Equivariant Chen-Ruan cohomology of orbifolds and Ruan's conjecture

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Chen-Ruan cohomology of a global quotient by a finite group

$\mathcal{X}$ is a global quotient of a complex variety Y by a finite group G, i.e., $\mathcal{X} = [Y/G]$ is the quotient orbifold which is a Deligne-Mumford stack.

Definition (Chen-Ruan cohomology)

As a vector space, $H^*_{CR}(\mathcal{X}) = \bigoplus_{[g]\in T} H^*(Y^g)^{C(g)}$, where T is the set of all conjugacy classes of G, Y^g is the g-invariant submanifold of Y and $C(g) \subset G$ is the centralizer of $g \in G$.
$H^*_{CR}(\mathcal{X})$ can be equipped with a triple pairing $< \gamma_1, \gamma_2, \gamma_3 >$ for $\gamma_i \in H^*(Y^{g_i})^{C(g_i)}$. This triple pairing defines a product by requiring that $< \gamma_1\gamma_2, \gamma_3, I_Y > = < \gamma_1, \gamma_2, \gamma_3 >$, where $I_Y \in H^*(Y)$ is the fundamental class of Y.

An equivalent definition

Instead of the original definition, we will focus on an equivalent definition proposed by Barbara Fantechi and Lothar Göttsche.

Definition

The vector space $H^*(Y, G)$ is defined as:

$$H^*(Y, G) := \bigoplus_{g \in G} H^*(Y^g)$$

For $g \in G$ and $\alpha \in H^*(Y, G)$, denote by $\alpha_g \in H^*(Y^g)$ the g-th direct summand of α.

Remark: G has an action over $H^*(Y, G)$: For $g, h \in G$, an left action by h induces an isomorphism $h : Y^g \to Y^{hgh^{-1}}$. Then h induces an isomorphic push-forward of cohomology classes $h_* : H^*(Y^g) \to H^*(Y^{hgh^{-1}})$
We define $h(\alpha_g) := (h_*\alpha_g)_{hgh^{-1}}$.

Age and Grading

Definition

Let Y be a manifold of dimension D with an action of a finite group G. For $g \in G$ and $y \in Y^g$, let $\lambda_1, \cdots, \lambda_D$ be the eigenvalues of the tangent map $T_g : T_{Y,y} \to T_{Y,y}$.
Write $\lambda_i = e^{2\pi i r_j}$, where r_j is a rational number within $[0, 1)$. The age of g in y is defined as the rational number $a(g, y) := \sum_{j=1}^{D} r_j$.

Remark: $a(g, y)$ only depends on the connected component of Z of Y^g in which y lies.

Definition

We define a rational grading on $H^*(Y, G)$ as follows: Let $g \in G$ and Z be a connected component of Y^g and $j : Z \hookrightarrow Y^g$ the inclusion. For $\alpha \in H^i(Z)$, $j_*\alpha$ has degree $i + 2a(g, Z)$.

Cup product

Definition

Define a bilinear map $\mu : H^*(Y, G) \times H^*(Y, G) \to H^*(Y, G)$ as follows: For $\alpha \in H^*(Y^g)$ and $\beta \in H^*(Y^h)$

$$\mu(\alpha, \beta) := i_*(\alpha|_{Y^{g,h}} \cdot \beta|_{Y^{g,h}} \cdot c(g, h))$$

where $i : Y^{g,h} \to Y^{gh}$ is the natural inclusion, $\alpha|_{Y^{g,h}}$, $\beta|_{Y^{g,h}}$ are the pull-backs of cohomology classes, and $c(g, h)$ is given by the top Chern class of a vector bundle $F(g, h)$ on $Y^{g,h}$.

Remark: It has been verified that μ preserves the grading in the previous definition. Moreover, μ is associative. Therefore, $H^*(Y, G)$ is a graded ring.

Orbifold cohomology

Definition

The orbifold cohomology of the orbifold $[Y/G]$ is the graded ring:

$$H_o^*([Y/G]) := H^*(Y, G)^G$$

i.e., the G-invariant part of $H^*(Y, G)$.

Remark: The new definition by Fantechi and Göttsche agress with the original defintion given by Chen and Ruan. We can also write the orbifold cohomology defined above as $H^*_{CR}([Y/G])$.

Now we can state a particular version of Ruan's conjecture.

