Purity of the affine Springer fibers

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December 27, 2025

The affine Springer fibers

Let $k = \mathbb{F}_q$, $F = k((\epsilon)) \supset \mathcal{O} = k[[\epsilon]]$, let $\mathrm{val} : F^\times \to \mathbb{Z}$ be a valuation normalized by $\mathrm{val}(\epsilon) = 1$.

Let $G = \mathrm{GL}_d$, with Lie algebra $\mathfrak{g}$. We have the affine Grassmannian $X = G((\epsilon))/G[[\epsilon]]$, which parametrizes lattices in F^d. It has the structure of an ind-k-scheme.

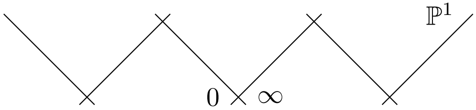
Let $\gamma \in \mathfrak{g}[[\epsilon]]$ be a regular semisimple element, the affine Springer fiber at γ is defined as

$$X_\gamma = \left\{[g] \in X \mid \mathrm{Ad}(g^{-1})\gamma \in \mathfrak{g}[[\epsilon]]\right\}.$$

It parametrizes lattices in F^d satisfying $\gamma L \subset L$.

Example

Let $G = \mathrm{GL}_2$, $\gamma = \begin{pmatrix} \epsilon & \\ & -\epsilon \end{pmatrix}$. Then $X_\gamma = \mathbb{Z} \times X_\gamma^0$, and X_γ^0 is an infinite chain of $\mathbb{P}^1$.



Example

Let $G = \mathrm{GL}_2$, $\gamma = \begin{pmatrix} & \epsilon \\ \epsilon^2 & \end{pmatrix}$. Then $X_\gamma = \mathbb{Z} \times X_\gamma^0$, and $X_\gamma^0 = \mathbb{P}^1$.

Motivation

The fundamental lemma of Langlands-Shelstad compares the orbital integrals of the form

$$\mathrm{O}_\gamma^G(\mathbb{1}_{\mathfrak{k}}) = \int_{G_\gamma(F)\backslash G(F)} \mathbb{1}_{\mathfrak{k}}(\mathrm{Ad}(g^{-1})\gamma)\frac{dg}{dt}$$

for a group G and its endoscopic groups, here

— G is a reductive group over a non-archimedean field F,

— $\gamma \in \mathfrak{g}(F)$ is a semisimple regular element,

— $\mathfrak{k} \subset \mathfrak{g}(F)$ is the Lie algebra of a maximal compact subgroup K of G,

— dg and dt are Haar measures on $G(F)$ and $T(F)$ respectively.

Observation: The orbital integrals can be interpreted as counting points on the affine Springer fibers.

Work of Goresky-Kottwitz-MacPherson

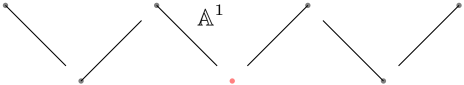
Conjecture (Goresky-Kottwitz-MacPherson)

The affine Springer fiber X_γ is cohomologically pure in the sense of Deligne, i.e. the eigenvalues of Fr_q acting on $H^i_{\text{ét}}(X_{\gamma,\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)$ are all algebraic numbers and have an absolute value $q^{i/2}$ under any embedding $\overline{\mathbb{Q}} \to \mathbb{C}$.

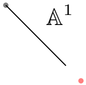
Assuming the purity hypothesis, they calculate the cohomology of X_γ for the split elements in a split reductive group $\mathbf{G}$, and deduce from it the fundamental lemma for $\mathbf{G}$ and its quasi-split outer forms. They verify the purity hypothesis in the equivalued case. In fact, they construct affine pavings for the affine Springers in this case.

Example

Let $G = \mathrm{GL}_2$. For $\gamma = \begin{pmatrix} \epsilon & \\ & -\epsilon \end{pmatrix}$, X_γ^0 admits the affine paving



For $\gamma' = \begin{pmatrix} & \epsilon \\ \epsilon^2 & \end{pmatrix}$, $X_{\gamma'}^0 = \mathbb{P}^1$ admits the affine paving



My Ph.D thesis

When I arrived at Orsay at 2006 as an ALGANT student, Ngô Bao Chau has finished the proof of the fundamental lemma for the unitary groups, in collaboration with Gérard Laumon, and is working on a proof for the general reductive groups. His approach is inspired by, but completely different from that of Goresky, Kottwitz and MacPherson.

On the other hand, Kottwitz's Ph. D student, Vincent Lucarelli, has succeeded in constructing affine pavings of X_γ for the non-equivalued split elements of PGL_3. Laumon proposed generalizing Lucarelli's work as my Ph. D thesis problem.

Theorem (Chen, 2011)

For $G = \mathrm{GL}_4$ and $\gamma \in \mathfrak{gl}_4(F)$ a split regular semisimple element, X_γ admits an affine paving, hence is cohomologically pure.

My recent work

After my thesis, I develop quite some techniques to construct affine pavings of X_γ, but it seems extremely difficult to push beyond GL_4.

On the other hand, for the non-equivalued totally ramified elements, i.e. elements γ with irreducible characteristic polynomial over F^{ur}, its seems hopeless to construct affine pavings of X_γ.

Nonetheless, by "deforming" the affine Springer fibers, I am able to show:

Theorem (Chen, 2024)

Let $\gamma \in \mathfrak{gl}_n(F)$ be a regular semisimple element with irreducible characteristic polynomial over F^{ur}, then X_γ is cohomologically pure.