Categorical Local Langlands Conjecture

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Part A.

Local Langlands Conjecture

◇ $W_{\mathbf{Q}_p}$ is the Weil group of $\mathbf{Q}_p$, i.e.,

$$\begin{array}{ccc} \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) & \twoheadrightarrow & \mathrm{Gal}(\mathbf{Q}_p^{\mathrm{nr}}/\mathbf{Q}_p) \simeq \widehat{\mathbf{Z}} \\ & & \| \\ \cup & & \overline{\langle \mathrm{Frob}_p \rangle} \\ & & \cup \\ W_{\mathbf{Q}_p} & \dashrightarrow & \langle \mathrm{Frob}_p \rangle \simeq \mathbf{Z} \end{array}$$

◇ ℓ is a prime distinct to p, and we fix an isomorphism $\overline{\mathbf{Q}}_\ell \simeq \mathbf{C}$.

◇ G is a connected quasi-split reductive group over $\mathbf{Q}_p$.

◇ $\widehat{G}$ is the Langlands dual group of G over $\overline{\mathbf{Q}}_\ell$, i.e., the connected split reductive group over $\overline{\mathbf{Q}}_\ell$ whose root datum is dual to that of $G_{\overline{\mathbf{Q}}_p}$.

Recall that **Local Artin Reciprocity** asserts an isomorphism of topological groups $\mathbf{Q}_p^\times \xrightarrow{\sim} W_{\mathbf{Q}_p}^{\mathrm{ab}}$.

- A **smooth representation** of $G(\mathbf{Q}_p)$ is a $\overline{\mathbf{Q}}_\ell$-vector space V together with a continuous $G(\mathbf{Q}_p)$-action $G(\mathbf{Q}_p) \times V \to V$.
- An ***L*-parameter** is a homomorphism $W_{\mathbf{Q}_p} \times \mathrm{SL}_2(\overline{\mathbf{Q}}_\ell) \to \widehat{G}(\overline{\mathbf{Q}}_\ell) \rtimes W_{\mathbf{Q}_p}$ of groups satisfying certain conditions.

Conjecture (Local Langlands Conjecture)

There exists a **surjective map with finite fibre**

$$\{\textit{isom. classes of sm. rep. of } G(\mathbf{Q}_p)\} \longrightarrow \{\textit{conj. classes of L-para.}\}$$

satisfying some properties.

Want to obtain: (1) a bijection
(2) sets $\rightsquigarrow$ categories

Part B.

Two Ideas from Geometric Representation Theory

The First Idea

Representations should be realized as **sheaves**. For example,

$$\mathrm{Rep}(G) \cong \mathrm{QCoh}([*/G]).$$

$$(\text{rep. of a grp.}) \cong (\text{shv. on a geom. obj.}).$$

In p-adic geometry (or say Local Langlands), we have

$$\mathcal{D}(G(\mathbf{Q}_p), \overline{\mathbf{Q}}_\ell) \cong \mathcal{D}_{\mathrm{lis}}([*/\underline{G(\mathbf{Q}_p)}], \overline{\mathbf{Q}}_\ell),$$

where $\mathcal{D}(G(\mathbf{Q}_p), \overline{\mathbf{Q}}_\ell)$ is the derived category of smooth representations of $G(\mathbf{Q}_p)$ over $\overline{\mathbf{Q}}_\ell$.

The Second Idea

If X is a complete smooth curve over $\mathbf{C}$, then we have the Betti Geometric Langlands correspondence

$$\mathrm{Shv}^{\mathrm{Betti}}_{\frac{1}{2},\mathrm{Nilp}}(\mathrm{Bun}_{G,X}) \xrightarrow{\sim} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}^{\mathrm{Betti}}_{\widehat{G}}).$$

Need to find (1) suitable geometric objects,
(2) suitable sheaf theories.

This inspires the Categorical Local Langlands Conjecture

$$\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbf{Q}}_\ell) \simeq \mathrm{IndCoh}(\mathrm{Par}_G).$$

Fargues–Scholze program realizes Local Langlands Program as Geometric Langalnds Program on Fargues–Fontaine curves.

Part C.

Sheaf Theories

- $\breve{\mathbf{Q}}_p := \widehat{\mathbf{Q}_p^{\mathrm{nr}}}$, the completion of the maximal unramified extension.
- $\sigma_{\mathbf{Q}_p} : \breve{\mathbf{Q}}_p \to \breve{\mathbf{Q}}_p$ the geometric Frobenius.
- $B(G) := G(\breve{\mathbf{Q}}_p)/\sigma_{\mathbf{Q}_p}$-conjugation (which can be endowed with a topology).

