Discontinuous Galerkin Methods

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PDEs with diffusion, continuous setting

- Poisson equation

$$-\Delta u = f \qquad \text{in } \Omega \qquad \text{with } f \in L^2(\Omega),$$
$$u = 0 \qquad \text{on } \Omega.$$

The weak formulation for the Poisson equation is

$$\text{Find } u \in V \text{ s.t. } a(u, v) = \int_\Omega fv \qquad \forall v \in V,$$

with

$$V := H_0^1 := \{v \in H^1(\Omega) \mid v|_{\partial\Omega} = 0\},$$

and bilinear form $a(v, w) : V \times V \to \mathbb{R}$ defined as

$$a(v, w) := \int_\Omega \nabla v \cdot \nabla w.$$

Using the Poincaré inequality, stating there exists C_Ω s.t. $\forall v \in H_0^1(\Omega)$,

$$\|v\|_{L^2(\Omega)} \leq C_\Omega \|\nabla v\|_{[L^2(\Omega)]^d},$$

we immediately obtain that $a(v, w)$ is coercive,

$$\|v\|_{H^1(\Omega)}^2 \leq (1 + C_\Omega) a(v, v) \qquad \forall v \in V.$$

The Lax-Milgram theorem then states that the weak formulation is well-posed.

Potential and diffusive flux

- The Poisson equation can be written in a mixed form as the first order system

$$\sigma + \nabla u = 0 \qquad \text{in } \Omega,$$
$$\nabla \cdot \sigma = f \qquad \text{in } \Omega,$$

with u called the potential and $\sigma = -\nabla u$ the diffusive flux.

- Definition. (Boundary averages and jumps). For a smooth function v, for all $F \in \mathcal{F}_h^b$ and for a.e. $x \in F$ we define the average and jump of v as

$$\{\!\{v\}\!\}_F(x) = [\![v]\!]_F(x) := v(x).$$

Potential and diffusive flux

Since the potential $u \in V = H_0^1(\Omega)$, $\forall T \in \mathcal{T}_h$ and all $F \in \mathcal{F}_T$ the trace $u_T := u_T|_F \in L^2(F)$.

The diffusive flux $\sigma \in H(\mathrm{div};\Omega) := \{v \in L^2(\Omega) \mid \nabla \cdot v \in L^2(\Omega)\}$.

Under the assumption $u \in W^{2,1}(\Omega)$ there holds $\sigma \in [W^{1,1}(\Omega)]^d$, hence $\forall T \in \mathcal{T}_h$ and $\forall F \in \mathcal{F}_T$, we have for $\sigma_T := \sigma|_T$ and $\sigma_{\partial T} := \sigma_T \cdot n_T$ at ∂T that the trace of σ satisfies $\sigma_{\partial T} \in L^1(F)$.

The trace $\sigma_{\partial T} \in L^2(F)$ if $u \in H^{\frac{3}{2}+\epsilon}(\Omega)$, $\epsilon > 0$.

- Lemma. (Jumps of potential and diffusive flux). Assume $u \in V \cap W^{2,1}(\Omega)$. Then there holds

$$
\begin{aligned}
[\![u]\!] &= 0 \qquad \forall F \in \mathcal{F}_h, \\
[\![\sigma]\!] \cdot n_F &= 0 \qquad \forall F \in \mathcal{F}_h^i.
\end{aligned}
$$

Heuristic derivation Symmetric Interior Penalty (SIP) method

Set $V_h := \mathbb{P}_d^k(\mathcal{T}_h)$ with $k \geq 1$ and assume that $\mathcal{T}_h$ is an admissible mesh sequence.

- Assumption. (Regularity of exact solution and space V_*). Assume that the exact solution u is such that

$$u \in V_* := V \cap H^2(\Omega).$$

  Note, this assumption is asserted if Ω is convex.

- Set $V_{*h} := V_* + V_h$.

Heuristic derivation SIP method

First, localize the gradient to mesh elements in the exact bilinear form a(v,w),

$$\forall (v, w_h) \in V_{*h} \times V_h, \quad a_h^{(0)}(v, w_h) := \int_\Omega \nabla_h v \cdot \nabla_h w_h = \sum_{T \in \mathcal{T}_h} \int_T \nabla v \cdot \nabla w_h.$$

To examine consistency, integrate by parts on each element

$$a_h^{(0)}(v, w_h) = -\sum_{T \in \mathcal{T}_h} \int_T (\Delta v) w_h + \sum_{T \in \mathcal{T}_h} \int_{\partial T} (\nabla v \cdot n_T) w_h.$$

Heuristic derivation SIP method

Use the relation

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T}(\nabla v\cdot n_T)w_h=\sum_{F\in\mathcal{F}_h^i}\int_F [\![(\nabla_h v)w_h]\!]\cdot n_F+\sum_{F\in\mathcal{F}_h^b}\int_F(\nabla v\cdot n_F)w_h,$$

since for all $F\in\mathcal{F}_h^i$ with $F=\partial T_1\cap\partial T_2$, we have $n_F=n_{T_1}=-n_{T_2}$.

