Branching problem for $\mathrm{GL}_2(\mathbb{Q}_p)$

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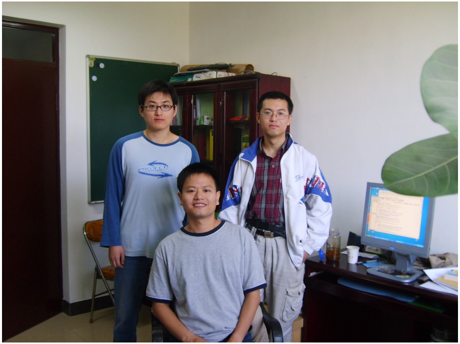
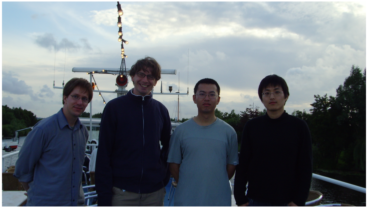
ALGANT and me



Chengyuan and me, with Ouyang in his office at Tsinghua University

In Paris, at Yongquan's wedding dinner party; 6 ALGANT alumni in total

ALGANT and me



Chengyuan and me with our mentors in Leiden.

Prof. Lenstra at his birthday boat trip.

Bas and Peter, at our graduation ceremony in Leiden.

Branching law

Let (G, H) be a pair of groups, where H is a subgroup of G.

Branching law: studies the behavior of the restriction $\Pi|_H$, where Π is an irreducible representation of G.

Example

- Mackey's formula (for complex representation of finite groups);
- Clebsch–Gordan decomposition theorem: $G := \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$, $H = \Delta\mathrm{SL}_2(\mathbb{C})$. Let $\pi_n := \mathrm{Sym}^n(\mathbb{C}^2)$. Then

$$(\pi_n \otimes \pi_m)|_H = \pi_{n+m} \oplus \pi_{n+m-2} \oplus \cdots \oplus \pi_{n-m}.$$

Features

- consider only finite dimensional representations;
- completely reducible;
- multiplicity one property in Clebsch–Gordan's theorem.

Multiplicity One Theorem

Let F be a finite extension of $\mathbb{Q}_p$. Consider $G = \mathrm{GL}_2(F)$, $H = L^\times$ (non-split torus) or T (split torus).

Theorem (Tunnell, Saito)

Let π be an infinite dimension irreducible complex representation of G. Let $\eta : H \to \mathbb{C}^\times$ be a character such that $\eta = \omega_\pi$. Then

1. $\dim(\mathrm{Hom}_H(\pi|_H, \eta)) \leq 1$;
2. description of $\mathrm{Hom}_H(\pi|_H, \eta) \neq 0$ in terms of local root number.

Theorem (Aizenbud–Gourevitch–Rallis–Schiffmann (Ann. of Math. 2010), Sun–Zhu (Ann. of Math. 2012))

Let F be a local field. For $G = \mathrm{GL}_{n+1}(F)$, $H = \mathrm{GL}_n(F)$

$$\dim \mathrm{Hom}_H(\Pi|_H, \pi) \leq 1.$$

The theorem also holds for orthogonal and unitary pairs.

Branching problem and Homological Branching

Let (G, H) be an interesting pair of p-adic groups. Let Π, π be irreducible complex representations of G, H respectively. We are interested in

the multiplicity space $\mathrm{Hom}_H(\Pi|_H, \pi)$, and its dimension (multiplicity)

Features

- consider infinite dimensional representations;
- not completely reducible;
- one has to consider the *correct* multiplicity space.

Prasad (ICM report, 2018): study the Euler–Poincaré characteristic

$$\mathrm{EP}(\Pi|_H, \pi) := \Sigma_{i=0}^{\infty}(-1)^i \dim \mathrm{Ext}^i_H(\Pi|_H, \pi).$$

At least in GGP case, we have finite dimensionality of Ext^i (Prasad, Aizenbud–Sayag) and vanishing of Ext^i for large i.

Mod p branching problem for p-adic groups

The mod p branching problem for p-adic group was first studied by Morra for the pair $(\mathrm{GL}_2(\mathbb{Q}_p), L^\times)$, where L is a quadratic extension of $\mathbb{Q}_p$.

Theorem (Morra, Math. Zeit. 2014)

Let π be an infinite dimensional admissible irreducible mod p representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. Let $\eta : L^\times \to \overline{\mathbb{F}}_p^\times$ be any character. Then

$$\mathrm{Hom}_{L^\times}(\pi|_{L^\times}, \eta) = 0.$$

Triple product case

Recent developements of Robin Zhang (IMRN2024) and Yikun Fan (preprint).

Irreducible mod p representations of $\mathrm{GL}_2(\mathbb{Q}_p)$

Theorem (Barthel–Livné (Duke Math. J. 1994),Breuil (Compos. Math. 2003))

The irreducible smooth $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ which admits a central character are the following:

1. $\chi \circ \det$, χ is a character of $\mathbb{Q}_p^\times$;
2. $\mathrm{St} \otimes \chi \circ \det$, St is the Steinberg representation;
3. (irreducible) principal series $\mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2$, $\chi_1 \neq \chi_2$;
4. supersingular representations: $\dfrac{\mathrm{c-Ind}_{\mathbb{Q}_p^\times \mathrm{GL}_2(\mathbb{Z}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathrm{Sym}^r \mathbb{F}^{\oplus 2}}{T} \otimes \chi \circ \det$, $0 \leq r \leq p-1$.

Main results

Theorem (Chen–W.)

Let π be a principal series or a special series of $G := \mathrm{GL}_2(\mathbb{Q}_p)$ with central character ζ.

1. Let η be a *generic* character of $L^\times$ such that $\eta|_{\mathbb{Q}_p^\times} = \zeta$. Then

$$\dim \mathrm{Ext}^i_{L^\times,\zeta}(\pi|_{L^\times}, \eta) = \begin{cases} 1 & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

   and

$$\dim \mathrm{Ext}^i_{L^\times,\zeta}(\eta, \pi|_{L^\times}) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

2. For any η, $\mathrm{Ext}^i_{L^\times,\zeta}(\pi|_{L^\times}, \eta)$ and $\mathrm{Ext}^i_{L^\times,\zeta}(\eta, \pi|_{L^\times})$ are finite dimensional for all i, and are non-zero for finitely many i, and

$$|\,\mathrm{EP}_\zeta(\pi|_{L^\times}, \eta)| = |\,\mathrm{EP}_\zeta(\eta, \pi|_{L^\times})| = 1.$$

Supersingular representations

For supersingular representations, we have the following theorem.

Theorem (Chen–W.)

Let π be a supersingular representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ with central character ζ and let η be a character of $L^\times$ such that $\eta|_{\mathbb{Q}_p^\times} = \zeta$. Then

$$\dim \mathrm{Ext}^i_{L^\times,\zeta}(\pi|_{L^\times}, \eta) = \begin{cases} 2 & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\dim \mathrm{Ext}^i_{L^\times,\zeta}(\eta, \pi|_{L^\times}) = \begin{cases} 2 & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Similar results hold for the diagonal torus T.

Thanks!