Galois cohomology and related invariants via (φ, τ)-modules.

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ALGANT group photo – 2017

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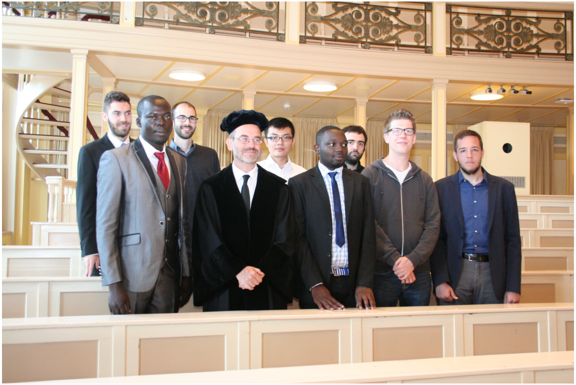

Notations

- Let p be an odd prime.
- $K/\mathbf{Q}_p$ a CDVF, with perfect residue field k.
- Fix $\pi \in K$ a uniformiser. $C_p := \widehat{\overline{K}}$ and $C_p^\flat$ its tilting.
- $\mathcal{O}_{C_p^\flat} \simeq \varprojlim_{x \mapsto x^p} \mathcal{O}_{\overline{K}} \ni \tilde{\pi} := (\sqrt[p^n]{\pi})_{n \in \mathbf{N}}, \varepsilon := (\zeta_{p^n})_{n \in \mathbf{N}}$.
- $K_\infty := \bigcup\limits_n K(\sqrt[p^n]{\pi})$ and $K_{p^\infty} := \bigcup\limits_n K(\zeta_{p^n})$.

Notations

$$\begin{tikzcd}
 & \overline{K} \arrow[d, dash, "G_L"] \arrow[ddl, dash, bend right, "G_{K_\infty}"'] \arrow[ddr, dash, bend left, "G_{K_{p^\infty}}"] & \\
 & L \arrow[dl, dash, "\overline{\langle\gamma\rangle}"'] \arrow[dr, dash, "\overline{\langle\tau\rangle}"] \arrow[dd, dash, "\hat{G}"] & \\
K_\infty \arrow[dr, dash] & & K_{p^\infty} \arrow[dl, dash, "\Gamma=\overline{\langle\gamma\rangle}"] \\
 & K &
\end{tikzcd}$$

- Fix γ with $\chi(\gamma) \in \mathbf{Z}_{>0}$ and τ such that $\tau(\tilde{\pi}) = \varepsilon\tilde{\pi}$.
- $\hat{G} \simeq \overline{\langle\tau\rangle} \rtimes \overline{\langle\gamma\rangle}$ with $\gamma\tau\gamma^{-1} = \tau^{\chi(\gamma)}$.

(φ, Γ)-modules

Definition (Fontaine[1])

An **étale** (φ, Γ)**-module** over $\mathbf{A}_{K_{p^\infty}}$ consists of

(1) an étale φ-module[2] $(D_{K_{p^\infty}}, \varphi_D)$ over $\mathbf{A}_{K_{p^\infty}}$;

(2) a φ_D-equivariant continuous[3] semilinear action Γ.

Theorem (Fontaine)

We have a category equivalence

$$\mathrm{Mod}_{\varphi,\Gamma}(\mathbf{A}_{K_{p^\infty}}) \simeq \mathrm{Rep}_{\mathbf{Z}_p}(G_K).$$

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[1] J.M. Fontaine, Représentations p-adiques des corps locaux (1 ère partie). The Grothendieck Festschrift: A Collection of Articles Written in Honor of the 60th Birthday of Alexander Grothendieck (1990).

[2] A φ-module is a finitely generated $\mathbf{A}_{K_{p^\infty}}$-module $D_{K_{p^\infty}}$ equipped with a $\varphi_{\mathbf{A}_{K_{p^\infty}}}$-semilinear operator φ_D over $D_{K_{p^\infty}}$ and étale means that the linearization $1 \otimes \varphi \colon \mathbf{A}_{K_{p^\infty}} \otimes_{\varphi, \mathbf{A}_{K_{p^\infty}}} D_{K_{p^\infty}} \to D_{K_{p^\infty}}$ is surjective.

[3] Being continuous for the natural topology of finitely generated $\mathbf{A}_{K_{p^\infty}}$-modules.

(φ, Γ)-module theory is one of the central tool in p-adic Hodge theory and has led to many progresses:

- The computation of Galois cohomology and Iwasawa cohomology (Herr, Cherbonnier-Colmez).
- The p-adic local Langlands correspondence for $GL_2(\mathbb{Q}_p)$ (Colmez).
- The study of p-adic L-functions and Iwasawa theory (Benois, Berger).
- Emerton-Gee stack.

