A Synthetic Approach for Computing Equivariant Slice Differentials

Yuchen Wu

University of California San Diego

December 2025

Plan for the talk

- Theory: **synthetic methods** for manipulating differentials in SS.
- Application: computing differentials in the **equivariant slice SS**.

Synthetic methods

Theorem (Kervaire invariant problem for $j = 6$, Lin–Wang–Xu)

There exists a closed framed manifold of dimension $2^6 - 2 = 126$ that cannot be converted to a homotopy sphere via framed surgery.

Synthetic methods

Theorem (Kervaire invariant problem for $j = 6$, Lin–Wang–Xu)

There exists a closed framed manifold of dimension $2^6 - 2 = 126$ that cannot be converted to a homotopy sphere via framed surgery.

Equivalently,

Theorem (Kervaire invariant problem for $j = 6$, Lin–Wang–Xu)

In $\mathrm{AdamsSS}(S^0)$, $h_6^2 \in E_2^{126,2}$ is a permanent cycle.

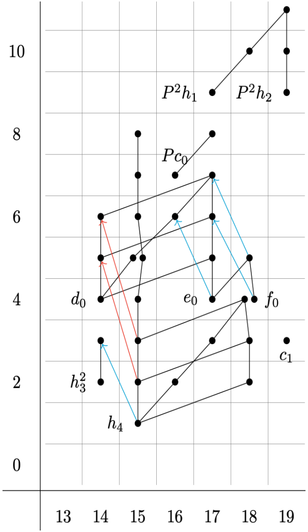


Figure: AdamsSS of S^0

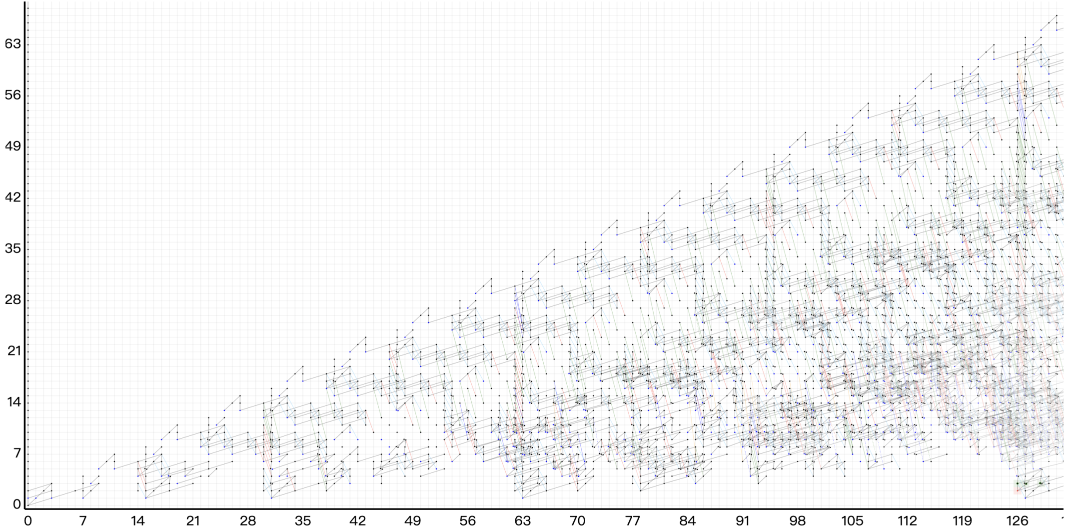


Figure: AdamsSS of S^0

Synthetic methods

Key ingredient: **hidden extensions**.