Moreover, at internal faces we have

$$[\![(\nabla_h v)w_h]\!]=\{\!\{\nabla_h v\}\!\}[\![w_h]\!]+[\![\nabla_h v]\!]\{\!\{w_h\}\!\}.$$

Combining all terms we obtain

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T}(\nabla v\cdot n_T)w_h=\sum_{F\in\mathcal{F}_h}\int_F\{\!\{\nabla_h v\}\!\}\cdot n_F[\![w_h]\!]+\sum_{F\in\mathcal{F}_h^i}\int_F[\![\nabla v]\!]\cdot n_F\{\!\{w_h\}\!\}.$$

Heuristic derivation SIP method

The bilinear form $a_h^{(0)}(v, w_h) : V_{*h} \times V_h \to \mathbb{R}$ then can be written as

$$a_h^{(0)}(v, w_h) = -\sum_{T \in \mathcal{T}_h} \int_T (\Delta v) w_h + \sum_{F \in \mathcal{F}_h} \int_F \{\!\{\nabla_h v\}\!\} \cdot n_F [\![w_h]\!]$$

$$+ \sum_{F \in \mathcal{F}_h^i} \int_F [\![\nabla_h v]\!] \cdot n_F \{\!\{w_h\}\!\}.$$

Heuristic derivation SIP method

To check consistency, set $v = u \in V_*$.

Use $-\Delta u = f$ and the fact that $[\![\sigma]\!] = -[\![\nabla u]\!] \cdot n_F = 0$ for $u \in V \cap W^{2,1}(\Omega)$, then

$$\forall w_h \in V_h, \qquad a_h^{(0)}(u, w_h) = \int_\Omega f w_h + \sum_{F \in \mathcal{F}_h} \int_F (\nabla u \cdot n_F)[\![w_h]\!].$$

To remove the last, inconsistent term we define $\forall (v, w_h) \in V_{*h} \times V$,

$$a_h^{(1)}(v, w_h) := \int_\Omega \nabla_h v \cdot \nabla_h w_h - \sum_{F \in \mathcal{F}_h} \int_F \{\!\{\nabla u\}\!\} \cdot n_F [\![w_h]\!].$$

Heuristic derivation SIP method

To preserve the symmetry of the bilinear form $a(v, w)$ also an extra term is added,

$$\sum_{F \in \mathcal{F}_h} \int_F [\![v]\!] \{\!\{\nabla_h w_h\}\!\}.$$

Note, this extra term does not influence the consistency since $[\![v]\!] = 0, \forall v \in H^1(\Omega)$.

The bilinear form for the Poisson equation $a_h^{cs} : V_{*h} \times V_h \to \mathbb{R}$ is then defined as

$$a_h^{cs}(v, w_h) := \int_\Omega \nabla_h v \cdot \nabla_h w_h - \sum_{F \in \mathcal{F}_h} \int_F \Big(\{\!\{\nabla v\}\!\} \cdot n_F [\![w_h]\!] + [\![v]\!] \{\!\{\nabla_h w_h\}\!\} \cdot n_F \Big),$$

and $a_h^{cs}(v_h, w_h)$ is symmetric on $V_h \times V_h$.

Consistency of $a_h^{cs}(v, w_h)$ follows directly from its construction.

Heuristic derivation SIP method

Using the following relation, obtained by integration by parts,

$$\sum_{T\in\mathcal{T}_h}\int_T \nabla v\cdot\nabla w_h = -\sum_{T\in\mathcal{T}_h}\int_T(\Delta v)w_h + \sum_{T\in\mathcal{T}_h}\int_{\partial T}(\nabla v\cdot n_T)w_h$$

$$= -\sum_{T\in\mathcal{T}_h}\int_T(\Delta v)w_h + \sum_{F\in\mathcal{F}_h}\{\!\{\nabla v\}\!\}\cdot n_F[\![w_h]\!] + \sum_{F\in\mathcal{F}_h^i}\int_F[\![\nabla v]\!]\cdot n_F\{\!\{w_h\}\!\},$$

we obtain the following equivalent bilinear form for $a_h^{CS}(v, w_h)$, namely

$$a_h^{CS}(v, w_h) = -\sum_{T\in\mathcal{T}_h}\int_T(\Delta v)w_h + \sum_{F\in\mathcal{F}_h^i}\int_F[\![\nabla v]\!]\cdot n_F\{\!\{w_h\}\!\}$$

$$-\sum_{F\in\mathcal{F}_h}\int_F[\![v]\!]\{\!\{\nabla_h w_h\}\!\}\cdot n_F.$$

Heuristic derivation SIP method

The discrete bilinear form a_h^{cs} is, however, not coercive since the last term in

$$a_h^{cs}(v_h, v_h) = \|\nabla_h v_h\|^2_{[L^2(\Omega)]^d} - 2 \sum_{F \in \mathcal{F}_h} \int_F \{\!\{\nabla_h v_h\}\!\} \cdot n_F [\![v_h]\!]$$

can have any sign.