Galois cohomology with (φ, Γ)-modules

Theorem (Herr[4])

Let $T \in \mathrm{Rep}_{\mathbf{Z}_p}(G_K)$ *and* $D_{K_{p^\infty}} \in \mathrm{Mod}_{\varphi,\Gamma}(\mathbf{A}_{K_{p^\infty}})$ *the associated* (φ, Γ)*-module. Then we have*

$$\left[D_{K_{p^\infty}} \xrightarrow{\varphi-1, \gamma-1} D_{K_{p^\infty}} \oplus D_{K_{p^\infty}} \xrightarrow{(\gamma-1)\ominus(\varphi-1)} D_{K_{p^\infty}} \right] \simeq R\Gamma(G_K, T).$$

Theorem (Colmez[5], Liu[6])

Similar results hold for overconvergent and Robba ring coefficients.

[4] L. Herr, Sur la cohomologie galoisienne des corps p-adiques, Bulletin de la SMF **126**, (1998).
[5] P. Colmez, Représentations triangulines de dimension 2. Astérisque, **319**, (2008).
[6] R. Liu, Cohomology and duality for (φ, Γ)-modules over the Robba ring. IMRN, (2008).

Breuil-Kisin modules

- $\mathfrak{S} := W(k)[[u]] \hookrightarrow A_{\mathrm{inf}} := W(\mathcal{O}_{C_p^\flat}); u \mapsto [\tilde{\pi}]$;
- $W(k)[[u]][1/u]^{\wedge p} =: \mathbf{A}_{K_\infty} \hookrightarrow \widetilde{\mathbf{A}} := W(C_p^\flat); u \mapsto [\tilde{\pi}]$.
- $\widetilde{\mathbf{A}}_L := \widetilde{\mathbf{A}}^{G_L}$.

Theorem (Fontaine, Kisin, Raynaud, Tate)

$$\mathrm{Mod}_\varphi^1(\mathfrak{S}) \simeq \mathrm{Rep}_{\mathbf{Z}_p}^{\mathrm{cris},\{0,1\}}(G_K) \simeq \mathrm{BT}/_{\mathcal{O}_K}$$

Breuil-Kisin module theory recently serves as foundational tools in integral p-adic Hodge theory, for example used in the following:

- Certain moduli theory of Galois representations (Kisin, Emerton-Gee).
- The unifying A_{inf}-cohomology and then prismatic cohomology theory developed by Scholze et al. (Bhatt, Morrow, Scholze).

(φ,τ)-modules

Definition (Caruso[7])

An (φ,τ)-**module** $(D_{K_\infty},\widetilde{D}_L)$ **over** $(\mathbf{A}_{K_\infty},\widetilde{\mathbf{A}}_L)$ consists of

(i) an étale φ-module D_{K_∞} over $\mathbf{A}_{K_\infty}$;

(ii) a τ-semi-linear endomorphism τ_D on $\widetilde{D}_L := \widetilde{\mathbf{A}}_L \otimes_{\mathbf{A}_{K_\infty}} D_{K_\infty}$ which commutes with $\varphi_{\widetilde{\mathbf{A}}_L} \otimes \varphi_D$ and such that:

$$\gamma \circ \tau_D = \tau_D^{\chi(\gamma)} \text{ on } D_{K_\infty}$$

Theorem (Caruso)

We have a category equivalence

$$\mathrm{Mod}_{\varphi,\tau}(\mathbf{A}_{K_\infty},\widetilde{\mathbf{A}}_L) \simeq \mathrm{Rep}_{\mathbf{Z}_p}(G_K).$$

[7] X. Caruso, Représentations galoisiennes p-adiques et (φ,τ)-modules, Duke Math. J. **162**, (2013).

Galois cohomology with (φ, τ)-modules

Theorem (Tavares Ribeiro[8])

Let $T \in \mathrm{Rep}_{\mathbf{Z}_p}(G_K)$ *and let* $\widetilde{D}_L = \left(\widetilde{\mathbf{A}} \otimes_{\mathbf{Z}_p} T\right)^{G_L}$*. Then we have*

$$\left[\widetilde{D}_L \xrightarrow{\tilde{\alpha}} \widetilde{D}_L \oplus \widetilde{D}_L \oplus \widetilde{D}_L \xrightarrow{\tilde{\beta}} \widetilde{D}_L \oplus \widetilde{D}_L \oplus \widetilde{D}_L \xrightarrow{\tilde{\eta}} \widetilde{D}_L\right] \simeq R\Gamma(G_K, T)$$

where $\tilde{\alpha} = \begin{pmatrix} \varphi - 1 \\ \gamma - 1 \\ \tau - 1 \end{pmatrix}$, $\tilde{\beta} = \begin{pmatrix} \gamma - 1 & 1 - \varphi & 0 \\ \tau - 1 & 0 & 1 - \varphi \\ 0 & \tau^{\chi(\gamma)} - 1 & \delta - \gamma \end{pmatrix}$, $\tilde{\eta} = \begin{pmatrix} \tau^{\chi(\gamma)} - 1 \\ \delta - \gamma \\ \varphi - 1 \end{pmatrix}^{\mathsf{t}}$

and $\delta = (\tau^{\chi(\gamma)} - 1)(\tau - 1)^{-1}$.

