Kummer log flat cohomology

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ALGANT Alumni in China

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图: Algant ceremony 2007 with Prof. Edixhoven

Prof. Edixhoven left us on 16 Jan 2022 at the age of 59. He had guided lots of algant students. The success of ALGANT is not possible without the selfless dedication of the algant faculties.

§1 Introduction

Log geometry provides the most efficient and systematic tools to treat degeneration and compactification, this is because certain classical maps behaves much better in log geometry, e.g.

semi-stable maps become log smooth.

More well-bahaving maps give the possibility of finer topologies in log world than in the non-log world. In early 1990's,

- Kazuhiro Fujiwara, Kazuya Kato and Chikara Nakayama developed a theory of (Kummer) log étale topology;
- Kazuya Kato developed a theory of Kummer log flat (kfl) topology.

In order to understand the cohomology in kfl top., one is led to the comparison b.t. kfl and fppf topology.

§2 Log Schemes

Roughly speaking, a log scheme "=" a scheme X + (an extra structure) log structure.

Definition

A **pre-log structure** on a scheme X is a pair $(M_X, M_X \xrightarrow{\alpha} \mathcal{O}_X)$ consisting of a (Zariski or étale) sheaf of monoids M_X and a homomor. $\alpha : M_X \to \mathcal{O}_X$ (of sheaves of monoids).

It is further called a **log structure**, if

$$\begin{array}{ccc} \alpha^{-1}(\mathcal{O}_X^\times) & \xrightarrow{\cong} & \mathcal{O}_X^\times \\ \big\downarrow & & \big\downarrow \\ M_X & \xrightarrow{\alpha} & \mathcal{O}_X \end{array}.$$

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Given a pre-log stru. (M_X, α), gives rise to a

$$\begin{array}{ccccc} \alpha^{-1}(\mathcal{O}_X^\times) & \longrightarrow & \mathcal{O}_X^\times & \xrightarrow{\iota} & \\ \downarrow & & \downarrow f & & \\ M_X & \longrightarrow & M_X^{\mathrm{a}} & \xrightarrow{\alpha^{\mathrm{a}}} & \mathcal{O}_X \end{array} \qquad (M_X \xrightarrow{\alpha} \mathcal{O}_X)$$

log stru. $(M_X^{\mathrm{a}}, \alpha^{\mathrm{a}})$, which we call the **log stru. ass. to** (M_X, α).

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A **morphism of log schemes** from $(X, (M_X, \alpha_X))$ to $(Y, (M_Y, \alpha_Y))$ consists of a morphism $f : X \to Y$ of schemes and a homomorphism $\lambda : f^{-1}M_Y \to M_X$ s.t. the following diagram commutes

$$\begin{array}{ccc} M_X & \xrightarrow{\alpha_X} & \mathcal{O}_X \\ \lambda \uparrow & & \uparrow \\ f^{-1}M_Y & \xrightarrow{f^{-1}(\alpha_Y)} & f^{-1}\mathcal{O}_Y \end{array} .$$

$\rightsquigarrow$ a category **Logsch**.

Call f **strict**, if $(f^{-1}M_Y \xrightarrow{\lambda} M_X \xrightarrow{\alpha_X} \mathcal{O}_X)^{\mathrm{a}} \xrightarrow{\cong} M_X$.

Definition

X log scheme, P monoid, $\mathbb{S}[P] := (\operatorname{Spec}\mathbb{Z}[P], (P \to \mathcal{O}_{\operatorname{Spec}\mathbb{Z}[P]})^{\mathrm{a}})$.
A **chart subordinated (sbd.) to** P of X is a strict morph. $X \to \mathbb{S}[P]$.
A log scheme X is called **fs**, if locally X admits a chart sbd. to an fs monoid.

Call a monoid P **fs**, if f.g. and $\mathbb{C}[P]$ normal ring (not nec. domain).
$\rightsquigarrow$ a full subcat. of fs log schemes **Logsch**$^{\mathrm{fs}} \subset$ **Logsch**.