To ensure coercivity a stabilization term penalising interface and boundary jumps is added to a_h^{cs}, resulting in the symmetric interior penalty (SIP) bilinear form $a_h^{SIP} : V_{*h} \times V_h \to \mathbb{R}$ for the Poisson equation,

$$a_h^{SIP}(v, w_h) := a_h^{cs}(v, w_h) + s_h(v, w_h),$$

with stabilization term

$$s_h(v, w_h) := \sum_{F \in \mathcal{F}_h} \frac{\eta}{h_F} \int_F [\![v]\!][\![w_h]\!],$$

with $\eta > 0$ a user defined penalty term, and h_F a local length scale, e.g. the diameter of a face $F \in \mathcal{F}_h$.

Heuristic derivation SIP method

Combining all terms gives the Symmetric Interior Penalty (SIP) bilinear form $a_h^{SIP} : V_{*h} \times V_h \to \mathbb{R}$ for the Poisson equation,

$$a_h^{SIP}(v, w_h) = \int_\Omega \nabla_h v \cdot \nabla_h w_h - \sum_{F \in \mathcal{F}_h} \int_F \left(\{\!\{\nabla_h v\}\!\} \cdot n_F [\![w_h]\!] + [\![v]\!] \{\!\{\nabla_h w_h\}\!\} \cdot n_F \right)$$

$$+ \sum_{F \in \mathcal{F}_h} \frac{\eta}{h_F} \int_F [\![v]\!][\![w_h]\!],$$

or equivalently

$$a_h^{SIP}(v, w_h) = - \sum_{T \in \mathcal{T}_h} \int_T (\Delta v) w_h + \sum_{F \in \mathcal{F}_h^i} \int_F [\![\nabla_h v]\!] \cdot n_F \{\!\{w_h\}\!\} - \sum_{F \in \mathcal{F}_h} \int_F [\![v]\!] \{\!\{\nabla_h w_h\}\!\} \cdot n_F$$

$$+ \sum_{F \in \mathcal{F}_h} \frac{\eta}{h_F} \int_F [\![v]\!][\![w_h]\!].$$

Note, for each element $T \in \mathcal{T}_h$, the stencil of a_h^{SIP} only consists of the element T and the direct neighbours connected to T at it's faces.

Discrete problem

The discrete DG problem for the Poisson equation with homogeneous boundary conditions is

$$\text{Find } u_h \in V_h \text{ s.t. } a_h^{SIP}(u_h, v_h) = \int_\Omega f v_h \qquad \forall v_h \in V_h.$$

Note, if the penalty parameter η is large enough then a_h^{SIP} is coercive on V_h.

The Lax-Milgram theorem then states that the discrete problem is well-posed.

Lemma. (Consistency). Assume $u \in V_*$. Then for all $v_h \in V_h$

$$a_h^{SIP}(u, v_h) = \int_\Omega f v_h.$$

Proof. Consistency of a_h^{SIP} follows directly from its construction.

Discrete problem

Remark. (Rough righthand sides)

At the continuous level the Poisson problem is well posed for a righthand side $f \in V' = H^{-1}(\Omega)$, the dual space of $H_0^1(\Omega)$, giving

$$a(u, v) = \langle f, v \rangle_{V', V} \qquad \forall v \in V.$$

Since the discrete space V_h is non-conforming in V it is not possible at the discrete level to take $\langle f, v_h \rangle_{V', V}$ as righthand side.

An alternative is to use a smoothing operator $I_h : V_h \to V_h \cap H_0^1(\Omega)$ and consider

$$\text{Find } u_h \in V_h \text{ s.t. } a_h^{SIP}(u_h, v_h) = \langle f, I_h v_h \rangle_{V', V} \qquad \forall v_h \in V_h,$$

but this formulation is no longer consistent.

