A survey on cohomological obstructions

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Philosophy

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In other words, rational solutions of equations correspond bijectively to rational points on the geometric object defined by those equations.

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(5) In 1970, Manin introduced an intermediate set

$$X(\mathbb{Q}) \subset \mathrm{OBS} \subset X(\mathbb{R}) \times \prod X(\mathbb{Q}_p)$$

using the Brauer group of X and class field theory. I will give a detailed construction (called the Brauer–Manin pairing) later.

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(2) Around 2000, Skorobogatov (Invent. Math. 1999) gives a negative answer via descent (and later Poonen (Ann. of Math. 2010) also shows the insufficiency). They constructed explicit examples with $\mathrm{OBS} \neq \varnothing$ but $X(\mathbb{Q}) = \varnothing$.

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(3) Harari, Skorobogatov, Stoll, Demarche, Cao, etc compared many obstruction sets.

(4) What about other fields?

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Why these fields? Because the analogue of

$$\mathbb{Q}/\mathbb{Z} \simeq \mathrm{Br}\ \mathbb{Q}_p \simeq H^2(\mathbb{Q}_p, \mathbb{Q}/\mathbb{Z}(1))$$

should be

$$H^{d+1}(K, \mathbb{Q}/\mathbb{Z}(d)) \simeq \mathbb{Q}/\mathbb{Z}$$

where K is a d-local field.

p-adic function fields

Does $H^{d+1}(X, \mathbb{Q}/\mathbb{Z}(d))$ work well for varieties when $d \geq 2$? We may use

$$H^{d+1}_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(d)) := H^0_{\mathrm{Zar}}(X, \mathcal{H}^{d+1}_{\text{ét}}(\mathbb{Q}/\mathbb{Z}(2)))$$

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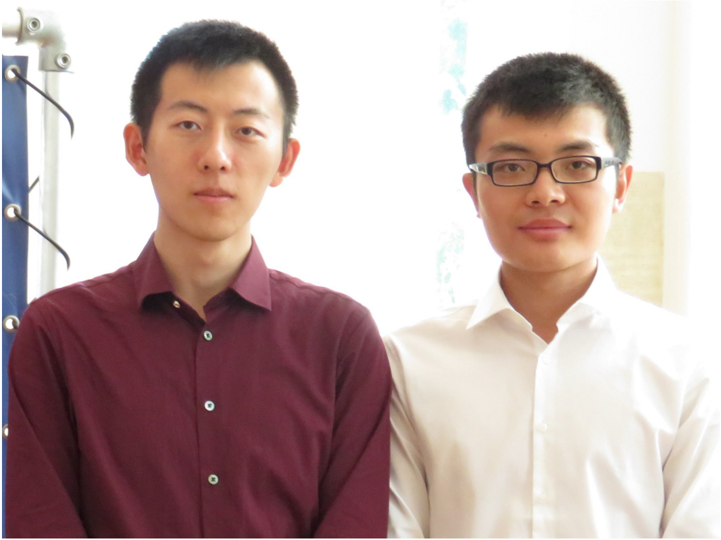
Theorem (Cao–T. 2026)

Let K be a p-adic function field (for instance $\mathbb{Q}_p(t)$). Let X be a smooth integral K-variety. The correct analogue of Manin's obstruction set in this setting is $X(\mathbf{A})^{H^3_{\mathrm{nr}}}$ instead of $X(\mathbf{A})^{H^3_{\text{ét}}}$.

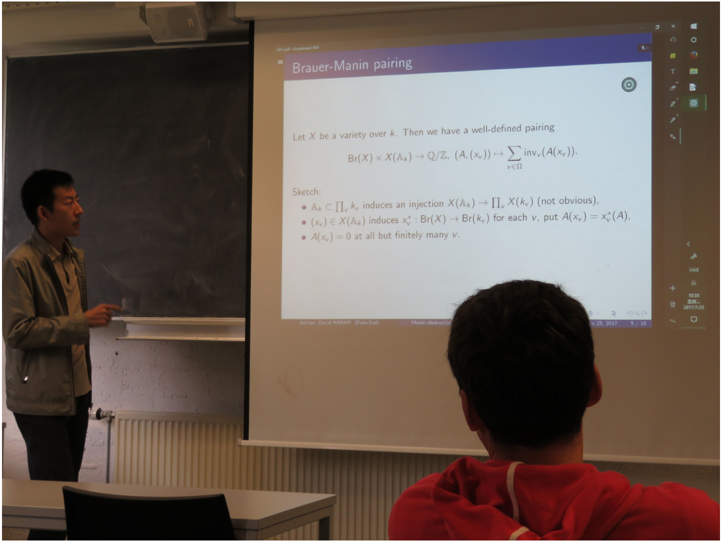
Raffaele, Salvador, Giada, and myself

The only one showing thumb, the only one published a paper in a top journal (Invent. Math.)

Algant graduation in Leiden, 2017



The promised construction



Thanks for your attention!