Automorphic side: the geometric object

Bun_G is the moduli stack of G-bundles over Fargues–Fontaine curves.

(a) Bun_G is an Artin v-stack, and $|\mathrm{Bun}_G| \cong B(G)$ topologically.

(b) Harder–Narasimhan stratification

$$\mathrm{Bun}_G = \coprod_{b \in B(G)} \mathrm{Bun}_G^b,$$

where each Bun_G^b is a locally closed v-substack of Bun_G.

- We can associate with each $b \in B(G)$ a reductive group G_b over E.
- G_b is an inner from of some Levi of G.
  (For example, if $b = 1$, then $G_1 = G$.)

Automorphic side: the sheaf theory

For any Artin v-stack X

$$\mathcal{D}_{\mathrm{lis}}(X, \overline{\mathbf{Q}}_\ell) = \text{the category of lisse-étale } \overline{\mathbf{Q}}_\ell\text{-sheaves on } X.$$

(a) $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G^b, \overline{\mathbf{Q}}_\ell) \simeq \mathcal{D}_{\mathrm{lis}}([*/\underline{G_b(\mathbf{Q}_p)}], \overline{\mathbf{Q}}_\ell) \simeq \mathcal{D}(G_b(\mathbf{Q}_p), \overline{\mathbf{Q}}_\ell)$, the derived category of smooth representations of $G_b(E)$.

(b) $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbf{Q}}_\ell)$ admits a semiorthogonal decomposition into $\mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G^b, \overline{\mathbf{Q}}_\ell)$'s.

Spectral side: the geometric object

Par_G is the moduli stack of ℓ-adically continuous L-parameters, and it is a QCA (derived) Artin stack.

Spectral side: the sheaf theory

$\mathrm{IndCoh}(\mathrm{Par}_G) = \mathrm{Ind}(\mathrm{Coh}(\mathrm{Par}_G))$.

Note that

$$\mathrm{Coh}(\mathrm{Par}_G) \subset \mathrm{QCoh}(\mathrm{Par}_G) \subset \mathrm{IndCoh}(\mathrm{Par}_G).$$

Part D.

Categorical Local Langlands Conjecture

- Fix a Whittaker datum (B, ψ), where $B = M \ltimes U \subset G$ a Borel and ψ a non-degenerate character on $U(\mathbf{Q}_p)$.
- $W_\psi := \text{c-Ind}_{U(\mathbf{Q}_p)}^{G(\mathbf{Q}_p)} \psi \in \mathcal{D}(G(\mathbf{Q}_p), \overline{\mathbf{Q}}_\ell)$ the Whittaker representation.
- $i_1 : \mathrm{Bun}_G^1 \hookrightarrow \mathrm{Bun}_G$ induces

$$\begin{tikzcd} \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G^1, \overline{\mathbf{Q}}_\ell) \arrow[r, "i_{1!}"] \arrow[d, phantom, "\wr"] & \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbf{Q}}_\ell) \\ \mathcal{D}(G(\mathbf{Q}_p), \overline{\mathbf{Q}}_\ell) & \end{tikzcd}$$

$$W_\psi \longmapsto i_{1!} W_\psi.$$

Theorem

There is a functor, which is called the **spectral action**,

$$\mathrm{QCoh}(\mathrm{Par}_G) \times \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbf{Q}}_\ell) \longrightarrow \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbf{Q}}_\ell), \quad (\mathscr{F}, A) \longmapsto \mathscr{F} * A.$$

The functor

$$a_\psi : \mathrm{QCoh}(\mathrm{Par}_G) \longrightarrow \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbf{Q}}_\ell), \quad \mathscr{F} \longmapsto \mathscr{F} * i_{1!} W_\psi$$

admits a right adjoint $c_\psi : \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbf{Q}}_\ell) \to \mathrm{QCoh}(\mathrm{Par}_G)$.

Conjecture (Categorical Local Langlands Conjecture)

The functor c_ψ induces an equivalence of symmetric monoidal stable ∞-categories

$$\mathbf{L}_\psi : \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbf{Q}}_\ell) \xrightarrow{\sim} \mathrm{QCoh}(\mathrm{Par}_G).$$

Thank You!