Other boundary conditions

- For the Poisson equation with inhomogeneous Dirichlet boundary conditions,

$$-\Delta u = f \qquad \text{in } \Omega,$$
$$u = g \qquad \text{at } \partial\Omega,$$

with $g \in H^{\frac{1}{2}}(\partial\Omega)$, we obtain the discrete problem

$$\text{Find } u_h \in V_h \text{ s.t.} \qquad a_h^{SIP}(u_h, v_h) = l_h^D(g; v_h) \qquad \forall v_h \in V_h,$$

with

$$l_h^D(g; v_h) := \int_\Omega f v_h - \int_{\partial\Omega} g \nabla_h v_h \cdot n + \sum_{F \in \mathcal{F}_h^b} \frac{\eta}{h_F} \int_F g v_h.$$

Other boundary conditions

Note the last two terms in l_h^D weakly enforce $u = g$ at $\partial\Omega$ since with the terms at $F \in \mathcal{F}_h^b$ we obtain

$$
\begin{aligned}
&-\sum_{F\in\mathcal{F}_h^b}\int_F [\![v]\!]\{\!\{\nabla_h w_h\}\!\}\cdot n_F + \sum_{F\in\mathcal{F}_h^b}\frac{\eta}{h_F}\int_F [\![v]\!][\![w_h]\!] \\
&= -\sum_{F\in\mathcal{F}_h^b}\int_F v\nabla_h w_h\cdot n + \sum_{F\in\mathcal{F}_h^b}\frac{\eta}{h_F}\int_F v w_h \\
&= -\sum_{F\in\mathcal{F}_h^b}\int_F g\nabla_h w_h\cdot n + \sum_{F\in\mathcal{F}_h^b}\frac{\eta}{h_F}\int_F g w_h
\end{aligned}
$$

which gives

$$
-\sum_{F\in\mathcal{F}_h^b}\int_F (v-g)\nabla_h w_h\cdot n + \sum_{F\in\mathcal{F}_h^b}\frac{\eta}{h_F}\int_F (v-g) w_h = 0 \qquad \forall w_h \in V_h.
$$

Since $w_h \in V_h$ is arbitrary, this implies $u(x) = g(x)$ for a.e. $x \in \partial\Omega$.

Other boundary conditions

For the Robin boundary condition

$$\gamma u + \nabla u \cdot n = g \qquad \text{on } \partial\Omega \text{ with } g \in L^2(\partial\Omega), \gamma \in L^\infty(\partial\Omega), \gamma \geq 0,$$

the discrete problem is

$$\text{Find } u_h \in V_h \text{ s.t. } a_h^R(u_h, v_h) = l_h^R(g; v_h) \qquad \forall v_h \in V_h,$$

where $\forall (v, w_h) \in V_{*h} \times V_h$,

$$a_h^R(v, w_h) := \int_\Omega \nabla_h v \cdot \nabla_h w_h - \sum_{F \in \mathcal{F}_h^i} \int_F \Big(\{\!\{\nabla_h v\}\!\} \cdot n_F [\![w_h]\!] + [\![v]\!] \{\!\{\nabla_h w_h\}\!\} \cdot n_F \Big)$$

$$+ \sum_{F \in \mathcal{F}_h^i} \frac{\eta}{h_F} \int_F [\![v]\!][\![w_h]\!] + \sum_{F \in \mathcal{F}_h^b} \int_F \gamma v w_h,$$

$$l_h^R(g; w_h) := \int_\Omega f w_h + \int_{\partial\Omega} g w_h.$$

Other boundary conditions

For Neumann boundary conditions ($\gamma = 0$) the data must satisfy the compatibility condition

$$\int_\Omega f = -\int_{\partial\Omega} g.$$

Discrete coercivity

Define for all $v \in V_{*h} = V_* + V_h$ the norm

$$|||v|||_{SIP} := \left(\|\nabla_h v\|^2_{[L^2(\Omega)]^d} + |v|^2_J \right)^{\frac{1}{2}},$$

with jump seminorm

$$|v|_J := (\eta^{-1} s_h(v, v))^{\frac{1}{2}} = \left(\sum_{F \in \mathcal{F}_h} \frac{1}{h_F} \|[\![v]\!]\|^2_{L^2(F)} \right)^{\frac{1}{2}}.$$

The norm $|||v|||_{SIP}$ is a norm on V_{*h} and also on $H^1(\mathcal{T}_h)$ if $|||v|||_{SIP} = 0$ implies $v = 0$.

Namely, if $|||v|||_{SIP} = 0$ then $\|\nabla_h v\|_{[L^2(\Omega)]^d} = 0$ and $|v|_J = 0$, which implies, respectively, that v is piecewise constant and the interface and boundary jumps are zero, hence $v = 0$ in Ω.

