p-adic measures and Fourier Analysis on $\mathbb{Q}_p$: towards a p-adic analog of Dirichlet series.

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Outline

p-adic interpolation of Riemann's ζ

p-adic Dirichlet series

Barsotti covectors and bivectors

A p-adic entire function

Functions and measures on $\mathbb{Q}_p$

Dieudonné formula

Perfectoid fields and Barsotti bivectors

The values of Riemann's ζ at non-positive integers are known since Euler to be rational numbers :

$$\zeta(0) = -1/2 \ , \ \zeta(-1) = -1/12 \ , \ \zeta(-2) = 0 \, ,$$

$$\zeta(-3) = 1/120 \ , \ \zeta(-4) = 0 \ , \ldots$$

In general we have

$$\zeta(-n) = (-1)^n B_{n+1}/n+1 \ , \ \forall \ n \in \mathbb{Z}_{<0} \, ,$$

where B_n is the n-th Bernouilli number.

For any positive integer r prime to p, there is a measure $\mu_r \in \mathscr{D}(\mathbb{Z}_p, \mathbb{Z}_p)$ such that for $k_0 = 0, 1, \ldots, p-2$, the analytic function of $s \in \mathbb{Z}_p$

$$\zeta_{p,k_0}(s) = \frac{1}{\langle r \rangle^{1-s}\omega(r)^{k_0} - 1} \int_{\mathbb{Z}_p^\times} \langle x \rangle^{-s} \omega(x)^{k_0-1} \mu_r(x)$$

interpolates the values $(1-p^{k-1})\zeta(1-k)$ for $k \equiv k_0 \mod (p-1)$.

Since $\langle x \rangle \in 1 + p\mathbb{Z}_p$, the integral is of the form

$$\sum_{n=0}^{\infty} \binom{s}{n} p^n y_n \;\;, \;\; \text{with } y_n \in \mathbb{Z}_p \;,$$

hence is analytic in $D(0, p^{\frac{p-2}{p-1}}) \subset \mathbb{A}^1_{\mathbb{Q}_p}$ ("open" disc of radius $p^{\frac{p-2}{p-1}}$).

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We define the *strip of width* $\rho > 0$ *around* $\mathbb{Q}_p$ as

$$\Sigma_\rho(\mathbb{Q}_p) := \bigcup_{a \in \mathbb{Q}_p} D(a, \rho) .$$

We consider the (Hopf) $\mathbb{Q}_p$-algebra $\mathcal{E}$ of power series in $\mathbb{Q}_p[[x]]$ which represent p-adically entire functions on $\mathbb{C}_p$ that restrict to bounded uniformly continuous maps $\mathbb{Q}_p \to \mathbb{Q}_p$, and its closed (Hopf) $\mathbb{Z}_p$-subalgebra $\mathcal{E}^\circ$ of functions bounded by 1 on $\mathbb{Q}_p$. Then $\mathcal{E}$ is endowed with the Fréchet topology of supnorms $|| - ||_\rho$ on $D(0, \rho)$, for any $\rho > 0$, and of the supnorm $|| - ||_\infty$ on $\mathbb{Q}_p$. The algebra $\mathcal{E}$ is similar to the algebra of complex trigonometric functions!

We would like to extend the definition of $\zeta_{p,k_0}(s)$ in a "canonical way" to a locally analytic function on a strip of positive width around $\mathbb{Q}_p$. Since a strip is not connected (in the sense of Berkovich) it is not clear what we mean by "canonical way".

Let $S = \mathbb{Z}[1/p]_{\geq 0}$; for any $a, b \in \mathbb{R}$, the set of $q \in S$ such that $v_p(q) > a$ but $q < b$ is finite.