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Definition

A morphism $u : Q \to P$ of fs monoids is **Kummer**, if inj. and $\mathbb{Q}$-surjective (i.e. $\forall a \in P$, $\exists n \in \mathbb{N}$ s.t. $a^n \in \operatorname{im}(u)$).
A morphism $f : X \to Y \in \mathbf{LogSch}^{\mathrm{fs}}$ is called **Kummer**, if $\forall x \in X$ the can. homom. of monoids

$$M_{Y,f(x)}/\mathcal{O}^{\times}_{Y,f(x)} \to M_{X,x}/\mathcal{O}^{\times}_{X,x}$$

is Kummer. Take geometric stalks in case of étale log stru.

Definition

$(X \xrightarrow{f} Y) \in \mathbf{LogSch}$, $(Q \xrightarrow{u} P) \in \mathbf{Mon}$ (the category of monoids).

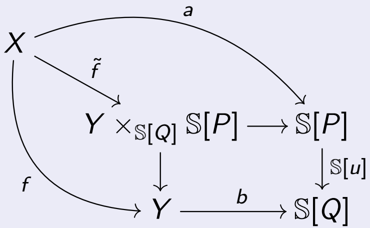
A **chart of** f **sbd. to** u is a triple $(X \xrightarrow{a} \mathbb{S}[P], Y \xrightarrow{b} \mathbb{S}[Q], Q \xrightarrow{u} P)$ with a (resp. b) a chart of X (resp. Y), s.t. the diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{a} & \mathbb{S}[P] \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathbb{S}[u]} \\ Y & \xrightarrow{b} & \mathbb{S}[Q] \end{array} \quad .$$

Definition

A morphism $f : X \to Y$ in $\textbf{Logsch}^{\mathrm{fs}}$ is called **log flat** (resp. **log smooth**, resp. **log étale**), if: classically fppf (classically étale, classically étale) locally on X and on Y , $\exists$ a chart $(X \xrightarrow{a} \mathbb{S}[P], Y \xrightarrow{b} \mathbb{S}[Q], Q \xrightarrow{u} P)$ s.t.

1. ★ P, Q are fs, u is inj. (resp. u is inj. with $|\mathrm{coker}(u^{\mathrm{gp}})_{\mathrm{tor}}| \in \mathcal{O}_X^\times$, resp. inj. with $\mathrm{coker}(u^{\mathrm{gp}})$ finite and $|\mathrm{coker}(u^{\mathrm{gp}})| \in \mathcal{O}_X^\times$);
2. the induced map



$\tilde{f}$ is classically flat (resp. smooth, resp. étale).

§3 Kummer log flat topology and cohomology

For $X \in \mathbf{Logsch}^{\mathrm{fs}}$, let (fs/X) be the cat. of fs log sch. over X.

Definition

A family $\{U_i \xrightarrow{f_i} X\}_i$ in $\mathbf{Logsch}^{\mathrm{fs}}$ is called a kfl cover of X, if

(i) f_i's are log flat and Kummer, and $\mathring{f}_i$'s are loc. f. p.

(ii) $X = \bigcup_i f_i(U_i)$.

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The Grothendieck top. on (fs/X) given by the kfl covers is called the **Kummer log flat** topology, and we get the kfl site $(\mathrm{fs}/X)_{\mathrm{kfl}}$.

The **classical fppf topology** on (fs/X) is given by families $\{U_i \xrightarrow{f_i} X\}_i$ with f_i's strict and $\{\mathring{f}_i\}$ a classical fppf cover, and we get the classical fppf sites $(\mathrm{fs}/X)_{\mathrm{fppf}}$.

We have an obvious map of sites

$$\varepsilon : (\mathrm{fs}/X)_{\mathrm{kfl}} \to (\mathrm{fs}/X)_{\mathrm{fppf}}.$$

Theorem (Kato, 1991)

1. *For* $X \in (\mathrm{fs}/S)$, $\mathrm{Mor}_S(-, X)$ *is a sheaf for the kfl top.*
2. **Kato's multip. group** $\mathbb{G}_{\mathrm{m,log}} : (\mathrm{fs}/X) \to \mathrm{Ab}, \quad U \mapsto \Gamma(U, M_U^{\mathrm{gp}})$ *is a sheaf for the kfl top.*

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In order to understand the cohomology for a sheaf F on $(\mathrm{fs}/X)_{\mathrm{kfl}}$, one is led to the higher direct images $R^i\varepsilon_*$, and use $H^i_{\mathrm{fl}}(-, R^j\varepsilon_*F) \Rightarrow H^{i+j}_{\mathrm{kfl}}(-, F)$.