Take $X \to Y$ a map between finite CW complexes, consider Adams SS:

$$\begin{array}{ccc} a \in E_2^{n,s}(X) & \longrightarrow & b \in E_2^{n,s}(Y) \\ \rightsquigarrow & & \rightsquigarrow \\ \alpha \in \pi_n^{\mathrm{st}}(X) & \longrightarrow & \beta \in \pi_n^{\mathrm{st}}(Y) \end{array}$$

$$\begin{array}{ccc} a \in E_2^{n,s}(X) & \longrightarrow & 0 \in E_2^{n,s}(Y) \\ \rightsquigarrow & \searrow \ \text{jump} = k & \\ & & b \in E_2^{n,s+k}(Y) \\ & & \rightsquigarrow \\ \alpha \in \pi_n^{\mathrm{st}}(X) & \longrightarrow & \beta \in \pi_n^{\mathrm{st}}(Y) \end{array}$$

For $f\colon X \to Y$ map between finite CW complexes, Lin, Wang and Xu establish

- generalized Mahowald trick:
  - translation between "extensions along f" and "diffs in $\mathrm{AdamsSS}(Cf)$"
- generalized Leibniz rule:
  - short diffs in X + extensions along f $\rightsquigarrow$ long diffs in Y

These provide enough information of $\mathrm{AdamsSS}(S^0)$ around $n = 126$.

To extend Lin–Wang–Xu's results beyond Adams-type SS, we use filtered spectra.

Roughly speaking, $\mathsf{Sp} = \mathsf{D}(R)$ where $R = \mathbb{S}$ is the sphere spectrum.

Definition (Filtered spectra)

$$\mathsf{FilSp} = \mathsf{Fun}(\mathbb{Z}_{\mathsf{poset}}, \mathsf{Sp}) = \{X\colon \cdots \to X(2) \to X(1) \to X(0) \to X(-1) \to \cdots\}.$$

To extend Lin–Wang–Xu's results beyond Adams-type SS, we use filtered spectra.

Roughly speaking, $\mathsf{Sp} = \mathsf{D}(R)$ where $R = \mathbb{S}$ is the sphere spectrum.

Definition (Filtered spectra)

$$\mathsf{FilSp} = \mathsf{Fun}(\mathbb{Z}_{\mathsf{poset}}, \mathsf{Sp}) = \{X\colon \cdots \to X(2) \to X(1) \to X(0) \to X(-1) \to \cdots\}.$$

For each $X \in \mathsf{FilSp}$, there is an associated SS $\{E_r^{n,s}(X)\}$ with

$$E_2^{n,s}(X) = \pi_n(X(n+s)/X(n+s+1)), \quad |d_r| = (-1, r).$$

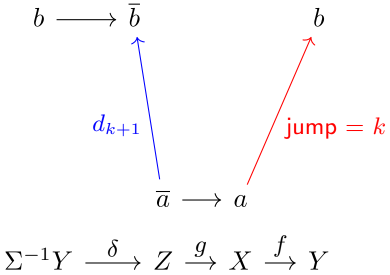
We can make sense of hidden extensions in this context.

Generalized Mahowald trick

We can translate between extensions and differentials.

Theorem (gen. Mahowald trick, Lin–Wang–Xu, W.)

Suppose $Z \xrightarrow{g} X \xrightarrow{f} Y$ *is a fiber sequences in* FilSp. *There is a correspondence between* *f-extension with filtration jump k* *and* *d_{k+1} differential in $E_\star^{*,*}(Z)$*.



Generalized Leibniz rule

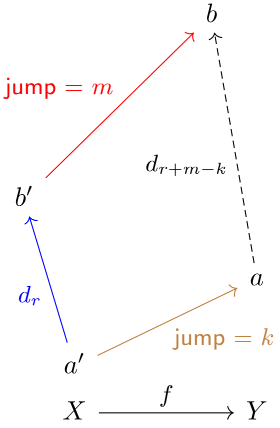
Short differentials and extensions can be merged to produce long differentials.

Theorem (gen. Leibniz rule, Lin–Wang–Xu, W.)

Let $f\colon X \to Y$ *be a map in* FilSp. *For* $a', b' \in E_2^{*,*}(X)$ *and* $a, b \in E_2^{*,*}(Y)$

1. $d_r(a') = b'$ *in* $E_\star^{*,*}(X)$.
2. *There is an* f*-extension from* a' *to* a *with filtration jump* k.
3. *There is an* f*-extension from* b' *to* b *with filtration jump* m.

We have $d_{r+m-k}(a) = b$ *in* $E_\star^{*,*}(Y)$.



