text{ for } d = 3.
\end{aligned}$$

Note, for the other faces F' of T, we have $n_T \cdot \sigma_{F'} = 0$ since $\sigma_{F'}$ is parallel to the face $F' \neq F$.

Continuous trace inequality

- Using the divergence theorem we obtain

$$\|v\|^2_{L^2(F)} = \int_F |v|^2 = \int_{\partial T} |v|^2(\sigma_F \cdot n_T) \qquad \text{(since } \sigma_F \cdot n_T = 1 \text{ at } F \text{ and}$$

$$\sigma_F \cdot n_T = 0 \text{ at } \partial T \backslash F)$$

$$= \int_T \nabla \cdot (|v|^2 \sigma_F)$$

$$= \int_T (2v\sigma_F \cdot \nabla v + |v|^2 \nabla \cdot \sigma_F).$$

Hence

$$\|v\|^2_{L^2(F)} \leq 2\|v\|_{L^2(T)} \|\sigma_F \cdot \nabla v\|_{L^2(T)} + \|\nabla \cdot \sigma_F\|_{L^\infty(T)} \|v\|^2_{L^2(T)}.$$

Continuous trace inequality

Since

$$\|\sigma_F\|_{[L^\infty(T)]^d} \le \frac{|F|_{d-1} h_T}{d|T|_d}, \qquad \text{because } a_F \text{ is the vertex opposite to } F,$$

$$\nabla \cdot \sigma_F = \frac{|F|_{d-1}}{|T|_d},$$

$$\|\sigma_F \cdot \nabla v\|_{L^2(T)} \le \|\sigma_F\|_{[L^\infty(T)]^d} \|\nabla v\|_{[L^2(T)]^d} \le \frac{|F|_{d-1} h_T}{d|T|_d} \|\nabla v\|_{[L^2(T)]^d},$$

we obtain the estimate

$$\begin{aligned}
\|v\|_{L^2(F)}^2 &\le 2\|v\|_{L^2(T)} \|\sigma_F \cdot \nabla v\|_{L^2(T)} + \nabla \cdot \sigma_F \|v\|_{L^2(T)}^2 \\
&\le \frac{|F|_{d-1} h_T}{d|T|_d} \Big(2\|\nabla v\|_{[L^2(T)]^d} + d h_T^{-1} \|v\|_{L^2(T)} \Big) \|v\|_{L^2(T)} \\
&\le \frac{1}{\rho_1} \Big(2\|\nabla v\|_{[L^2(T)]^d} + d h_T^{-1} \|v\|_{L^2(T)} \Big) \|v\|_{L^2(T)}, \quad \text{using } \frac{|F|_{d-1}}{|T|_d} \le \frac{d}{\rho_1 h_T}.
\end{aligned}$$

Continuous trace inequality

- If $\mathcal{T}_h$ is a general mesh use the subdivision into a matching simplicial mesh.

  For each $F' \in \mathscr{F}_F$, let T' denote the simplex in $\mathscr{C}_T$ of which F' is a face.

  Applying the continuous trace inequality for F' and T' yields,

$$\|v\|^2_{L^2(F')} \leq \frac{1}{\rho_1}\Big(2\|\nabla v\|_{[L^2(T')]^d} + dh_{T'}^{-1}\|v\|_{L^2(T')}\Big)\|v\|_{L^2(T')}.$$

  From the mesh regularity we have $h_{T'} \geq \rho_2 h_T$ and $\rho_2 \leq 1$, which gives $\frac{1}{h_{T'}} \leq \frac{1}{\rho_2 h_T}$, and

$$\|v\|^2_{L^2(F')} \leq \frac{1}{\rho_1 \rho_2}\Big(2\|\nabla v\|_{[L^2(T')]^d} + dh_T^{-1}\|v\|_{L^2(T')}\Big)\|v\|_{L^2(T')}.$$

  Hence, after summing $F' \in \mathscr{F}_F$ and using the fact that $T' \in \mathscr{C}_T$ appears at most $(d+1)$-times gives

$$\|v\|^2_{L^2(F)} \leq \frac{d+1}{\rho_1 \rho_2}\Big(2\|\nabla v\|_{[L^2(T)]^d} + dh_T^{-1}\|v\|_{L^2(T)}\Big)\|v\|_{L^2(T)}.$$

Comparison of $\|\cdot\|_{L^p(T)}$- and $\|\cdot\|_{L^q(T)}$-norms

- Lemma. (Comparison of $\|\cdot\|_{L^p(T)}$- and $\|\cdot\|_{L^q(T)}$-norms). Let $\mathcal{T}_{\mathcal{H}}$ be a shape- and contact-regular mesh sequence with parameter ρ.