There is a family of functions $\{G_q\}_{q \in S}$ in $\mathcal{E}^\circ$ such that

1. $G_0 = 1$ and $||G_q||_\infty = 1$, $\forall q \in S$;
2. for any $\rho, \varepsilon, b > 0$, there exists a such that

   $$||G_q(x)||_\rho < \varepsilon$$

   if $v_p(q) < a$ or $q > b$. (In particular, for any $q < b$, except a finite number of such q's.)
3. $$G_q(x+y) = \sum_{q_1+q_2=q} G_{q_1}(x) G_{q_2}(y) \ , \ \ \forall q \in S$$

   (an unconditionally convergent sum for the supnorm on $D(0,\rho) \times D(0,\rho)$, any $\rho > 0$).

A *p-adic Dirichlet series* is a sum of the form

$$L(s) = \sum_{q \in S} a_q G_q(s)$$

which converges unconditionally in a strip of positive width around $\mathbb{Q}_p$.

Our task is now to show that this definition is reasonable.

Let

$$F(T) = \exp(\sum_{i=0}^{\infty} T^{p^i}/p^i) \in 1 + T\mathbb{Z}_{(p)}[[T]]$$

(similar to $\exp(T)$) be the (p-adic) *Artin-Hasse exponential* series and let

$$E(T) \in T + T^2\mathbb{Z}_{(p)}[[T]]$$

(similar to $\log(1+T)$) be its inverse in the sense that

$$F(E(T)) = 1 + T \text{ and } E(F(T) - 1) = T \ .$$

Then E is called the (p-adic) *Artin-Hasse logarithm*.

Lemma

Let K be any perfectoid field (either p-adic or of characteristic p). The power series F and E establish inverse homeomorphisms

$$K^{\circ\circ} \xrightarrow{\sim} 1 + K^{\circ\circ} \ , \ \ \lambda \longmapsto F(\lambda) \ ,$$

$$1 + K^{\circ\circ} \xrightarrow{\sim} K^{\circ\circ} \ , \ \ c \longmapsto E(c-1) \ .$$

Here is my main result of (canonical, locally) analytic continuation of a multiplicative family of additive characters.

Theorem

For any perfectoid extension $C/\mathbb{Q}_p$ and any $c \in 1 + C^{\flat\circ\circ}$, let $\lambda \in C^{\flat\circ\circ}$ be such that $c = F(\lambda)$ (i.e. $\lambda = E(c-1)$). Then the character $s \longmapsto (c^\sharp)^s$ may be continued as

$$(c^s)^\sharp = \sum_{q \in S} G_q(s)(\lambda^q)^\sharp \ ,$$

an analytic function of s on a strip of positive width around $\mathbb{Q}_p$ in the analytic C-line.

Notice that, for a given $x = c^\sharp \in C^\circ$, the named extension depends on the choice of $c \in C^{\flat\circ}$.

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We consider the direct limit of the polynomial addition laws of Witt vectors of length n, via the Verschiebung map $W_n \to W_{n+1}$

$$V : (x_{-n}, \dots, x_{-1}, x_0) \to (0, x_{-n}, \dots, x_{-1}, x_0) \ .$$

This direct limit is the $\mathbb{Z}_p$-formal group CW of *Witt covectors*. (The projective limit $W_{n+1} \to W_n$,

$$(x_0, x_1, \dots, x_{n+1}) \mapsto (x_0, x_1, \dots, x_n) \ ,$$

produces instead the algebraic group W of Witt vectors.)

We consider in fact Witt bivectors $x = (\dots, x_{-1}, x_0; x_1, \dots)$, infinite in both directions.

To make sense of the previous definitions we define a graduation and a linear topology in the polynomial ring

$$\mathscr{P} := \mathbb{Z}_{(p)}[..., x_{-m}, \ldots, x_{-1}, x_0]$$

(resp.

$$\mathscr{P} := \mathbb{Z}_{(p)}[..., x_{-m}, \ldots, x_{-1}; x_0, x_1, \ldots]\ )\ ,$$

as follows. We attribute to each x_i the *weight* p^i so that $\mathscr{P}$ becomes the S-graded ring

$$\mathscr{P} = \bigoplus_{s \in S} \mathscr{P}_s = \{\sum_{s \in S} a_s(\underline{x}) T^s\}$$

where $a_s(\underline{x}) \in \mathscr{P}$ is a polynomial of pure weight s.