Equivariant slice SS

We use these tools to study $H\,\mathrm{SliceSS}(\mathrm{BP}^{((G))}\langle m\rangle)$, here

- $G = C_{2^n}$, $H \subset G$.
- $m \in \mathbb{Z}_{\geq 1}$.
- $\mathrm{BP}^{((G))}\langle m\rangle$: **Hill–Hopkins–Ravenel theories**. We write $\mathrm{BP}^{((C_2))} = \mathrm{BP}_{\mathbb{R}}$.
- $H\,\mathrm{SliceSS}$: **equivariant slice spectral sequence**.

Equivariant slice SS

We use these tools to study $H\,\mathrm{SliceSS}(\mathrm{BP}^{((G))}\langle m\rangle)$, here

- $G = C_{2^n}$, $H \subset G$.
- $m \in \mathbb{Z}_{\geq 1}$.
- $\mathrm{BP}^{((G))}\langle m\rangle$: **Hill–Hopkins–Ravenel theories**. We write $\mathrm{BP}^{((C_2))} = \mathrm{BP}_{\mathbb{R}}$.
- $H\,\mathrm{SliceSS}$: **equivariant slice spectral sequence**.

Definitions are complicated.

Motivations:

- Geometric topology: partial computation of $C_8\,\mathrm{SliceSS}(\mathrm{BP}^{((C_8))}\langle 1\rangle)$ solves Kervaire invariant problem (in negative) for $j \geq 7$.
- Chromatic homotopy: For $h = 2^{n-1}m$, $\mathrm{BP}^{((C_{2^n}))}\langle m\rangle$ is a "model" for Lubin–Tate theory E_h with C_{2^n} action. Conjecturally, combining all these data together yields $\pi_*(\mathbb{S})_{\widehat{2}}$.
- Arithmetic geometry: "Higher height analog" of crystalline / prismatic cohomology.

For $G = C_2$, all $C_2\,\mathrm{SliceSS}(\mathrm{BP}_{\mathbb{R}}\langle m\rangle)$ are fully understood.

The E_2 page of $C_2\,\mathrm{SliceSS}(\mathrm{BP}_{\mathbb{R}}\langle m\rangle)$ is $H\underline{\mathbb{Z}}_\star[\bar{t}_1, \dots, \bar{t}_m]$, where $H\underline{\mathbb{Z}}_\star = \mathbb{Z}[a_\sigma, u_{2\sigma}]/(2a_\sigma) + \text{negative cone}.$

All diffs are generated by $d_3(u_{2\sigma}), d_7(u_{4\sigma}), d_{15}(u_{8\sigma}), \dots, d_{2^{m+1}-1}(u_{2^m\sigma})$.

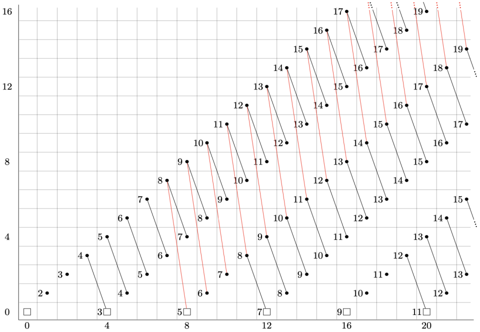


Figure: C_2 SliceSS of $\mathrm{BP}_{\mathbb{R}}\langle 2\rangle$

For $G = C_4$ things get quite involved:

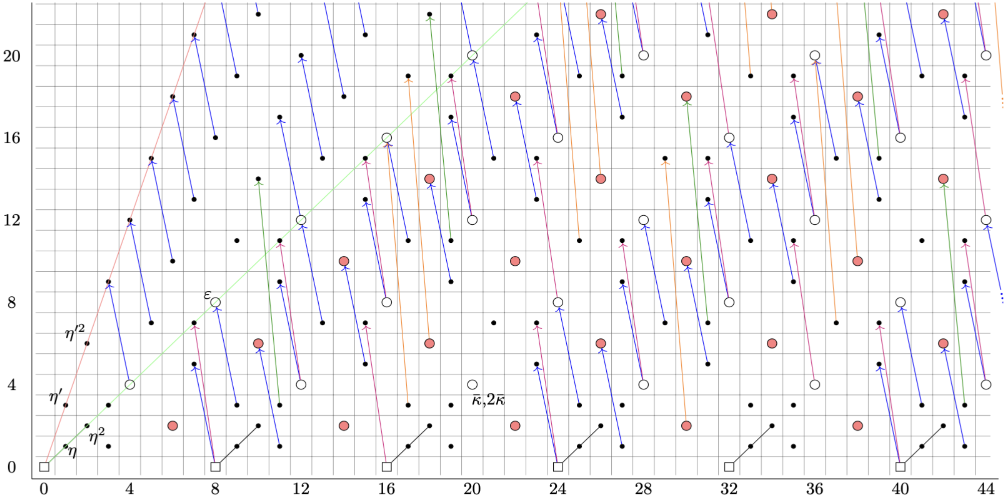


Figure: C_4 SliceSS of $\mathrm{BP}^{((C_4))}\langle 1\rangle$

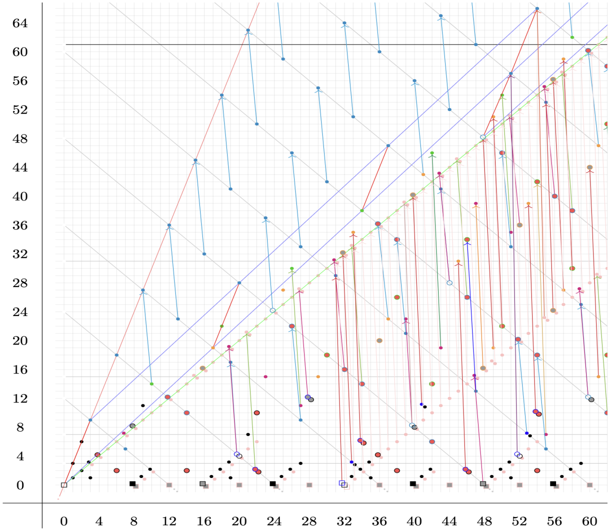


Figure: C_4 SliceSS of $\mathrm{BP}^{((C_4))}\langle 2\rangle$

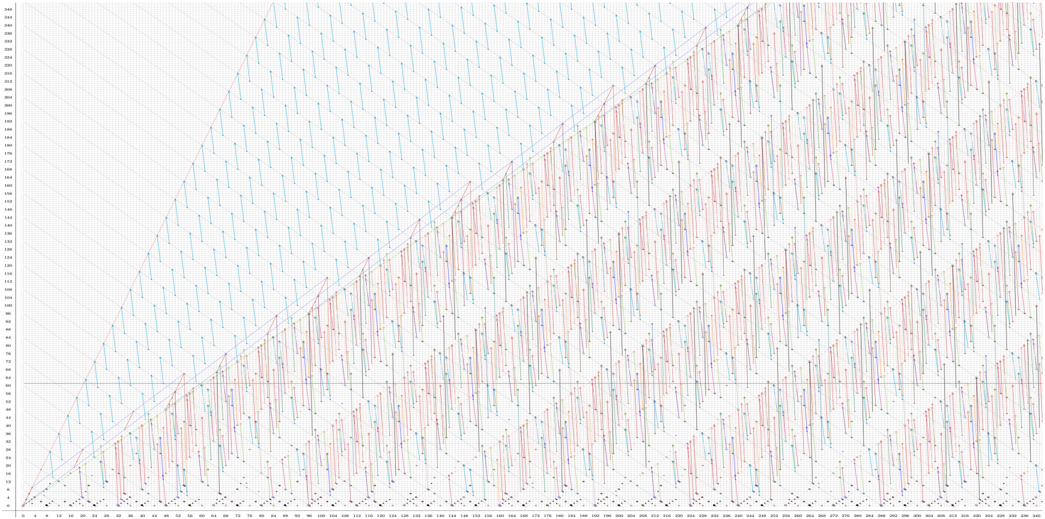


Figure: C_4 SliceSS of $\mathrm{BP}^{((C_4))}\langle 2\rangle$

We work with $X = \mathrm{BP}^{((C_4))}\langle m \rangle$ for all $m \in \mathbb{Z}_{\geq 1}$.