[8] F. Tavares Ribeiro, An explicit formula for the Hilbert symbol of a formal group, Annales de l'Institut Fourier **61**, (2011).

Cohomology of (φ,τ)-modules

Theorem (Z.[9])

Let $T \in \mathrm{Rep}_{\mathbf{Z}_p}(G_K)$ and $(D_{K_\infty}, \widetilde{D}_L) \in \mathrm{Mod}_{\varphi,\tau}(\mathbf{A}_{K_\infty}, \widetilde{\mathbf{A}}_L)$ the associated (φ,τ)-module. Then we have

$$\left[D_{K_\infty} \xrightarrow{\varphi-1,\tau_D-1} D_{K_\infty} \oplus \widetilde{D}_{L,0} \xrightarrow{(\tau_D-1)\ominus(\varphi-1)} \widetilde{D}_{L,0}\right] \simeq R\Gamma(G_K, T).$$

where $\widetilde{D}_{L,0} = \{x \in \widetilde{D}_L;\, \gamma(x) = x + \tau_D(x) + \cdots + \tau_D^{\chi(\gamma)-1}(x)\}$.

Theorem (Gao-Z.[10])

(1) Similar results hold for overconvergent and Robba ring coefficients.

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[9] L. Zhao, Herr complex of (φ,τ)-modules, JNT. **271**.

[10] H. Gao, L. Zhao, Cohomology of (φ,τ)-modules, Advances in Math. **476**.

(2) Let $\nabla_\tau := (\log \tau^{p^n})/p^n$ for $n \gg 0$ be the Lie-algebra operator with respect to the τ-action, and define

$$N_\nabla := \frac{1}{p\mathfrak{t}} \cdot \nabla_\tau$$

where $\mathfrak{t}$ is a certain "normalizing" element. Then

$$\Big[\mathbf{D}^\dagger_{\mathrm{rig},K_\infty} \xrightarrow{(\varphi-1,\, N_\nabla)} \mathbf{D}^\dagger_{\mathrm{rig},K_\infty} \oplus \mathbf{D}^\dagger_{\mathrm{rig},K_\infty} \xrightarrow{(-N_\nabla,\, \frac{pE(u)}{E(0)}(\varphi-1))} \mathbf{D}^\dagger_{\mathrm{rig},K_\infty}$$

computes $\varinjlim_n H^i(G_{K(\pi_n)}, V) = \bigcup_n H^i(G_{K(\pi_n)}, V)$.

$$\Big[\mathbf{D}^\dagger_{\mathrm{rig},K_{p^\infty}} \xrightarrow{(\varphi-1,\, \nabla_\gamma)} \mathbf{D}^\dagger_{\mathrm{rig},K_{p^\infty}} \oplus \mathbf{D}^\dagger_{\mathrm{rig},K_{p^\infty}} \xrightarrow{(-\nabla_\gamma,\, \varphi-1)} \mathbf{D}^\dagger_{\mathrm{rig},K_{p^\infty}}\Big].$$

computes $\varinjlim_n H^i(G_{K(\mu_n)}, V) = \bigcup_n H^i(G_{K(\mu_n)}, V)$.

Definition (Du-Z.)

A **(effective) prismatic** $(\varphi, \hat{G})$**-modules** is a triple $(\mathfrak{M}, \varphi_{\mathfrak{M}}, \tau_{\mathfrak{M}})$:

(1) $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ is a Breuil-Kisin module.