For $s \in S_{>0}$ and $m \in \mathbb{Z}$ we consider the ideal $\mathcal{I}_{s,m}$ of $\mathscr{P}$ generated by the monomials in the x_i's of pure weight s and divisible by some x_i with $i \leq m$.

We let $\widehat{\mathscr{P}}$ be the completion of $\mathscr{P}$ in the topology with a basis of open ideals $\{\mathcal{I}_{s,m}\}_{s,m}$. Then $\widehat{\mathscr{P}}$ is a Hopf algebra in a suitable category of linearly topologized $\mathbb{Z}_p$-modules which represents the additive formal group CW (resp. biv). Addition in CW is described by a power series

$$\Phi(x_0, x_{-1}, \dots; y_0, y_{-1}, \dots) \in \widehat{\mathscr{P}} \widehat{\otimes}^{\mathrm{u}}_{\mathbb{Z}_p} \widehat{\mathscr{P}} .$$

The same power series, up to a shift of coefficients defines the addition in biv.

The addition law of Witt covectors $x = (\ldots, x_{-1}, x_0)$ is trivialized in terms of the *ghost components of covectors*

$$x^{(i)} = \sum_{n \leq i} p^{n-i} x_n^{p^{i-n}} = x_i + p^{-1} x_{i-1}^p + \ldots \ ,$$

for $i = 0, -1, \ldots$ (limits of the usual ghost components of Witt vectors divided by p^i) namely

$$x + y = z \Leftrightarrow (x^{(i)} + y^{(i)} = z^{(i)} \quad \text{for any} \quad i = 0, 1, \ldots) \ .$$

Notice that this trivialization takes place only after restriction of CW to $\mathbb{Q}_p$-algebras. The same formulas, up to a shift in the indices hold in biv.

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The function Ψ_p

A prime p is fixed all over. We consider the formal solution

$$\Psi(T) = \Psi_p(T) = T + \sum_{i=2}^{\infty} a_i T^i \in \mathbb{Z}[[T]] \ ,$$

of the functional equation

$$(*) \qquad \sum_{j=0}^{\infty} p^{-j} \Psi(p^j T)^{p^j} = T \ .$$

This function was discovered in my thesis (Padova 1974 - Ann. Sc. Norm. Sup. 1975). The following facts were proven there.

1. $\Psi_p \in \mathcal{E}^\circ$;
2. for any $i \in \mathbb{Z}$ and $x \in \mathbb{Q}_p$, if we define

$$x_{-i} := \Psi_p(p^i x) \mod p \in \mathbb{F}_p$$

then

$$x = \sum_{i>>-\infty}^{\infty} [x_i]\, p^i \in \mathbb{Q}\mathrm{W}(\mathbb{F}_p) = \mathbb{Q}_p \ ,$$

where $[t]$, for $t \in \mathbb{F}_p$, is the Teichmüller representative of t in $\mathrm{W}(\mathbb{F}_p) = \mathbb{Z}_p$.
3. Ψ_p trivializes the addition law of Witt covectors with entries in the Fréchet algebra $\mathbb{Q}_p\{x, y\}$ of entire functions of x and y, in the sense that

$$\Psi(x+y) = \Phi(\Psi(x), \Psi(px), \dots; \Psi(y), \Psi(py), \dots) \ ,$$

where

$$\Phi(x_0, x_{-1}, \dots; y_0, y_{-1}, \dots) \in \mathbb{Z}_{(p)}[[x_0, x_{-1}, \dots; y_0, y_{-1}, \dots]]$$

is the power series giving the addition of covectors. In our case, for $x_0 = \Psi(x)$, $x_{-1} = \Psi(px)$, $x_{-2} = \Psi(p^2 x)$, ..., we get

$$x^{(i)} = p^{-i} x \ ,$$

so the statement is clear.

