On the Weil representation

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References: Weil Representations

L.Barthel, *Local Howe correspondence for groups of similitudes*, J. Reine Angew. Math., 414 (1991), 207-220.

P. Cartier, *Quantum Mechanical Commutation Relations and Theta Functions*,Proc. of Symp. Pure Mathematics, 9, Amer. Math. Soc., 1966, pp. 361-38.

G.Lion, M.Vergne, *The Weil representation Maslov index and Theta series*, Progress in Math., vol.6, Birkhauser, Boston, 1980.

. Moeglin, M.-F. Vignéras, and J.-L. Waldspurger, *Correspondances de Howe sur un corps p-adique*, Lecture Notes in Math. 1291, Springer-Verlag, Berlin, 1987.

References: Weil Representations

P.Perrin, *Représentations de Schrödinger. Indice de Maslov et groupe metaplectique*, in "Non commutative Harmonic Analysis and Lie Groups", Lect. Notes Math. 880 (1981), 370-407.

R.R.Rao, *On some explicit formulas in the theory of the Weil representation*, Pacific J. Math. 157 (1993), 335-371.

S.S.Kudla, *Notes on the local theta correspondence*, preprint, available at http://www.math.utotonto.ca/ skudla/castle.pdf, 1996.

A.Weil, *Sur certains groupes d'opérateurs unitaires*, Acta Math. 111 (1964), 143-211.

Notation and conventions

- F: the usual real field.
- ψ: a non-trivial unitary character of F.
- $(W, \langle , \rangle)$: a symplectic vector space of dimension $2m$ over F.
- $\mathrm{Sp}(W)$: the corresponding symplectic group.
- $\mu_n = \{t \in \mathbb{C} \mid t^n = 1\}$.

Heisenberg group

The Heisenberg group $\mathrm{H}(W)$, attached to W and F, is a topological group $W \oplus F$, with the law

$$(w,t)(w',t') = (w+w', t+t'+\tfrac{\langle w,w'\rangle}{2})$$

where $w, w' \in W, t, t' \in F$.

Stone-von Neumann's theorem

Theorem (Stone-von Neumann's theorem)

There exists a unique (up to isomorphism) unitary irreducible complex representation of $\mathrm{H}(W)$ *with central character* ψ.

Let us call it the Heisenberg representation, and denote it by π_ψ.
$(*)$ According to Weil's work, this representation can be extended to a projective representation of $\mathrm{Sp}(W)$, and then to an actual representation of a $\mathbb{C}^\times$-covering group over $\mathrm{Sp}(W)$.

Weil index

Let (Q, V) be a non-degenerate quadratic vector space on F.

Theorem (André Weil)

There exists a unique root of unity of degree 8, called the **Weil index** *attached to* $\psi(Q)$*, denoted by* $\gamma_\psi(Q)$*, such that*

$$\mathcal{F}(\psi(Q)dv) = \gamma_\psi(Q)|\rho|_F^{-\frac{1}{2}}\psi(Q^*)^{-1}dv^*,$$

for $\psi(Q)dv \in S^*(V)$*, and* $\psi(Q^*)^{-1}dv^* \in S^*(V^*)$.

Remark

The Weil index only depends on the Witt class of (Q, V) *and* ψ.

Weil index: Example

- $F = \mathbb{R}$.
- $\psi_0(t) = e^{2\pi it}$.
- $\psi = \psi_0^e$, for some $e \in F^\times$.

Example (Rao Prop. A.10)

If $V = F$ and $Q(x) = x^2$, then $\gamma_\psi(Q) = \psi_0(\frac{\text{sign } e}{8})$.

Lemma ([Rao, Def.A.6])

$\gamma_\psi(Q) = \epsilon(Q)\gamma(\psi)^{\dim V}\gamma(\det Q, \psi)$.

2-cocycle I

- $W = X \oplus X^*$: a complete polarization.
- $Q(g_1, g_2) = Q(X^*, X^* g_2^{-1}, X^* g_1)$: the corresponding Leray invariant.
- Define $\widetilde{c}_{X^*}(g_1, g_2) = \gamma_\psi(Q(g_1, g_2)/2)$.