Discrete coercivity

- Lemma. (Bound on consistency term). For all $(v, w_h) \in V_{*h} \times V_h$,

$$\Big|\sum_{F\in\mathcal{F}_h}\int_F \{\!\{\nabla_h v\}\!\}\cdot n_F[\![w_h]\!]\Big| \leq \Big(\sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T} h_F\|\nabla v|_T\cdot n_F\|^2_{L^2(F)}\Big)^{\frac{1}{2}}|w_h|_J.$$

Proof. For all $F \in \mathcal{F}_h^i$ with $F \in \partial T_1 \cap \partial T_2$, and $a_i = (\nabla v)|_{T_i}\cdot n_F$, $i \in \{1,2\}$ the Cauchy-Schwarz inequality yields

$$\begin{aligned}\int_F \{\!\{\nabla_h v\}\!\}\cdot n_F[\![w_h]\!] &= \int_F \frac{1}{2}(a_1+a_2)[\![w_h]\!] \\ &\leq \Big(\frac{1}{2}h_F(\|a_1\|^2_{L^2(F)}+\|a_2\|^2_{L^2(F)})\Big)^{\frac{1}{2}} h_F^{-\frac{1}{2}}\|[\![w_h]\!]\|_{L^2(F)} \\ &= \Big(\frac{1}{2}h_F(\|\nabla v|_{T_1}\cdot n_{T_1}\|^2_{L^2(F)}+\|\nabla v|_{T_2}\cdot n_{T_2}\|^2_{L^2(F)})\Big)^{\frac{1}{2}} h_F^{-\frac{1}{2}}\|[\![w_h]\!]\|_{L^2(F)}.\end{aligned}$$

Discrete coercivity

Moreover, $\forall F \in \mathcal{F}^b$ with $F = \partial T \cap \Omega$,

$$\int_F \{\!\{\nabla_h v\}\!\} \cdot n_F [\![w_h]\!] \leq h_F^{\frac{1}{2}} \|\nabla v|_T \cdot n_F\|_{L^2(F)} h_F^{-\frac{1}{2}} \|[\![w_h]\!]\|_{L^2(F)}.$$

Summing over mesh faces, using the Cauchy-Schwarz inequality gives the result.

Discrete coercivity

- Lemma. (Discrete coercivity) For all $\eta > \overline{\eta} := C_{tr}^2 N_\partial$, the SIP bilinear form a_h^{SIP} is coercive on V_h w.r.p. to the SIP-norm $||| \cdot |||_{SIP}$,

$$\forall v_h \in V_h, \qquad a_h^{SIP}(v_h, v_h) \geq C_\eta \ ||| \ v_h |||_{SIP}^2,$$

with $C_\eta := (\eta - C_{tr}^2 N_\partial)(1+\eta)^{-1}$, C_{tr} the coefficient in the discrete trace inequality, and N_∂ the maximum number of faces connected to an element $T \in \mathcal{T}_h$.

Proof.

Let $v_h \in V_h$. Since $\forall T \in \mathcal{T}_h$ and all $F \in \mathcal{F}_T$, $h_F \leq h_T$ we obtain using the discrete trace inequality

$$\sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} h_F \|\nabla v_h|_T \cdot n_F\|_{L^2(F)}^2 \leq \sum_{T \in \mathcal{T}_h} h_T \|\nabla v_h|_T \cdot n_F\|_{L^2(\partial T)}^2$$
$$\leq C_{tr}^2 N_\partial \|\nabla_h v_h\|_{[L^2(\Omega)]^d}^2.$$

Discrete coercivity

Hence,

$$\Big|\sum_{F\in\mathcal{F}_h}\int_F \{\!\{\nabla_h v\}\!\}\cdot n_F [\![w_h]\!]\Big| \le \Big(\sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T} h_F \|\nabla v|_T\cdot n_F\|^2_{L^2(F)}\Big)^{\frac{1}{2}} |w_h|_J$$

$$\le C_{tr}^2 N_\partial \|\nabla_h v_h\|^2_{[L^2(\Omega)]^d} |w_h|_J.$$

Thus

$$a_h^{SIP}(v_h, v_h) \ge \|\nabla_h v_h\|^2_{[L^2(\Omega)]^d} - 2C_{tr} N_\partial^{\frac{1}{2}} \|\nabla_h v_h\|_{[L^2(\Omega)]^d} |v_h|_J + \eta |v_h|_J^2.$$

Use the inequality: Let β be a positive real number, and let $\eta > \beta^2$ then for all $x, y \in \mathbb{R}$

$$x^2 - 2\beta xy + \eta y^2 \ge \frac{\eta - \beta^2}{1+\eta}(x^2 + y^2).$$

Note, this inequality can be obtained by expanding

$$(x - \beta y)^2 + (\beta x - \eta y)^2 \ge 0.$$

Discrete coercivity

Set $\beta = C_{tr} N_\partial^{\frac{1}{2}}$, $x = \|\nabla_h v_h\|_{[L^2(\Omega)]^d}$, and $y = |v_h|_J$, then we obtain

$$a_h^{SIP}(v_h, v_h) \geq C_h \, ||| \, v_h |||_{SIP}^2$$

Remark. (Poincaré inequality using the SIP-norm). There exists a σ_2, independent of h, s.t.

$$\forall v_h \in V_h, \qquad \|v_h\|_{L^2(\Omega)} \leq \sigma_2 \, ||| \, v_h \, |||_{SIP} \, .$$

