I came, I saw, and I counted

A line counting story on smooth cubic surfaces

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Table of Contents

Enumerative geometry: what is it?

- Enumerative geometry ≈ The Hilbert's 15th Problem

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- Enumerative geometry $\approx$ The Hilbert's 15th Problem

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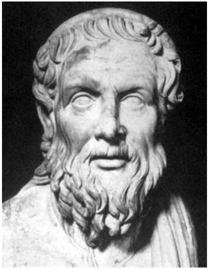
- Typical question: how many X satisfy condition Y?

One of the first enumerative problems

-  (taken from Wiki)

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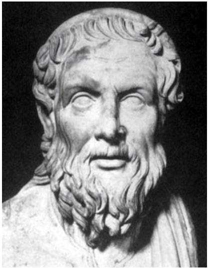
-  (taken from Wiki)
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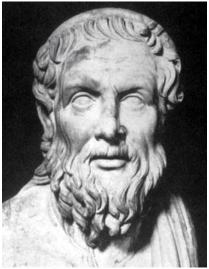
-  (taken from Wiki)
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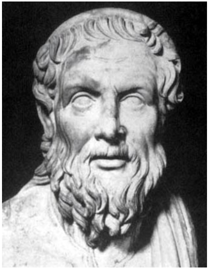
-  (taken from Wiki)
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- Some extra conditions are required, just like we extend from $\mathbb{R}$ to $\mathbb{C}$ for the fundamental theorem of algebra

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What about singular conics?

- Cohomology method: regard the line and the conic as global sections (s_1, s_2) of $\mathcal{O}_{\mathbb{P}^2}(1)$ and $\mathcal{O}_{\mathbb{P}^2}(2)$ over $\mathbb{P}^2_{\mathbb{C}}$

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- This will give us the desired number 2

Lines on a smooth cubic surface over $\mathbb{C}$

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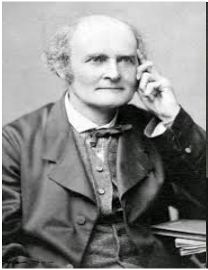
- (taken from Wiki)

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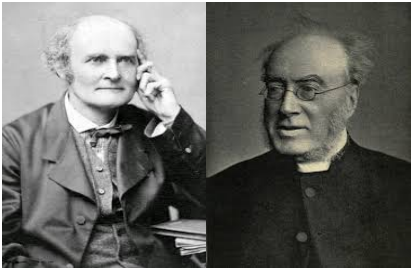
- (taken from Wiki)
- [PB07]In the letter to Salmon, Cayley said there could only be finitely many of lines on a smooth cubic surface. Then Salmon proved the number 27.

Classical approaches

- [Gat21] The Fermat cubic $X = V_+(x_0^3 + x_1^3 + x_2^3 + x_3^3) \subseteq \mathbb{P}^3_{\mathbb{C}}$ contains 27 lines, represented by the following matrices
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- Compute $e(Sym^3\mathcal{S}^*)$ using algebraic topology. This is independent of the choice of cubic surface!

Lines on a smooth cubic surface over $\mathbb{R}$: part 1

- In 1858 (Xianfeng 9th year, during the Taiping rebellion), Swiss mathematician Ludwig Schläfli [Sch58] proved the number of real lines can be 3,7,15, and 27 depending on surfaces.

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- [PB07]He was the first to study real cubics, and actually classified real surfaces, depending on the number of real lines and real tritangents.

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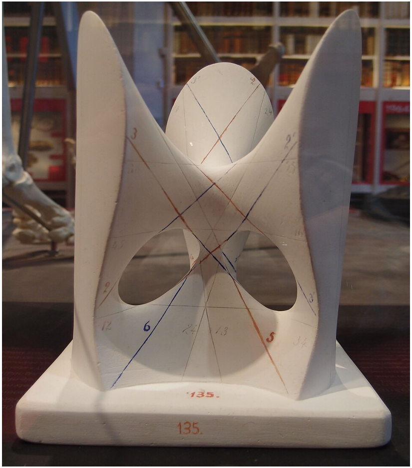
- Segre, Benedetti–Silhol[BS95], Okonek–Teleman[OT14], Finashin–Kharlamov[FK13], Horev–Solomon[HS12] proved the number of hyperbolic lines minus the number of elliptic lines is 3.

A beautiful model of the Clebsch surface



(Taken from Google. I lost my photo of a similar model taken in the math department of Universitat Regensburg)

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- $e(Sym^3\mathcal{S}^*) = 3$, i.e. the weighted count of real lines remains constant[Wic19].