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Notice that there exists a measure given by

$$E(T) = E(\Delta_1 - \Delta_0) \in \mathbb{Z}_p[[T]] = \mathscr{D}(\mathbb{Z}_p, \mathbb{Z}_p) \ ,$$

a converging sum in $\mathscr{D}(\mathbb{Z}_p, \mathbb{Z}_p)$.

We now study the ring $\mathscr{D}_{\mathrm{unif}}\,(\mathbb{Q}_p, \mathbb{Z}_p)$ of measures defined on *uniformly measurable sets* that is subsets of $\mathbb{Q}_p$ which are unions of closed discs of the same radius. It is the $(p, T = \Delta_1 - \Delta_0)$-adic completion of $\mathbb{Z}_p[\mathbb{Q}_p] = \mathbb{Z}_p[\Delta_a\,,\ a \in \mathbb{Q}_p]$. We define the *canonical measure* on $\mathbb{Q}_p$ as

$$\mu_{\mathrm{can}} := \lim_{n\to\infty} E(\Delta_{p^{-n}} - \Delta_0)^{p^n}$$

a convergent expression in $\mathscr{D}_{\mathrm{unif}}\,(\mathbb{Q}_p, \mathbb{Z}_p)$.

So

$$\lim_{n\to+\infty} F(\mu_{\mathrm{can}}^{p^{i-n}})^{p^n} = \Delta_{p^i} \ , \ \forall\, i$$

and, for any $x \in \mathbb{Q}_p$,

$$\lim_{n\to\infty} \exp(x \sum_{i=0}^{\infty} \mu_{\mathrm{can}}^{p^{i-n}} / p^{i-n}) = \lim_{n\to+\infty} F(\mu_{\mathrm{can}}^{p^{-n}})^{p^n x} = \Delta_x \ .$$

For any $q \in S$, the measure μ_{can}^q (= q^{th}-power of μ_{can}) is well-defined in $\mathscr{D}_{\mathrm{unif}}(\mathbb{Q}_p, \mathbb{Z}_p)$ and our formula of analytic continuation says that Δ_x, for $x \in \mathbb{Q}_p$, can also be calculated as

$$\Delta_x = \sum_{q \in S} G_q(x) \mu_{\mathrm{can}}^q \ .$$

This is the analog over $\mathbb{Q}_p$ of the formula

$$\Delta_x = (1+T)^x = \sum_{n=0}^{\infty} \binom{x}{n} T^n \ , \ \text{for } x \in \mathbb{Z}_p \ .$$

Any bounded uniformly continuous function $f : \mathbb{Q}_p \to \mathbb{Q}_p$ admits the generalized Amice-Fourier expansion

$$f = \sum_{q \in S} \Big(\int_{\mathbb{Q}_p} f \, \mu_{\mathrm{can}}^q \Big) \, G_q$$

which converges for the supnorm on $\mathbb{Q}_p$.

The tautological formula for $\mu \in \mathscr{D}(\mathbb{Z}_p, \mathbb{Z}_p)$

$$\mu = \sum_{n=0}^{\infty} \left(\int_{\mathbb{Z}_p} \binom{x}{n} d\mu(x) \right) T^n \ ,$$

has the analog for $\mu \in \mathscr{D}_{\mathrm{unif}}\ (\mathbb{Q}_p, \mathbb{Z}_p)$

$$\mu = \sum_{q \in S} \left(\int_{\mathbb{Q}_p} G_q(x) d\mu(x) \right) \mu_{\mathrm{can}}^q \ .$$

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We recall the affine group functor

$$\mathrm{U} : R \longmapsto (1 + TR[[T]], \cdot)$$

whose addition law is

$$(a_1, a_2, \dots) \oplus (b_1, b_2, \dots) = (c_1, c_2, \dots)$$

if

$$(1 + \sum_{i=1}^{\infty} a_i T^i)(1 + \sum_{i=1}^{\infty} b_i T^i) = 1 + \sum_{i=1}^{\infty} c_i T^i \ .$$