(2) $\tau_{\mathfrak{M}}$ a τ-semilinear endomorphism of $\widehat{\mathfrak{M}} := A_{\mathrm{st}}^{(2)} \otimes_{\mathfrak{S}} \mathfrak{M}$ whose linearization is a φ-equivariant isomorphism:

$$1 \otimes \tau_{\mathfrak{M}} : A_{\mathrm{st}}^{(2)} \otimes_{\tau, \mathfrak{S}} \mathfrak{M} \longrightarrow \widehat{\mathfrak{M}}$$

such that

$$\gamma \circ \tau_{\mathfrak{M}} = \tau_{\mathfrak{M}}^{\chi(g)} \text{ over } \mathfrak{M} \subset \widehat{\mathfrak{M}}.$$

It is called **semistable** if it satisfies

$$(\tau_{\mathfrak{M}} - 1_{\mathfrak{M}})(\mathfrak{M}) \subset \left(A_{\mathrm{st}}^{(2)} \cap W(\mathfrak{m}_{C_p}^{b})\right) \otimes_{\mathfrak{S}} \mathfrak{M},$$

A semistable one is called **crystalline** if

$$\tau_{\mathfrak{M}}(\mathfrak{M}) \subset A^{(2)} \otimes_{\mathfrak{S}} \mathfrak{M}.$$

Theorem (Du-Z. in progress)

We have category equivalences

$$\mathrm{Mod}^{\mathrm{cris}}_{\varphi,\tau}(\mathfrak{S}, A^{(2)}) \simeq \mathrm{Rep}^{\mathrm{cris},\geq 0}_{\mathbf{Z}_p}(G_K)$$

and

$$\mathrm{Mod}^{\mathrm{st}}_{\varphi,\tau}(\mathfrak{S}, A^{(2)}_{\mathrm{st}}) \simeq \mathrm{Rep}^{\mathrm{st},\geq 0}_{\mathbf{Z}_p}(G_K).$$

Theorem (Du-Z. in progress)

(1) Let $T \in \mathrm{Rep}^{\mathrm{cris},\geq 0}_{\mathbf{Z}_p}(G_K)$ and $\mathfrak{M}$ the corresponding prismatic $(\varphi, \hat{G})$-module. Define

$$(A^{(2)} \otimes_{\mathfrak{S}} \mathfrak{M})_0 := \{x \in A^{(2)} \otimes_{\mathfrak{S}} \mathfrak{M};\ \gamma(x) = (1 + \tau_{\mathfrak{M}} + \cdots + \tau_{\mathfrak{M}}^{\chi(\gamma)-1})(x)\}$$

and similarly $(A^{(2)} \otimes_{\mathfrak{S}} \mathfrak{M})_0$. Then we have

$$\left[\mathfrak{M} \xrightarrow{\varphi-1, \tau_{\mathfrak{M}}-1} \mathfrak{M} \oplus (A^{(2)} \otimes_{\mathfrak{S}} \mathfrak{M})_0 \xrightarrow{\tau_{\mathfrak{M}}-1 \ominus \varphi-1} (A^{(2)} \otimes_{\mathfrak{S}} \mathfrak{M})_0\right] [1/p]$$

computes $H^i_f(G_K, T[1/p])$ after inverting p.

Let $T \in \mathrm{Rep}^{\mathrm{st},\geq 0}_{\mathbf{Z}_p}(G_K)$ and $\mathfrak{M}$ the corresponding prismatic $(\varphi, \hat{G})$-module.

$$\left[\mathfrak{M} \xrightarrow{\varphi-1, \tau_{\mathfrak{M}}-1} \mathfrak{M} \oplus (A^{(2)}_{\mathrm{st}} \otimes_{\mathfrak{S}} \mathfrak{M})_0 \xrightarrow{\tau_{\mathfrak{M}}-1 \ominus \varphi-1} (A^{(2)}_{\mathrm{st}} \otimes_{\mathfrak{S}} \mathfrak{M})_0\right]$$

computes $H^i_g(G_K, T[1/p])$ after inverting p.

(2) Let $T \in \mathrm{Rep}^{\mathrm{cris},\geq 0}_{\mathbf{Z}_p}(G_K)$ and $\mathfrak{M}$ the corresponding prismatic $(\varphi, \hat{G})$-module. We have

$$\left[\mathfrak{M} \xrightarrow{\tau_{\mathfrak{M}}-1} (A^{(2)} \otimes_{\mathfrak{S}} \mathfrak{M})_0\right] \simeq R\Gamma((\mathcal{O}_K)_{\Delta}, \mathfrak{M}_{\Delta}).$$

and taking Frobenius mapping fiber computes the cohomology for the corresponding F-gauge

$$\left[\mathfrak{M} \xrightarrow{\tau_{\mathfrak{M}}-1} (A^{(2)} \otimes_{\mathfrak{S}} \mathfrak{M})_0\right]^{\varphi=1} \simeq R\Gamma((\mathcal{O}_K)^{\mathrm{syn}}, \mathcal{E}(\mathfrak{M}_{\Delta})).$$

Remark

A. Abhinandan constructed a 3-term complex using Wach modules that computes $H^i_f(G_K, T[1/p])$ for $T \in \mathrm{Rep}^{\mathrm{cris}}_{\mathbf{Z}_p}(G_K)$ when $K/\mathbf{Q}_p$ is unramified.

Thank you!