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Theorem (Kato 1991)

Assume that X is locally noetherian.

1. $R^1\varepsilon_*\mathbb{G}_{\mathrm{m,log}} = 0$.
2. *Let G be either a finite flat gp. sch. or a smooth affine gp. sch. over $\mathring{X}$ (the underlying scheme of X), and endow G with the induced log struc. from X. Then*

$$R^1\varepsilon_*G \cong \varinjlim_n \mathcal{H}om_X(\mathbb{Z}/n\mathbb{Z}(1), G) \otimes (\mathbb{G}_{\mathrm{m,log}}/\mathbb{G}_{\mathrm{m}}).$$

Theorem (Zhao, J. Inst. Math. Jussieu 2023 + Arxiv preprint 2022)

1. *The affine condition from Kato's theorem can be removed.*
2. *Assume that G is as in one of the following cases:*
   - (i) *G is smooth and affine over $\mathring{X}$, and has geometrically connected fibers;*
   - (ii) *G is finite and flat over $\mathring{X}$;*
   - (iii) *G is an ex. of an ab. sch. A by a torus T over $\mathring{X}$.*

   Then
   - (a) $R^2\varepsilon_*G \cong \varinjlim_n R^2\varepsilon_*G[n]$;
   - (b) *l prime, $R^2\varepsilon_*G[l^r]$ is supported on the locus where l is invertible;*
   - (c) *if n is invertible on S, then*

$$R^2\varepsilon_*G[n] \cong G[n](-2) \otimes \bigwedge^2(\mathbb{G}_{\mathrm{m,log}}/\mathbb{G}_{\mathrm{m}}).$$

3. $R^i\varepsilon_*\mathbb{G}_{\mathrm{m}} \xrightarrow{\cong} R^i\varepsilon_*\mathbb{G}_{\mathrm{m,log}}$ *for $i > 1$.*
4. *For G a smooth gp. sch., the higher direct images $R^i\varepsilon_*G$ are always torsion.*

Theorem (Zhao, Math. Proc. Camb. Philos. Soc. 2025)

G a gp. sch. over $\mathring{X}$ which is étale locally iso. to $\mathbb{Z}^r$. Then:

(1) $R^1\varepsilon_* G = 0$.

(2) *Let $i > 1$. For each prime number l, let U_l be the locus on X on which l is invertible and ${}_lj : U_l \hookrightarrow X$ the corresponding strict open immersion. Then*

$$R^i\varepsilon_* G$$
$$\cong \bigoplus_{l \text{ prime}} R^{i-1}\varepsilon_{\mathrm{fppf}*}(G \otimes_{\mathbb{Z}} \mathbb{Q}_l/\mathbb{Z}_l)$$
$$\cong \bigoplus_{l \text{ prime}} {}_lj_{\mathrm{fppf}!}({}_lj_{\mathrm{kfl}}^{-1} G \otimes_{\mathbb{Z}} \mathbb{Q}_l/\mathbb{Z}_l(-i+1) \otimes_{\mathbb{Z}} \bigwedge^{i-1}(\mathbb{G}_{\mathrm{m,log}}/\mathbb{G}_{\mathrm{m}})_{(U_l)_{\mathrm{fl}}}).$$

An example

R a DVR with finite residue field, and choose a uniformizer π, $S = \operatorname{Spec} R$ endowed with the canonical log stru. $R^\times \times \pi^{\mathbb{N}} = R\backslash\{0\} \to R$. With the help from the 1st theorem, we can compute

$$H^1_{\mathrm{kfl}}(S, \mathbb{G}_{\mathrm{m}}) = \mathbb{Q}/\mathbb{Z}$$
$$H^2_{\mathrm{kfl}}(S, \mathbb{G}_{\mathrm{m}}) = \mathbb{Q}/\mathbb{Z}$$
$$H^2_{\mathrm{kfl}}(S, \mathbb{G}_{\mathrm{m}}) \xrightarrow{\cong} H^2_{\mathrm{fl}}(\operatorname{Spec} K, \mathbb{G}_{\mathrm{m}}) = \mathrm{Br}(K)$$

End

Thank you for you attention!