  Let $1 \leq p, q, \leq \infty$ be two real numbers. Then for all $h \in \mathcal{H}$, all $v_h \in \mathbb{P}_d^k(\mathcal{T}_h)$ and all $T \in \mathcal{T}_h$,

$$\|v_h\|_{L^p(T)} \leq C_{inv,p,q} h_T^{d(\frac{1}{p}-\frac{1}{q})} \|v_h\|_{L^q(T)}$$

  where $C_{inv,p,q}$ only depends on ρ, d, k, p and q.

Comparison of $\|\cdot\|_{L^p(T)}$- and $\|\cdot\|_{L^q(T)}$-norms

- Proof. Since $v_h \in \mathbb{P}_d^k(\mathcal{T}_h)$, we can use that all norms are equivalent in a finite dimensional space,

$$
\begin{aligned}
&\|\widehat{v}_h\|_{L^p(\widehat{T})} \leq \widehat{C}\|\widehat{v}_h\|_{L^q(\widehat{T})} \\
\Leftrightarrow &\Big(\int_T |v_h(x)|^p \frac{1}{\det B_T} dx\Big)^{\frac{1}{p}} \leq \widehat{C}\Big(\int_T |v_h(x)|^q \frac{1}{\det B_T} dx\Big)^{\frac{1}{q}} \\
\Leftrightarrow &\left(\frac{\mathrm{Vol}(\widehat{T})}{\mathrm{Vol}(T)}\right)^{\frac{1}{p}} \|v_h\|_{L^p(T)} \leq \widehat{C}\left(\frac{\mathrm{Vol}(\widehat{T})}{\mathrm{Vol}(T)}\right)^{\frac{1}{q}} \|v_h\|_{L^q(T)} \\
\Leftrightarrow &\|v_h\|_{L^p(T)} \leq \widehat{C}\left(\frac{\mathrm{Vol}(T)}{\mathrm{Vol}(\widehat{T})}\right)^{\frac{1}{p}-\frac{1}{q}} \|v_h\|_{L^q(T)} \\
\Rightarrow &\|v_h\|_{L^p(T)} \leq C_{inv,p,q} h_T^{d(\frac{1}{p}-\frac{1}{q})} \|v_h\|_{L^q(T)}.
\end{aligned}
$$

Discrete trace inequality in $L^p(F)$

- Lemma. (Discrete trace inequality in $L^p(F)$). Let $\mathcal{T}_{\mathcal{H}}$ be a shape- and contact-regular mesh sequence with parameter ρ.

  Let $1 \le p, q, \le \infty$ be two real numbers. Then for all $h \in \mathcal{H}$, all $v_h \in \mathbb{P}_d^k(\mathcal{T}_h)$ and all $T \in \mathcal{T}_h$, and all $F \in \mathcal{T}_T$,

  $$h_T^{\frac{1}{p}} \|v_h\|_{L^p(F)} \le C_{tr,p} \|v_h\|_{L^p(T)},$$

  where $C_{tr,p}$ only depends on ρ, d, k and p.

Discrete trace inequality in $L^p(F)$

- Proof. Combine the discrete trace inequality with the relation between $L^p$and L^q-norms, then

$$
\begin{aligned}
h_T^{\frac{1}{p}} \|v_h\|_{L^p(F)} &\le C_{inv,p,2} h_T^{\frac{1}{p}} \delta_F^{(d-1)(\frac{1}{p}-\frac{1}{2})} \|v_h\|_{L^2(F)} && \text{(use relation between } L^p \text{ and } L^q \text{ norms} \\
& && \text{with } q = 2 \text{ for a face } F \text{ with } \dim(F) = d-1) \\
&\le C_{inv,p,2} C_{tr} h_T^{\frac{1}{p}-\frac{1}{2}} \delta_F^{(d-1)(\frac{1}{p}-\frac{1}{2})} \|v_h\|_{L^2(T)} && \text{(use discrete trace inequality)} \\
&\le C_{inv,p,2} C_{tr} C_{inv,2,p} h_T^{\frac{1}{p}-\frac{1}{2}} \delta_F^{(d-1)(\frac{1}{p}-\frac{1}{2})} h_T^{d(\frac{1}{2}-\frac{1}{p})} \|v_h\|_{L^p(T)} && \\
& && \text{(use relation between } L^p \text{and } L^q\text{-norms)} \\
&\le C_{tr,p} \|v_h\|_{L^p(T)} && \text{using } \delta_F \cong h_T.
\end{aligned}
$$

Polynomial approximation

Since $u_h \in V_h$ we obtain from the error bound for the variational equation the relation

$$\inf_{y_h \in V_h} |||u - y_h||| \leq |||u - u_h||| \leq C \inf_{y_h \in V_h} |||u - y_h|||_*, \tag{10}$$

hence we need a bound for the approximation error on the righthand side of (10).

