Smallest Brauer subgroup obstructing the Hasse principle

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ALGANT Master: 2006-2008 Padova, Italy & Orsay, France

Central simple algebras and Brauer groups

- a CSA(central simple algebra) A is a finite dimensional k-algebra, having no non-trivial 2-sided ideal, with center k
- Wedderburn: $A \simeq M_n(D)$ matrix algebra, for some division algebra D with center k
- Brauer equivalence: $A \sim A' \Leftrightarrow D = D'$ with $A \simeq M_n(D)$ and $A' \simeq M_{n'}(D')$
- Brauer group $\mathrm{Br}(k) = (\{CSAs\}/\sim, \otimes_k)$
  neutral element: $M_n(k)$; inverse: A^{op} since $A \otimes A^{\mathrm{op}} = M_{d^2}(k)$ with $d = \dim_k A$
- 2-torsion $\mathrm{Br}(k)[2]$ is given by classes of quaternion algebras $(a, b) = \mathrm{Vect}_k(1, i, j, ij)$ with $i^2 = a, j^2 = b, ij = -ji$
- For number fields k:

Proposition (global class field theory)

We have an exact sequence $0 \to \mathrm{Br}(k) \to \bigoplus_{v\in\Omega} \mathrm{Br}(k_v) \stackrel{\sum_v \mathrm{inv}_v}{\to} \mathbb{Q}/\mathbb{Z} \to 0,$ *where* $\mathrm{inv}_v : \mathrm{Br}(k_v) \to \mathbb{Q}/\mathbb{Z}$ *is an isomorphism for p-adic fields, monomorphism otherwise.*

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Brauer groups

- Cohomological interpretation: $\mathrm{Br}(k) = \mathrm{H}^2_{\text{ét}}(\mathrm{Spec}(\mathrm{k}), \mathbb{G}_m)$
- Extension to schemes: $\mathrm{Br}(X) = \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$

Theorem

When X is (proper) rationally connected, then $\mathrm{Br}(X)/\mathrm{Br}(k)$ is finite.

- rationally connected means any 2 geometric points can be connected by a rational curve
  $\forall P, Q \in X(\mathbb{C}),\ \exists f : \mathbb{P}^1_{\mathbb{C}} \to X_{\mathbb{C}}$ *s.t.* $P, Q \in f(\mathbb{P}^1)$
- Not easy to compute $\mathrm{Br}(X)/\mathrm{Br}(k)$.

Question 1 ("inverse Brauer problem")

For a given finite abelian group B, does there exist a rationally connected variety $X_{/K}$ such that $\mathrm{Br}(X)/\mathrm{Br}(k) = B$?

- Known cases: groups of small order or certain p-groups

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Hasse principle

- k number field, $v \in \Omega_k = \{\text{places/primes of } k\}$, $k \subset k_v$ completion (example $\mathbb{Q} \subset \mathbb{Q}_p$)
- X varieties defined over k, injection $X(k) \hookrightarrow \prod_{v \in \Omega} X(k_v)$

- Violations of Hasse principle:
  - intersections of quadrics in $\mathbb{P}^4$ (i.e. del Pezzo surfaces of degree 4)
  - cubic curves $\subset \mathbb{P}^2$ (i.e. genus $g = 1 > 0$)

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- 1970s, Manin made use of $\mathrm{Br}(X)$
- Brauer–Manin pairing

$$X(\mathbf{A}_k) \times \mathrm{Br}(X) \to \mathbb{Q}/\mathbb{Z}$$
$$((x_v), \beta) \mapsto \langle (x_v), \beta \rangle := \sum_{v \in \Omega} \mathrm{inv}_v(\beta(x_v))$$

where $\mathrm{inv}_v : \mathrm{Br}(k_v) \hookrightarrow \mathbb{Q}/\mathbb{Z}$ local invariant (local class field theory)

- $X(\mathbf{A}_k)^{\mathrm{Br}} = \{(x_v) \in X(\mathbf{A}_k);\ (x_v) \perp \beta,\ \forall \beta \in \mathrm{Br}(X)\}$
- Fact. $X(k) \subset X(\mathbf{A}_k)^{\mathrm{Br}} \subset X(\mathbf{A}_k)$
(global class field theory)
- Brauer–Manin obstruction to Hasse principle (explaining the violation of HP)
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Conjecture

When the Brauer group completely controls the violation of Hasse principle?