Then Dieudonné formula is the T-adic formula in $\mathbb{Z}_{(p)}[x_0, x_1, \dots][[T]]$

$$\prod_{i=0}^{\infty} F(x_i T^{p^i}) = \exp \sum_{i=0}^{\infty} x^{(i)} T^{p^i} = 1 + \sum_{i=1}^{\infty} g_i(x_0, x_1, \dots, x_{\ell(i)}) T^i ,$$

where $x^{(i)}$ is the ghost component of the Witt vector $(x_0, x_1, \dots)$, divided by p^i, and, $\forall z > 0$,

$$\ell(z) = \lfloor \log_p z \rfloor \quad \text{(lower integral part).}$$

Here

$$g_i(x_0, x_1, \dots, x_{\ell(i)}) \in \mathbb{Z}_{(p)}[x_0, x_1, \dots, x_{\ell(i)}]$$

Dieudonné formula gives a closed immersion

$$\mathrm{W} \hookrightarrow \mathrm{U} \ , \ \ (x_0, x_1, \dots) \longmapsto (g_1, g_2, \dots) \ .$$

We extend the formula to bivectors

$$x = (\dots, x_{-1}; x_0, x_1, \dots)\ .$$

In order to better understand the extended Dieudonné formula, we have at least two choices :

1. to interpret the variable "T" as a measure on $\mathbb{Q}_p$ in a way which differs from the common use in Iwasawa theory and to consider the Fréchet $\mathbb{Q}_p$-algebra $\mathcal{E}^\circ \widehat{\otimes}_{\mathbb{Z}_p} \mathscr{D}(\mathbb{Q}_p, \mathbb{Z}_p)$;
2. to consider entire functions with values in a perfectoid field C of characteristic 0 and to regard T as $\alpha^\sharp$ for some $\alpha \in 1 + C^{\flat\circ\circ}$.

In both cases the generalized Dieudonné formula specializes to give interesting results.

We get in $\widehat{\mathscr{P}}$ the *extended Dieudonné formula*

$$\prod_{i=-\infty}^{\infty} F(x_i T^{p^i}) = \sum_{q \in S} g_q(\ldots, x_{\ell(q)-1}, x_{\ell(q)})\, T^q \ ,$$

where

$$g_q(\ldots, x_{\ell(q)-1}, x_{\ell(q)}) := \lim_{n \to +\infty} g_{p^n q}(x_{-n}, \ldots, x_{\ell(q)-1}, x_{\ell(q)}) \ .$$

The formula should include

$$= \exp \sum_{i=-\infty}^{\infty} x^{(i)} T^{p^i} \ ,$$

but this does not converge in $\widehat{\mathscr{P}}$.

Our understanding of the Witt polynomials permits us to replace in the previous formula the Witt bivector

$$x = (\dots, x_{-1}; x_0, x_1, \dots)$$

by

$$(\dots, \Psi(px); \Psi(x), \Psi(p^{-1}x), \dots)$$

whose ghost components are $x^{(i)} = p^{-i}x$, for any $i \in \mathbb{Z}$.

We then define

$$G_q(x) := g_q(\dots, \Psi(p^{1-\ell(q)}x), \Psi(p^{-\ell(q)}x))$$

for any $q \in S$.

Theorem

The specialization $x_i \mapsto \Psi_p(p^{-i}x)$, i.e. $x^{(i)} \mapsto p^{-i}x$, transforms the functions $g_q(\dots, x_{\ell(q)-1}, x_{\ell(q)})$, for any $q \in S'$, into functions $G_q(x) \in \mathcal{E}^\circ$ with the properties listed above.