Boundedness

- Define on V_h the norm

$$|||v|||_{SIP,*} := \left(|||\, v \,|||_{SIP}^2 + \sum_{T \in \mathcal{T}_h} h_T \|\nabla v|_T \cdot n_T\|_{L^2(\partial T)}^2 \right)^{\frac{1}{2}}.$$

- Lemma. (Boundedness) There is a C_{bnd}, independent of h, s.t.

$$\forall (v, w_h) \in V_{*h} \times V_h, \qquad a_h^{SIP}(v, w_h) \leq C_{bnd} \,|||\, v \,|||_{SIP,*} \,|||w_h\,|||_{SIP} .$$

Boundedness

Proof. Let $(v, w_h) \in V_{*h} \times V_h$. Observe that

$$
\begin{aligned}
a_h^{SIP}(v, w_h) &= \int_\Omega \nabla_h v \cdot \nabla_h w_h - \sum_{F \in \mathcal{F}_h} \int_F \{\!\{\nabla_h v\}\!\} \cdot n_F [\![w_h]\!] \\
&- \sum_{F \in \mathcal{F}_h} \int_F [\![v]\!] \cdot \{\!\{\nabla_h w_h\}\!\} \cdot n_T + \sum_{F \in \mathcal{F}_h} \frac{\eta}{h_F} \int_F [\![v]\!][\![w_h]\!] \\
&= \mathscr{T}_1 + \mathscr{T}_2 + \mathscr{T}_3 + \mathscr{T}_4.
\end{aligned}
$$

Using the Cauchy-Schwarz inequality yields the estimates

$$
\begin{aligned}
|\mathscr{T}_1| &\le \|\nabla_h v\|_{[L^2(\Omega)]^d} \|\nabla_h w_h\|_{[L^2(\Omega)]^d}, \\
|\mathscr{T}_4| &\le \eta |v|_J |w_h|_J.
\end{aligned}
$$

Using the bound on the consistency term and $h_F \leq h_T$, we obtain

$$
\begin{aligned}
|\mathscr{T}_2| &= \Big| \sum_{F \in \mathcal{F}_h} \int_F \{\!\{\nabla_h v\}\!\} \cdot n_F [\![w_h]\!] \Big| \\
&\leq \Big(\sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} h_F \|\nabla v|_T \cdot n_F\|^2_{L^2(F)} \Big)^{\frac{1}{2}} |w_h|_J \quad \text{(bound on consistency term)} \\
&\leq \Big(\sum_{T \in \mathcal{T}_h} h_T \|\nabla v|_T \cdot n_T\|^2_{L^2(\partial T)} \Big)^{\frac{1}{2}} |w_h|_J \qquad \text{(since } h_F \leq h_T\text{)} \\
&\leq |||v|||_{SIP,*} \, |||w_h|||_{SIP} \, .
\end{aligned}
$$

Boundedness

Finally,

$$
\begin{aligned}
|\mathscr{T}_3| &= \Big| \sum_{F \in \mathcal{F}_h} \int_F [\![v]\!] \{\!\{\nabla_h w_h\}\!\} \cdot n_F \Big| \\
&\leq |v|_J \Big(\sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} h_F \|\nabla w_h|_T \cdot n_F\|^2_{L^2(F)} \Big)^{\frac{1}{2}} && \text{(similar to bound on consistency term)} \\
&\leq C_{tr} N_\partial^{\frac{1}{2}} |v|_J \|\nabla_h w_h\|_{[L^2(\Omega)]^d} && \text{(use discrete trace theorem)} \\
&\leq C_{tr} N_\partial^{\frac{1}{2}} \, |||\, v \,|||_{SIP,*} \, |||w_h \,|||_{SIP} \, .
\end{aligned}
$$

Collecting all terms gives the bound with $C_{bnd} = 2 + \eta + C_{tr} N_\partial^{\frac{1}{2}}$.

$|||\cdot|||_{SIP}$-norm error estimate

Using the consistency, boundedness and coercivity of a_h^{SIP} we immediately obtain the following error estimate.

- Theorem 1. ($|||\cdot|||_{SIP}$-norm estimate). Let $u \in V_*$ solve

$$\text{Find } u \in V \text{ s.t. } a(u,v) = \int_\Omega fv \quad \forall v \in V.$$

Let $u_h \in V_h$ solve

$$\text{Find } u_h \in V_h \text{ s.t. } a_h^{SIP}(u_h, v_h) = \int_\Omega f v_h \quad \forall v_h \in V_h.$$

Then there is a C, independent of h, s.t.

$$|||u - u_h|||_{SIP} \leq C \inf_{v_h \in V_h} |||u - v_h|||_{SIP,*} \,.$$

$|||\cdot|||_{SIP}$-norm error estimate

- Corollary. (Convergence rate in $|||\cdot|||_{SIP}$-norm). In addition to the assumptions for Theorem 1, assume $u \in H^{k+1}(\Omega)$. Then there holds

$$|||u - u_h|||_{SIP} \leq Ch^k \|u\|_{H^{k+1}(\Omega)}.$$

  This error estimate is optimal, both for the broken gradient and the jump seminorm.