Then:

- $\widetilde{c}_{X^*}(-,-)$ defines a non-trivial class of order 2 in $\mathrm{H}^2(\mathrm{Sp}(W), \mu_8)$.
- Let $\mathrm{Mp}(W)$ denote the associated Metaplectic group.

Metaplectic group

The set:

$$\mathrm{Mp}(W) = \{(g, t) \mid g \in \mathrm{Sp}(W), t \in \mu_8\}.$$

The group law:

$$(g_1, t_1)(g_2, t_2) = (g_1 g_2, \widetilde{c}_{X^*}(g_1, g_2) t_1 t_2).$$

There exists an exact sequence:

$$1 \longrightarrow \mu_8 \longrightarrow \mathrm{Mp}(W) \longrightarrow \mathrm{Sp}(W) \longrightarrow 1.$$

2-cocycle II

Rao and Perrin define:

$$m_{X^*}: \mathrm{Sp}(W) \longrightarrow \mu_8; \tag{1}$$

$$g \longmapsto \gamma(x(g), \psi^{\frac{1}{2}})^{-1} \gamma(\psi^{\frac{1}{2}})^{-j(g)}. \tag{2}$$

$$\overline{c}_{X^*}(g_1, g_2) = m_{X^*}(g_1 g_2)^{-1} m_{X^*}(g_1) m_{X^*}(g_2) \widetilde{c}_{X^*}(g_1, g_2). \tag{3}$$

Then:

- $\overline{c}_{X^*}$ defines a 2-cocycle on $\mathrm{Sp}(W)$ with values in μ_2.
- $\overline{c}_{X^*}(g_1, g_2) = (x(g_1), x(g_2))_F (-x(g_1)x(g_2), x(g_1 g_2))_F$
$$((\text{-}1)^t, \det(2q))_F (-1, -1)_F^{\frac{t(t-1)}{2}} \epsilon(2q).$$

2-cocycle: Example

Example

If $\dim W = 2$, $\mathrm{Sp}(W) \simeq \mathrm{SL}_2(F)$. For $g_1, g_2, g_3 = g_1 g_2 \in \mathrm{SL}_2(F)$ with $g_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$, we have:

(1) $x(g_1) = \begin{cases} d_1 F^{\times 2} & \text{if } c_1 = 0 \\ c_1 F^{\times 2} & \text{if } c_1 \neq 0 \end{cases}$;

(2) $\overline{c}_{X^*}(g_1, g_2) = (x(g_1), x(g_2))_F (-x(g_1)x(g_2), x(g_3))_F$.

Remark

If we take $g_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, *then* $\overline{c}_{X^*}(g_1, g_2) = (x(g_1), x(g_2))_F = (-1, x(g_2))_F$, *which is not continuous on the variable* g_2.

Schrödinger model

By Weil's result, π_ψ can extend to be a unitary representation of $\mathrm{Mp}(W) \ltimes \mathrm{H}(W)$. The representation π_ψ can be realized on $L^2(X)$ by the following formulas:

$$\pi_\psi[(x,0)\cdot(x^*,0)\cdot(0,k)]f(y) = \psi(k + \langle x+y, x^*\rangle)f(x+y), \quad (4)$$

$$\pi_\psi[\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, t]f(y) = t\psi(\tfrac{1}{2}\langle y, yb\rangle)f(y), \quad (5)$$

$$\pi_\psi[\begin{pmatrix} a & 0 \\ 0 & a^{*-1} \end{pmatrix}, t]f(y) = t|\det(a)|^{1/2}f(ya), \quad (6)$$

$$\pi_\psi([\omega, t])f(y) = t\int_{X^*} \psi(\langle y, y^*\rangle)f(y^*\omega^{-1})dy^*. \quad (7)$$

Lattice model

If $\psi = \psi_0$, $L = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_m \oplus \mathbb{Z}e_1^* \oplus \cdots \oplus \mathbb{Z}e_m^*$, we let $\mathcal{H}(L)$ be the space of measurable functions $f : W \longrightarrow \mathbb{C}$ such that

(i) $f(l + w) = \psi(-\frac{\langle x_l, x_l^* \rangle}{2} - \frac{\langle l, w \rangle}{2})f(w)$, for all $l = x_l + x_l^* \in L = (L \cap X) \oplus (L \cap X^*)$, almost all $w \in W$;

(ii) $\int_{L \backslash W} ||f(w)||^2 dw < +\infty$.