We also specialize T to $\mu_{\mathrm{can}} \in \mathscr{D}_{\mathrm{unif}}(\mathbb{Q}_p, \mathbb{Z}_p)$. So in the ring $\mathcal{E}^\circ \widehat{\otimes}^{\mathrm{u}}_{\mathbb{Z}_p} \mathscr{D}_{\mathrm{unif}}(\mathbb{Q}_p, \mathbb{Z}_p)$ we get

$$\prod_{i=-\infty}^{\infty} F(\Psi(p^{-i}x)\mu_{\mathrm{can}}^{p^i}) = \sum_{q \in S} G_q(x)\mu_{\mathrm{can}}^{q} \; .$$

For any $x \in \mathbb{Q}_p$ the previous equality continues into

$$\lim_{n \to \infty} \exp(x \sum_{i=0}^{\infty} \mu_{\mathrm{can}}^{p^{i-n}} / p^{i-n}) = \lim_{n \to +\infty} F(\mu_{\mathrm{can}}^{p^{-n}})^{p^n x} = \Delta_x \; .$$

So, we have shown that the additive character

$$\mathbb{Q}_p \longrightarrow \mathscr{D}_{\mathrm{unif}}\left(\mathbb{Q}_p, \mathbb{Z}_p\right) \quad , \quad x \longmapsto \Delta_x$$

belongs to $\mathcal{E}^\circ \widehat{\otimes}_{\mathbb{Z}_p} \mathscr{D}_{\mathrm{unif}}\left(\mathbb{Q}_p, \mathbb{Z}_p\right)$.
In order to better understand the extended Dieudonné formula, we have at least two choices :

1. to interpret the variable "T" as the measure μ_{can} on $\mathbb{Q}_p$ and to view the formula as giving an analytic continuation of the character $x \longmapsto \Delta_x$;
2. to consider entire functions with values in a perfectoid field C of characteristic 0, to interpret T^q as $(\lambda^q)^\sharp$ for some $\lambda \in C^{\flat\circ}$ and to interpret the formula as giving an analytic continuation of $s \longmapsto (F(\lambda)^\sharp)^s$ to s in a strip around $\mathbb{Q}_p$.

In both cases the generalized Dieudonné formula specializes to give interesting results.

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For a perfect complete valued ring (R, v_R) of characteristic p, Candilera and Cristante modified Barsotti's definitions as follows :

1. the $\mathbb{Q}_p$-algebra

$$\mathrm{W}(R)[1/p] = \{(\ldots, 0, x_{-n}, \ldots, x_{-1}; x_0, x_1 \ldots) \mid x_i \in R\}$$

   of *special bivectors* ;

2. the $\mathbb{Q}_p$-vector space

$$\mathrm{biv} R = \{x = (\ldots, x_{-n}, \ldots, x_{-1}; x_0, x_1 \ldots) \mid x_i \in R \text{ such that } \liminf_{i \to -\infty} v_R(x_i) > 0\}$$

   of *bivectors* ;

3. the $\mathbb{Q}_p$-algebra $\mathrm{Biv} R$ of *Bivectors*.

For $x \in \mathrm{biv} R$ we have

$$x = (\dots, x_{-2}, x_{-1}; x_0, x_1 \dots) = \sum_{i \in \mathbb{Z}} [x_i^{p^{-i}}] p^i \ .$$

Then $\mathrm{Biv} R$ is the completion of $\mathbb{Q}\mathrm{W}(R)$ (and of $\mathrm{biv} R$) in the topology of the valuation

$$w(x) = \min\{p^{-i} v_R(x_i) + i\} \ .$$

We have

$$\mathrm{W}(R)[1/p] \subset \mathrm{biv} R \subset \mathrm{Biv} R$$

We denote by φ the Frobenius and by V the Verschiebung.

