Syntomic cohomology and arithmetic applications

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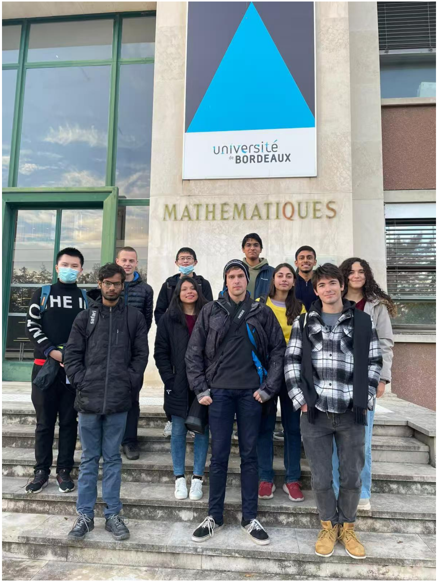

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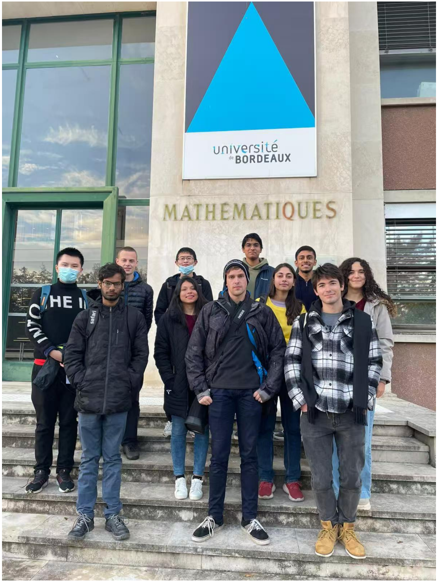

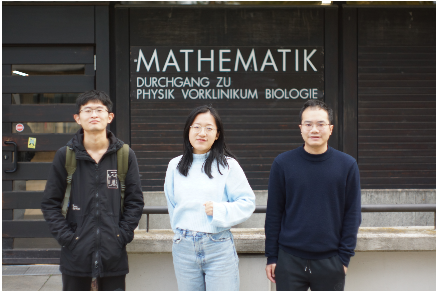

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Definition (Fontaine-Messing)

Let X be a smooth variety over $\mathbb{F}_p$, the syntomic complex of X is

$$\mathrm{Syn}(X,r) := [\mathrm{Fil}^n_N R\Gamma_{\mathrm{crys}}(X/\mathbb{Z}_p) \xrightarrow{1-\varphi/p^r} R\Gamma_{\mathrm{crys}}(X/\mathbb{Z}_p)]$$

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'Cut the cohomology using the filtration and Frobenius together'.

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Let X be a smooth variety over $\mathbb{F}_p$, there is an exact sequence in X_{syn}

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Theorem (Fontaine-Messing, Bloch-Kato)

Let $\mathfrak{X}$ be a smooth p-adic formal scheme over $\mathbb{Z}_p$. For $i \leq r$, we have isomorphisms

$$H^i_{\text{ét}}(\mathfrak{X}_\eta, \mathbb{Q}_p(r)) \cong H^i_{\text{syn}}(\mathfrak{X}_s, r)_{\mathbb{Q}}$$

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This gives a crystalline study of p-adic étale cohomology.

Link with flat cohomology

Let k be a finitely generated in char $p > 0$ and $f : X \to \operatorname{Spec} k$ be a projective smooth variety.

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Theorem (L-Qin)

Let G be a p-divisible group over X. Then $R^i f_{\mathrm{fppf}*} G$ is isogeneous to a p-divisible group H^i. Moreover, we have a natural isomorphism of F-isocrystals:

$$\mathcal{M}^{cr}(H^i)_{\mathbb{Q}} \cong R^i f_{\mathrm{crys}\,*} \mathcal{M}^{cr}(G)_{\mathbb{Q},[0,1]}$$

where $\mathcal{M}^{cr}(-)$ is the covariant crystalline Dieudonne functor.

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Theorem (L-Qin)

The finiteness of exponent of $\mathrm{Br}(X_{k^s})^{G_k}[p^\infty]$ is equivalent to the Tate conjecture $T^1(X,l)$ for X.

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Example

Let $\mathbb{D}_C$ be the one dimensional open unit disk over C.

$$H^n_{\text{proét}}(\mathbb{D}_C, \mathbb{Q}_p) = \begin{cases} \mathbb{Q}_p & n = 0 \\ \mathcal{O}(\mathbb{D}_C)/C(-1) & n = 1 \\ 0 & others \end{cases}$$

$$H^n_{\text{proét},c}(\mathbb{D}_C, \mathbb{Q}_p) \begin{cases} \mathcal{O}(\partial\mathbb{D}_C)/\mathcal{O}(\mathbb{D}_C)(-1) \oplus \mathbb{Q}_p(-1) & n = 2 \\ 0 & others \end{cases}$$

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There is no naive Poincare duality by degree reason.

Link with p-adic étale cohomology

Using the syntomic cohomology, one can attach a (quasi-coherent) sheaf $\mathcal{E}_{\text{proét}}(X, r)$ over the Fargues-Fontaine curve which 'represents' the pro-étale cohomology. Colmez-Gilles-Niziol proved a 'Poincare duality on the Fargues-Fontaine curve':

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Theorem (CGN)

Let X be a smooth connected Stein space over K of dimension d. Assume $r, r' \geq 2d$ and $s = r + r' - d$. Then there is a natural quasi-isomorphism of complexes of G_K-equivariant sheaves

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But there is a genuiue duality for arithmetic cohomology

Theorem (L)

Let X be a smooth partially proper variety over K where $[K, \mathbb{Q}_p] < \infty$.

1. *There is a quasi-isomorphism in $D(\mathbb{Q}_{p,\square})$ (solid $\mathbb{Q}_p$-vector spaces) induced by the arithmetic trace map*

$$\gamma_{\text{proét}} : R\Gamma_{\text{proét}}(X, \mathbb{Q}_p(j)) \xrightarrow{\simeq} \mathbb{D}_{\mathbb{Q}_p}(R\Gamma_{\text{proét},c}(X, \mathbb{Q}_p(d+1-j))[2d+2])$$

where $\mathbb{D}_{\mathbb{Q}_p}(-) := R\,\mathrm{Hom}_{\mathbb{Q}_p}(-, \mathbb{Q}_p)$.

2. *Assume further that X is a Stein space. Then the cohomology and compactly supported cohomology groups are classical. Moreover $\gamma_{\text{proét}}$ induces isomorphisms of topological vector spaces*

$$H^i_{\text{proét}}(X, \mathbb{Q}_p(j)) \simeq H^{2d+2-i}_{\text{proét},c}(X, \mathbb{Q}_p(d+1-j))^*,$$

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Thank you!