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- After Marc Hoyois's thesis[Hoy15], Marc Levine, and at the same time, Kirsten Wickelgren and Jesse Kass started to build $\mathbb{A}^1$-enumerative geometry, which takes values in the Grothendieck-Witt ring $GW(k)$

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  3. $\langle a \rangle + \langle b \rangle = \langle a + b \rangle + \langle ab(a + b) \rangle$ for $a, b \in k^\times, a + b \neq 0$.
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- $GW(\mathbb{C}) \cong \mathbb{Z}$ by taking the rank
- $GW(\mathbb{R}) \cong \mathbb{Z} \times \mathbb{Z}$ by taking the rank and signature
- For a separable field extension $k \subseteq E$, we have a map $Tr_{E/k} : GW(E) \to GW(k), \beta \mapsto Tr_{E/k} \circ \beta$ where $\beta : V \times V \to E$ is a bilinear form.

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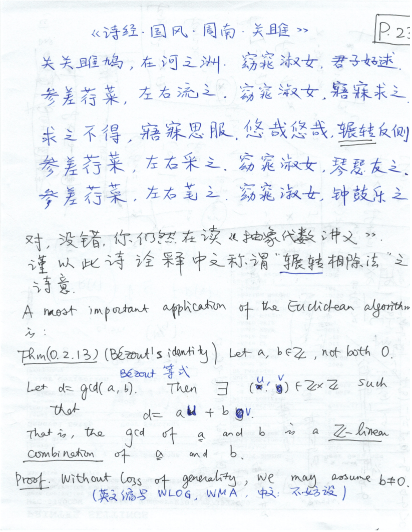
- This recovers the $\mathbb{C}, \mathbb{R}$ cases!

How did I get to know the program?

I strongly suggest Prof. Liang's webpage. There are many helpful suggestions and inspiring stories. This is where I first read about interesting things as a 1st year student. Also I knew the existence of ALGANT there.

2017/2018: Autumn semester, SUSTC, Shenzhen


《诗经·国风·周南·关雎》 P.23

关关雎鸠，在河之洲。窈窕淑女，君子好逑。
参差荇菜，左右流之。窈窕淑女，寤寐求之。
求之不得，寤寐思服。悠哉悠哉，辗转反侧。
参差荇菜，左右采之。窈窕淑女，琴瑟友之。
参差荇菜，左右芼之。窈窕淑女，钟鼓乐之。

对，没错，你仍然在读《抽象代数讲义》。
谨以此诗诠释中文称谓"辗转相除法"之诗意。

A most important application of the Euclidean algorithm is:

Thm (0.2.13) (Bézout's identity) Let $a, b \in \mathbb{Z}$, not both 0.
(Bézout 等式)
Let $d = \gcd(a, b)$. Then $\exists\ (u, v) \in \mathbb{Z} \times \mathbb{Z}$ such that $d = au + bv$.
That is, the gcd of a and b is a $\mathbb{Z}$-linear combination of a and b.

Proof. Without loss of generality, we may assume $b \neq 0$.
(英文缩写 WLOG, WMA, 中文：不妨设)

In Abstract Algebra Lecture notes

2020-2021: Regensburg

- Pure remote unfortunately

2020-2021: Regensburg

- Pure remote unfortunately
- I stayed in China for the winter semester, failed to go to Regensburg for the spring semester

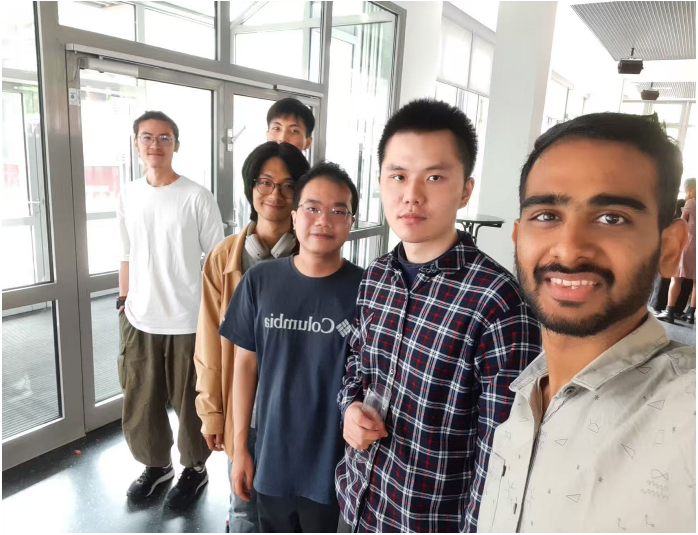
2020-2021: Regensburg

- Pure remote unfortunately
- I stayed in China for the winter semester, failed to go to Regensburg for the spring semester
- I paid for nonrefundable accomodation for the spring semester

2021-2022: Milano

11.2021 After the Welcoming Ceremony in Milano.

07.2022: Essen



Some participants in this conference are in this photo!

Thanks

Many thanks again for your attention, to the organizers and to the ALGANT program.

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