We now pick any perfectoid field (C, v_C) of characteristic 0 and $R = C^{\flat\circ}$ equipped with the valuation $v_C^\flat$ such that

$$v_C^\flat(y) = v_C(y^\sharp) \ , \ \forall\, y \in C^\flat \ .$$

Candilera and Cristante (Annali SNS Pisa (1995)) prove that there is a continuous surjection

$$\Theta_C : \mathrm{Biv} C^{\flat\circ} \longrightarrow C \ , \ \ x \longmapsto \sum_{i \in \mathbb{Z}} (x_i^\sharp)^{p^{-i}} p^i \ .$$

They also extend a theorem of Barsotti (Annali SNS Pisa (1964)) as follows :

Theorem

Let R be a perfect ring of characteristic p equipped with a rank 1 valuation v. If $y \in \mathrm{biv} R$ is of the form $y = (\ldots, y_0; y_0, y_0, \ldots)$, with $v(y_0) > 0$, then

$$\exp(y) = [F(y_0)] \ .$$

Conversely if $x = [x_0]$, with $v(x_0 - 1) > 0$, then

$$\log(1 - x) = (\ldots, E(x_0); E(x_0), E(x_0), \ldots) \ .$$

Notice that

$$y = (\dots, y_0; y_0, y_0, \dots) = \sum_{i \in \mathbb{Z}} [y_0^{p^{-i}}] p^i \ .$$

Now we pick any $c \in 1 + C^{\flat\circ\circ}$ so that $\log[c]$ converges in $\mathrm{Biv} C^{\flat\circ}$ and the theorem applies. We have

$$[c] = \exp(\sum_{i \in \mathbb{Z}} [\lambda^{p^{-i}}] p^i) \ ,$$

where $\lambda = E(c-1)$. We apply Θ_C and get an equality in C

$$c^\sharp = \exp \sum_{i \in \mathbb{Z}} (\lambda^{p^{-i}})^\sharp p^i \ .$$

So, if $s \in \mathbb{Q}_p$, we have

$$(c^s)^\sharp = \sum_{q \in S} G_q(s)(\lambda^q)^\sharp \ ,$$

which we may think of as a "branch of $s \longmapsto (c^\sharp)^s$". But the r.h.s. makes sense in a full strip around $\mathbb{Q}_p$! This is our formula of analytic continuation for any branch of the additive character

$$\mathbb{Q}_p \longrightarrow 1 + C^{\flat\circ\circ} \quad , \quad s \longmapsto (c^\sharp)^s \ .$$

For example, if $C \supset \mathbb{Q}_p(p^{1/p^\infty}, \zeta_{p^\infty})$ and

$$\lambda = (0, p^{1/p} \bmod pC^\circ, p^{1/p^2} \bmod pC^\circ, ...) =$$

$$(p, \zeta_p p^{1/p}, \zeta_{p^2} p^{1/p^2}, ...) \in C^{\flat\circ\circ} \ ,$$

we have $\lambda^\sharp = p \in C^{\circ\circ}$ and

$$\exp((\ldots,\lambda,\lambda;\lambda,\ldots)) = \exp\sum_{i\in\mathbb{Z}}[\lambda^{p^{-i}}]p^i =$$

$$[F(\lambda)] = \lim_{n\to\infty} F(\zeta_{p^n} p^{1/p^n})^{p^n} =$$

$$\lim_{n\to\infty} \exp(\sum_{i=0}^{\infty} p^{-i}\zeta_{p^{n-i}} p^{p^{i-n}})^{p^n} \ .$$

The sum

$$\pi(\lambda) = \sum_{i\in\mathbb{Z}} \zeta_{p^i} p^{i+(1/p^i)}$$

is convergent in $C^{\circ\circ}$. The character $s \longmapsto \exp\pi(\lambda)s$ extends as $s \longmapsto \sum_{q\in S} G_q(s)\zeta^q p^q$ on a strip around $\mathbb{Q}_p$, where if $q = Q/p^n$,

$$\zeta^q = \zeta_{p^n}^Q \ .$$

THANK YOU FOR YOUR ATTENTION !!