- Lemma. (Uniform equivalence of $|||\cdot|||_{SIP}$ and $|||\cdot|||_{SIP,*}$ norms on V_h).

  The $|||\cdot|||_{SIP}$- and $|||\cdot|||_{SIP,*}$-norms are uniformly equivalent on V_h,

$$C_{SIP}\, |||\, v_h|||_{SIP,*} \leq |||v_h|||_{SIP} \leq |||v_h\, |||_{SIP,*} \qquad \forall v_h \in V_h,$$

  with C_{SIP} independent of h.

  Proof. The upper bound is immediate. The lower bounds follows from the trace inequality and the uniform bound on N_∂,

$$|||v_h|||^2_{SIP,*} = |||v_h\, |||^2_{SIP} + \sum_{T\in\mathcal{T}_h} h_T \|\nabla v|_T \cdot n_T\|^2_{L^2(\partial T)}$$

$$= |||v_h\, |||^2_{SIP} + \sum_{T\in\mathcal{T}_h} C^2_{tr} N_\partial \|\nabla v\|^2_{[L^2(T)]^2} \qquad \text{(use discrete trace inequality)}$$

$$\leq C\, |||\, v_h\, |||^2_{SIP}\, .$$

Error analysis using the $|||\cdot|||_{SIP,*}$-norm

A consequence of the norm equivalence between $|||v_h|||_{SIP}$ and $|||v_h|||_{SIP,*}$ is discrete coercivity on V_h in the form

$$\forall v_h \in V_h, \qquad a_h^{SIP}(v_h, v_h) \geq C'_\eta \, |||\, v_h|||^2_{SIP,*},$$

with C'_η independent of h, and the boundedness of a_h^{SIP} on $V_{*h} \times V_h$

$$\forall (v, w_h) \in V_{*h} \times V_h, \qquad a_h^{SIP}(v, w_h) \leq C'_{bnd} \, |||\, v\, |||_{SIP,*} \, |||w_h\, |||_{SIP,*} \, .$$

Using the Lax-Milgram theorem we obtain then immediately convergence and error estimates in the $|||v_h|||_{SIP,*}$-norm.

Error analysis using the $|||\cdot|||_{SIP,*}$-norm

- Theorem. ($|||\cdot|||_{SIP,*}$-norm error estimate). Under the assumptions of Theorem 1, there is a C, independent of h, s.t.

$$|||u-u_h|||_{SIP,*} \leq C \inf_{v_h \in V_h} |||u-v_h|||_{SIP,*} \,.$$

- Corollary. (Convergence rate in $|||\cdot|||_{SIP,*}$-norm). Under the assumptions of Theorem 1, assume that $u \in H^{k+1}(\Omega)$. Then there holds

$$|||u-u_h|||_{SIP,*} \leq Ch^k \|u\|_{H^{k+1}(\Omega)},$$

with C independent of h.

Note, the convergence rate in the $|||v_h|||_{SIP}$ and $|||v_h|||_{SIP,*}$ norm is the same.

The $|||v_h|||_{SIP,*}$-norm estimate, however, also provides bounds on the convergence of the normal gradient at mesh element boundaries.

L^2-norm error estimate

- Using the broken Poincaré inequality,

$$\forall v \in H^1(\mathcal{T}_h), \qquad \|v\|_{L^2(\Omega)} \leq \sigma_2' \, ||| \, v |||_{SIP},$$

with σ_2' independent of h, see Brenner (2003), we obtain the $L^2(\Omega)$-error estimate

$$\|u - u_h\|_{L^2(\Omega)} \leq \sigma_2' C h^k \|u\|_{H^{k+1}(\Omega)}.$$

This estimate is, however, suboptimal by one power of h.

L^2-norm error estimate

To obtain a better L^2-error estimate we use a duality argument, also called Aubin-Nitsche argument.

- Definition. (Elliptic regularity). Elliptic regularity holds for the weak formulation of the Poisson equation if there is a C_{ell}, only depending on Ω, s.t. for all $\psi \in L^2(\Omega)$, the solution of

$$\text{Find } \zeta \in H_0^1(\Omega) \text{ s.t. } a(\zeta, v) = \int_\Omega \psi v \qquad \forall v \in H_0^1(\Omega),$$

  is in V_* and satisfies

$$\|\zeta\|_{H^2(\Omega)} \leq C_{ell} \|\psi\|_{L^2(\Omega)}.$$

  Elliptic regularity can be asserted if the polygonal domain Ω is convex. See Grisvard (1992).