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When X is the following varieties, then
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- *(Colliot-Thélène 1980) rationally connected varieties*
- *(Skorobogatov 2001) smooth projective curves*

- For curves C of genus 1, the conjecture is true when $Ш(\mathrm{Jac}(C), k)$ is finite.
- Known cases include Châtelet surfaces (Colliot-Thélène–Sansuc–Swinnerton-Dyer 1987):

$$y^2 - az^2 = P(x)$$

where $a \in k^*,\ P \in k[x],\ \deg(P) = 4$

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An example (Châtelet surface)

- Châtelet surface X over $k = \mathbb{Q}$

$$y^2 + 3z^2 = -(x^2 - 6)(x^2 - 5) \quad \subset \mathbb{A}^3$$

- $X(\mathbb{R}) \neq \varnothing$ easy; $X(\mathbb{Q}_p) \neq \varnothing (\forall p \neq 3)$ by Hensel's lemma; $X(\mathbb{Q}_3) \neq \varnothing$ by some more work.
- class of quaternion algebra $\beta = (6 - x^2, -3) \in \mathrm{Br}(\mathbb{Q}(X))$
- Grothendieck: "purity thm for Br"$\Rightarrow \beta \in \mathrm{Br}(X) \subset \mathrm{Br}(\mathbb{Q}(X))$
(i.e. this algebra extends from the generic point $\mathbb{Q}(X)$ to an Azumaya algebra over the scheme X)
- $\forall p \neq 3,\ \forall x_p \in X(\mathbb{Q}_p),\ \mathrm{inv}_p(\beta(x_p)) = 0$
$p = 3,\ \forall x_3 \in X(\mathbb{Q}_3),\ \mathrm{inv}_3(\beta(x_3)) = \frac{1}{2} \in \mathbb{Q}/\mathbb{Z}$
$v = \infty,\ \forall x_\infty \in X(\mathbb{R}),\ \mathrm{inv}_\infty(\beta(x_\infty)) = 0$
- $\Rightarrow \forall (x_v) \in X(\mathbf{A}_\mathbb{Q}),\ \langle (x_v), \beta \rangle = \frac{1}{2} \neq 0$
- This means $X(\mathbf{A}_\mathbb{Q})^{\mathrm{Br}} = \varnothing$ and therefore $X(\mathbb{Q}) = \varnothing$
- In this case, a single element β obstructs the Hasse principle. Indeed $\mathrm{Br}(X)/\mathrm{Br}(\mathbb{Q}) \simeq \mathbb{Z}/2\mathbb{Z}$ is generated by β.

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Compactness argument

- If X is proper such that $X(\mathbf{A}_k) \neq \varnothing$, then $X(\mathbf{A}_k) = \prod X(k_v)$ is compact
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Main result

Theorem (Yongqi Liang & Yufan Liu 2025^+)

Given a pair of finite abelian groups $0 \neq B_0 \subseteq B$, then

1. *"inverse Brauer problem": There exists a rationally connected variety X defined by a normic equation*

$$\mathrm{N}_{K/k}(\mathrm{z}) = P(x)$$

   such that $\mathrm{Br}(X)/\mathrm{Br}(k) = B$.

2. *More over, we may further require that B_0 is the smallest subgroup that obstructs the Hasse principle.*
   In other words, X violates the Hasse principle, and for any subgroup $B' \subseteq B$, the subset $X(\mathbf{A}_k)^{B'} = \varnothing$ if and only if $B_0 \subseteq B'$.

Thank you for your attention!

2008 graduates:
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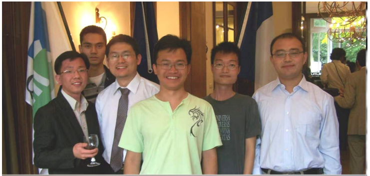

20 years later