The optimality of the error estimate is classified as:

- (Optimality, quasi-optimality and suboptimality of the error estimate).

  i) Optimal, if $|||\cdot||| = |||\cdot|||_*$.

  ii) Quasi-optimal, if the norms $|||\cdot|||$ and $|||\cdot|||_*$ are different, but the lower and upper bounds in (10) converge for smooth u at the same rate as $h \to 0$.

  iii) Suboptimal, if the upper bound in (10) converges at a slower rate than the lower bound.

Polynomial approximation

- (Optimal polynomial approximation). The mesh sequence $\mathcal{T}_{\mathcal{H}}$ has optimal polynomial approximation properties if for all $h \in \mathcal{H}$, all $T \in \mathcal{T}_h$ and all polynomial degrees k, there is a linear interpolation operator $I_T^k : L^2(T) \to \mathbb{P}_d^k(T)$ s.t. $\forall s \in \{0, \cdots, k+1\}$ and all $v \in H^s(T)$ there holds

$$|v - I_T^k v|_{H^m(T)} \leq C_{app} h_T^{s-m} |v|_{H^s(T)} \qquad \forall m \in \{0, \ldots, s\},$$

  where C_{app} is independent of both T and h.

- (Admissible mesh sequences). A mesh sequence $\mathcal{T}_{\mathcal{H}}$ is admissible if it is shape- and contact-regular, and if it has optimal polynomial approximation properties.

- Lemma. (Optimality of L^2-orthogonal projection). Let $\mathcal{T}_{\mathcal{H}}$ be an admissible mesh sequence.

  Let π_h be the L^2-orthogonal projection onto $\mathbb{P}_d^k$. Then $\forall s \in \{0, \cdots, k+1\}$ and all $v \in H^s(T)$, we have

$$|v - \pi_h v|_{H^m(T)} \leq C'_{app} h_T^{s-m} |v|_{H^s(T)} \qquad \forall m \in \{0, \cdots, s\}.$$

  where C_{app} is independent of both T and h.

Polynomial approximation

Proof. For $m = 0$, we have since $\pi_h : L^2(T) \to \mathbb{P}_d^k$ is the L^2-orthogonal projection that

$$\|v - \pi_h v\|_{L^2(T)} \leq \|v - I_T^k v\|_{L^2(\Omega)} \leq C_{app} h_T^s |v|_{H^s(T)}$$

For $m \geq 1$, use m-times the inverse inequality, together with the triangle inequality

$$\begin{aligned}
|v - \pi_h v|_{H^m(T)} &\leq |v - I_T^k v|_{H^m(T)} + |I_T^k v - \pi_h v|_{H^m(T)} && \text{(triangle inequality)} \\
&\leq |v - I_T^k v|_{H^m(T)} + C' h_T^{-m} \|I_T^k v - \pi_h v\|_{L^2(T)} && \text{(use } m\text{-times inverse inequality)} \\
&\leq |v - I_T^k v|_{H^m(T)} + C' h_T^{-m} \|v - I_T^k v\|_{L^2(T)} + C' h_T^{-m} \|v - \pi_h v\|_{L^2(T)} && \\
&\leq |v - I_T^k v|_{H^m(T)} + 2C' h_T^{-m} \|v - I_T^k v\|_{L^2(T)} && \text{(using } m = 0 \text{ case)} \\
&\leq C'_{app} h_T^{s-m} |v|_{H^s(T)} && \text{(use optimal polynomial approximation error).}
\end{aligned}$$

Polynomial approximation

- Lemma. (Polynomial approximation on mesh faces). Let $\mathcal{T}_{\mathcal{H}}$ be an admissible mesh sequence.

  Let π_h be the L^2-orthogonal projection onto $\mathbb{P}_d^k$. Then for all $s \in \{1, \cdots, k+1\}$ and all $v \in H^s(T)$, we have

  $$\|v - \pi_h v\|_{L^2(F)} \leq C''_{app} h_T^{s-\frac{1}{2}} |v|_{H^s(T)},$$

  and if $s \geq 2$,

  $$\|\nabla(v - \pi_h v)|_T \cdot n_T\|_{L^2(F)} \leq C''_{app} h_T^{s-\frac{3}{2}} |v|_{H^s(T)},$$

  where C'_{app}, C''_{app} are independent of both T and h.