L^2-norm error estimate

- Theorem 2. ($L^2(\Omega)$ error estimate). Let $u \in V_*$ solve

$$\text{Find } u \in V \text{ s.t. } a(u,v) = \int_\Omega fv \quad \forall v \in V.$$

Let $u_h \in V_h$ solve

$$\text{Find } u_h \in V_h \text{ s.t. } a_h^{SIP}(u_h, v_h) = \int_\Omega f v_h \quad \forall v_h \in V_h.$$

Assume elliptic regularity. Then there is a C, independent of h, s.t.

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch \,|||\, u - u_h \,|||_{SIP,*} \,.$$

L^2-norm error estimate

Proof. Consider the auxiliary problem

$$\text{Find } \zeta \in H_0^1(\Omega) \text{ s.t. } a(\zeta, v) = \int_\Omega (u - u_h) v \qquad \forall v \in H_0^1(\Omega),$$

and use elliptic regularity to obtain

$$\|\zeta\|_{H^2(\Omega)} \leq C_{ell} \|u - u_h\|_{L^2(\Omega)}.$$

L^2-norm error estimate

Since $\zeta \in V_* \cap H^2(\Omega)$, we have

$$[\![\nabla\zeta]\!] \cdot n_F = 0, \quad \forall F \in \mathcal{F}_h^i, \quad \text{and} \quad [\![\zeta]\!] = 0, \quad \forall F \in \mathcal{F}_h.$$

Using the integrated by parts form of a_h^{SIP},

$$a_h^{SIP}(v, w_h) = -\sum_{T\in\mathcal{T}_h} \int_T (\Delta v) w_h + \sum_{F\in\mathcal{F}_h^i} \int_F [\![\nabla_h v]\!] \cdot n_F \{\!\{w_h\}\!\} - \sum_{F\in\mathcal{F}_h} \int_F [\![v]\!] \{\!\{\nabla_h w_h\}\!\} \cdot n_F$$
$$+ \sum_{F\in\mathcal{F}_h} \int_F \frac{\eta}{h_F} \int_F [\![v]\!][\![w_h]\!].$$

we obtain then

$$a_h^{SIP}(\zeta, u - u_h) = -\int_\Omega (\Delta\zeta)(u - u_h).$$

L^2-norm error estimate

Since $-\Delta\zeta = u - u_h$ and a_h^{SIP} is symmetric we have

$$a_h^{SIP}(u - u_h, \zeta) = a_h^{SIP}(\zeta, u - u_h)$$
$$= -\int_\Omega (\Delta\zeta)(u - u_h)$$
$$= \|u - u_h\|^2_{L^2(\Omega)}.$$

Furthermore, since consistency implies Galerkin orthogonality,

$$a_h^{SIP}(u - u_h, w_h) = 0 \quad \forall w_h \in V_h,$$

and with π_h^1, the L^2-orthogonal projection onto $\mathbb{P}_d^1(\mathcal{T}_h) \subset V_h$, which exists since $k \geq 1$, we have

$$a_h^{SIP}(u - u_h, \pi_h^1\zeta) = 0.$$

L^2-norm error estimate

Using the boundedness of a_h^{SIP} on $V_{*h} \times V_{*h}$, since

$$a_h^{SIP}(v, w) \lesssim |||v|||_{SIP,*} \, |||w|||_{SIP,*} \quad \forall v, w \in V_{*h},$$

the approximation property of π_h^1 in the SIP-norm, and the regularity of ζ, we obtain

$$\begin{aligned}
\|u - u_h\|_{L^2(\Omega)}^2 &= a_h^{SIP}(u - u_h, \zeta - \pi_h^1 \zeta) \\
&\lesssim |||u - u_h|||_{SIP,*} \, |||\zeta - \pi_h^1 \zeta|||_{SIP,*} && \text{(use boundedness)} \\
&\lesssim |||u - u_h|||_{SIP,*} \, h \|\zeta\|_{H^2(\mathcal{T}_h)} && \text{(approximation property of } \pi_h^1) \\
&\lesssim |||u - u_h|||_{SIP,*} \, h C_{ell} \|u - u_h\|_{L^2(\Omega)} && \text{(elliptic regularity)},
\end{aligned}$$

which gives

$$\|u - u_h\|_{L^2(\Omega)} \leq C h \|u - u_h\|_{SIP,*}.$$

L^2-norm error estimate

- Corollary. (Convergence rate in L^2-norm). In addition to the assumptions in Theorem 2, assume $u \in H^{k+1}(\Omega)$, then there holds

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch^{k+1}\|u\|_{H^{k+1}(\Omega)},$$

  with C independent of h.

  Note this error estimate is optimal in h and the symmetry of a_h^{SIP} was important to obtain this result.

- (Adjoint consistency). The property

$$a_h^{SIP}(u - u_h, \zeta) = \int_\Omega (-\Delta\zeta)(u - u_h),$$

  which results from symmetry and consistency is called adjoint consistency.