We fully understand $C_2 \,\mathrm{SliceSS}(\mathrm{BP}_{\mathbb{R}}\langle m \rangle)$. Its differentials are generated by $d_3(u_{2\sigma}), d_7(u_{4\sigma}), d_{15}(u_{8\sigma}), \ldots, d_{2^{m+1}-1}(u_{2^m\sigma})$.

We work with $X = \mathrm{BP}^{((C_4))}\langle m\rangle$ for all $m \in \mathbb{Z}_{\geq 1}$.

We fully understand $C_2\,\mathrm{SliceSS}(\mathrm{BP}_{\mathbb{R}}\langle m\rangle)$. Its differentials are generated by $d_3(u_{2\sigma}), d_7(u_{4\sigma}), d_{15}(u_{8\sigma}), \ldots, d_{2^{m+1}-1}(u_{2^m\sigma})$.

$\rightsquigarrow$ Two sources for short diffs in $C_4\,\mathrm{SliceSS}(\mathrm{BP}^{((C_4))}\langle m\rangle)$:

1. Transfer differentials: $d_r \leftrightsquigarrow d_r$ between $C_2\,\mathrm{SliceSS}(\mathrm{BP}_{\mathbb{R}}\langle 2m\rangle)$ and $C_2\,\mathrm{SliceSS}(\mathrm{BP}^{((C_4))}\langle m\rangle)$.
2. Sheared differentials: $d_r \leftrightsquigarrow d_{2r-1}$ between $C_2\,\mathrm{SliceSS}(\mathrm{BP}_{\mathbb{R}}\langle m\rangle)$ and the region of $C_4\,\mathrm{SliceSS}(\mathrm{BP}^{((C_4))}\langle m\rangle)$ above the line of slope 1.

Using GLR and GMT, we can “splice together” the h-th transfer differential $(1 \leq h \leq 2m)$ and the q-th sheared differentials $(1 \leq q \leq m)$ to get new families of long differentials.

Using GLR and GMT, we can "splice together" the h-th transfer differential $(1 \leq h \leq 2m)$ and the q-th sheared differentials $(1 \leq q \leq m)$ to get new families of long differentials.

Theorem (Exotic transfer paradigm, W.)

Suppose for $1 \leq h \leq 2m$, there exists a homogeneous polynomial P and two classes x, c in the E_2-page of the C_2-slice SS of $\mathrm{BP}^{((C_4))}\langle m \rangle$ so that

$$x \cdot \bar{v}_h = P(\bar{t}_1 \gamma \bar{t}_1, \ldots, \bar{t}_{2^m-1} \gamma \bar{t}_{2^m-1}) + c + \gamma(c) \mod (2, \bar{v}_1, \ldots, \bar{v}_{h-1}).$$

Then for each $1 \leq q \leq m$ and $j \geq 0$, there is a differential in the C_4-slice SS of $\mathrm{BP}^{((C_4))}\langle m \rangle$

$$d_{2^{h+1}+2^{q+2}-5}(\mathrm{tr}(x a_{\sigma_2}) u_{2^{h-1}\lambda_1} u_{k\sigma}) = P(\bar{\mathfrak{d}}_1, \ldots, \bar{\mathfrak{d}}_{2^m-1}) \bar{\mathfrak{d}}_{2^q-1} u_{2^{q+1}j\sigma} a_{(2^h+2^q-1)\lambda_1} a_{(2^{q+1}-2)\sigma}$$

here $k = 2^{q+1}j + 2^q - p - 1$.