Then the Heisenberg representation π_ψ can be realized on $\mathcal{H}(L)$ by the following formulas:

$$\pi_\psi([w', t])f(w) = \psi(t + \frac{\langle w, w' \rangle}{2})f(w + w')$$

for $w, w' \in W$, $t \in \mathbb{R}$.

Fock model

If $\psi = \psi_0$, we let $\mathcal{H}_F$ denote the Fock space of holomorphic functions f on $\mathbb{C}^m$ such that

$$\|f\|^2 = \int_{\mathbb{C}^m} \mid f(w) \mid^2 e^{-\pi\|w\|^2} d(w) < +\infty.$$

The Heisenberg representation of $\mathrm{H}(V)$ associated to ψ can be realized on $\mathcal{H}_F$ by the following formulas:

$$\pi_\psi([w', t])f(w) = \psi(t)e^{-\frac{\pi}{2}\|w'\|^2 - \pi w \overline{w'}^T} f(w + w')$$

for $w, w' \in V \simeq W$, $t \in \mathbb{R}$.

Siegel modular forms

Compare the three standard realizations of the Weil representation—the Schrödinger, lattice and Fock models—and then produce Siegel modular forms. This viewpoint goes back to

G. Mackey, Infinite-dimensional group representations and their applications, C.I.M.E., Edizioni Cremonese, Rome 1971, 221—330.

Example 1

(1) $\theta(z) \overset{\text{Def.}}{=} \sum_{n\in\mathbb{Z}} e^{2\pi i z n^2}$.

(2) $\Gamma_0(4)^{\kappa} \overset{\text{Def.}}{=} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \Gamma_\theta \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \Gamma_0(4) \sqcup \Gamma_0(4)\omega h(2)$.

(3) $\nu_\theta(r) =$

$$\begin{cases} \left(\frac{c}{d}\right)\epsilon_d^{-1}, & r = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4), \\ \left(\frac{c}{d}\right)\epsilon_d^{-1} e^{-\frac{i\pi}{4}} \overline{c}_{X^*}(r, \omega h(2)), & r = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \omega h(2) \in \Gamma_0(4)\omega h(2). \end{cases}$$

Proposition

$\theta(rz) = \nu_\theta(r)\sqrt{cz+d}\,\theta(z)$, *for* $r \in \Gamma_0(4)^\kappa$.

Remark

For $r \in \Gamma_0(4)$, *it recovers the classical formula of Shimura; following Lion—Vergne [LiVe], we extend it to the slightly larger group* $\Gamma_0(4)^\kappa$.

Example 2

(1) $\theta'_{1/2}(z) = \sum_{n\in\mathbb{Z}} e^{i\pi(n+\frac{1}{2})^2 z}$ (fermionic theta function).

(2) $\nu_{\theta'}(r) =$

$$\begin{cases} \left(\frac{2c}{d}\right)\epsilon_d^{-1}\left(\frac{-1}{d}\right)^{b/2} i^{b/2}, & r = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2), \\ \left(\frac{2c}{d}\right)\epsilon_d^{-1} i^{b/2}\left(\frac{-1}{d}\right)^{b/2} e^{\frac{i\pi}{4}}, & r = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma(2)u(1). \end{cases}$$

Proposition

$\theta'^{1/2}(rz) = \nu_{\theta'}(r)\sqrt{cz+d}\,\theta'^{1/2}(z)$, for $r \in \Gamma_0(2)$.

Remark

This means that Zagier's fermionic theta function is a $1/2$-modular form for $\Gamma_0(2)$, not merely for $\Gamma(2)$ or $\Gamma_0(4)$.

Thank you for your attention!