Corollary (Exotic transfer differentials)

There is a family of C_4-slice differentials for $\mathrm{BP}^{((C_4))}\langle m\rangle$:

$$\begin{aligned} & d_{2^{2m+1}+2^{m+2}-5}\big(\operatorname{tr}(\bar{t}_{2^m-1}^{2^m-1} a_{\sigma_2}) u_{2^{2m-1}\lambda_1} u_{(2^{m+1}j+2^{m+1}-2^{2m}-1)\sigma}\big) \\ = \; & \bar{\mathfrak{d}}_{2^m-1}^{2^m+1} u_{2^{m+1}j\sigma} a_{(2^{2m}+2^m-1)\lambda_1} a_{(2^{m+1}-2)\sigma} \end{aligned}$$

from splicing the longest transfer diff from $\mathrm{BP}_{\mathbb{R}}\langle 2n\rangle$ and the longest sheared diff from $\mathrm{BP}_{\mathbb{R}}\langle n\rangle$;

Corollary (Exotic transfer differentials)

There is a family of C_4-slice differentials for $\mathrm{BP}^{((C_4))}\langle m\rangle$:

$$d_{2^{2m+1}+2^{m+2}-5}\big(\operatorname{tr}(\bar{t}_{2^m-1}^{2^m-1}a_{\sigma_2})u_{2^{2m-1}\lambda_1}u_{(2^{m+1}j+2^{m+1}-2^{2m}-1)\sigma}\big) = \bar{\mathfrak{d}}_{2^m-1}^{2^m+1}u_{2^{m+1}j\sigma}a_{(2^{2m}+2^m-1)\lambda_1}a_{(2^{m+1}-2)\sigma}$$

from splicing the longest transfer diff from $\mathrm{BP}_{\mathbb{R}}\langle 2n\rangle$ and the longest sheared diff from $\mathrm{BP}_{\mathbb{R}}\langle n\rangle$;

If $2 \leq m \leq 4$, there is another family of C_4-slice differentials for $\mathrm{BP}^{((C_4))}\langle m\rangle$:

$$d_{2^{2m}+2^{m+1}-5}\big(\operatorname{tr}(\bar{t}_1 a_{\sigma_2})u_{2^{2m-2}\lambda_1}u_{(2^m j+2^{m-1}-2^{2m-2}-1)\sigma}\big) = \bar{\mathfrak{d}}_{2^{m-1}-1}^{2^{m-1}}\bar{\mathfrak{d}}_{2^m-1}u_{2^m j\sigma}a_{(2^{2m-1}+2^{m-1}-1)\lambda_1}a_{(2^m-2)\sigma}$$

coming from the second longest pair.

Corollary (Exotic transfer differentials)

There is a family of C_4-slice differentials for $\mathrm{BP}^{((C_4))}\langle m \rangle$:

$$d_{2^{2m+1}+2^{m+2}-5}\big(\operatorname{tr}(\bar{t}_{2^m-1}^{2^m-1} a_{\sigma_2}) u_{2^{2m-1}\lambda_1} u_{(2^{m+1}j+2^{m+1}-2^{2m}-1)\sigma}\big) = \bar{\mathfrak{d}}_{2^m-1}^{2^m+1} u_{2^{m+1}j\sigma} a_{(2^{2m}+2^m-1)\lambda_1} a_{(2^{m+1}-2)\sigma}$$

from splicing the longest transfer diff from $\mathrm{BP}_{\mathbb{R}}\langle 2n \rangle$ and the longest sheared diff from $\mathrm{BP}_{\mathbb{R}}\langle n \rangle$;

If $2 \leq m \leq 4$, there is another family of C_4-slice differentials for $\mathrm{BP}^{((C_4))}\langle m \rangle$:

$$d_{2^{2m}+2^{m+1}-5}\big(\operatorname{tr}(\bar{t}_1 a_{\sigma_2}) u_{2^{2m-2}\lambda_1} u_{(2^m j+2^{m-1}-2^{2m-2}-1)\sigma}\big) = \bar{\mathfrak{d}}_{2^{m-1}-1}^{2^{m-1}} \bar{\mathfrak{d}}_{2^m-1} u_{2^m j\sigma} a_{(2^{2m-1}+2^{m-1}-1)\lambda_1} a_{(2^m-2)\sigma}$$

coming from the second longest pair.

In known cases, this recovers d_{11} in $C_4\,\mathrm{SliceSS}(\mathrm{BP}^{((C_4))}\langle 1 \rangle)$ as well as d_{43} and d_{19} in $C_4\,\mathrm{SliceSS}(\mathrm{BP}^{((C_4))}\langle 2 \rangle)$, difficult to obtain before.

Thanks!