Conjecture: Let Y be a complex variety with an action of a finite group G. $\mathcal{X} = [Y/G]$ is the quotient orbifold and a projective variety Z is a crepant resolution of singularities of Y/G.

It is conjectured that the orbifold cohomology of $\mathcal{X}$ coincides with the small quantum cohomology of Z as a ring.

$\mathbb{C}^*$-equivariant orbifold cohomology

A $\mathbb{C}^*$-action on a stack $\mathcal{X}$ naturally induces a $\mathbb{C}^*$-action on $\mathcal{I}\mathcal{X}$, the inertia stack of $\mathcal{X}$. The $\mathbb{C}^*$-equivariant Chen-Ruan cohomology of $\mathcal{X}$, $H^*_{CR,\mathbb{C}^*}(\mathcal{X})$ is defined as follows:

$$H^*_{CR,\mathbb{C}^*}(\mathcal{X}) \simeq H^*_{\mathbb{C}^*}(\mathcal{I}\mathcal{X})$$

as a vector space with the grading shifted by the age as before, and the cup product deformed by an "equivariant version" of the Chern class $c(g,h)$.

Symplectic case

We now focus on a particular case when $Y = V \simeq \mathbb{C}^{2n}$ is a symplectic vector space, i.e., V is equipped with a non-degenerate skew-symmetric bilnear form $\omega \in \Lambda^2 V^*$ and $G \subset Sp(V)$ is a finite subgroup of the group of symplectic isomorphisms.
By the previous discussions on $\mathbb{C}^*$-equivariant case,

$$H^*_{CR,\mathbb{C}^*}([Y/G]) \simeq H^*_{CR}([(Y \times_{\mathbb{C}^*} \mathbb{E}\mathbb{C}^*)/G]) \simeq (\bigoplus_{g\in G} H^*((Y \times_{\mathbb{C}^*} \mathbb{E}\mathbb{C}^*)^g))^G$$
$$\simeq (\bigoplus_{g\in G} H^*(Y^g \times_{\mathbb{C}^*} \mathbb{E}\mathbb{C}^*))^G \simeq (\bigoplus_{g\in G} H^*_{\mathbb{C}^*}(Y^g))^G \tag{1}$$

Notice that Y^g is a sub-vector space of Y and thus contractible. $H^*_{\mathbb{C}^*}(Y^g) \simeq H^*_{\mathbb{C}^*}(\{\bullet\}) \simeq H^*(\mathbb{CP}^\infty) \simeq \mathbb{C}[\lambda]$, i.e., a polynomial ring generated by one element $\lambda \in H^2(\mathbb{CP}^\infty)$.

the structure of the $\mathbb{C}^*$ equivariant Chen-Ruan cohomology

Therefore, $H^*_{CR,\mathbb{C}^*}([Y/G]) \simeq (\bigoplus_{g\in G} \mathbb{C}[\lambda])^G \simeq \mathbb{C}[\lambda] \otimes_{\mathbb{C}} (\bigoplus_{g\in G} 1_g)^G \simeq \mathbb{C}[\lambda] \otimes_{\mathbb{C}} Z[G]$, where $Z[G]$ is the center of the group algebra $\mathbb{C}[G]$, because G acts over $\bigoplus_{g\in G} 1_g = \mathbb{C}[G]$ by conjugation.

It is clear that $\mathbb{C}[\lambda] \otimes_{\mathbb{C}} Z[G]$ is a subalgebra of $\mathbb{C}[\lambda] \otimes_{\mathbb{C}} \mathbb{C}[G]$. We define the grading and cup product on the latter and then restrict it to $H^*_{CR,\mathbb{C}^*}([Y/G])$.

The grading of $\lambda^t \otimes 1_g$ is defined as $2t + 2a(g)$ (In this particular case, the age $a(g,y)$ does not depend on $y \in Y^g$, thus we just write $a(g,y)$ as $a(g)$).

The cup product of $H^*_{CR,\mathbb{C}^*}([Y/G])$ is given by $(\lambda^{t_1} \otimes 1_g) \cup (\lambda^{t_2} \otimes 1_h) = c(g,h)\lambda^{t_1+t_2} \otimes 1_{gh}$ where $c(g,h)$ is simply a number. $c(g,h) = 1$ if and only if $a(g) + a(h) = a(gh)$ and $c(g,h) = 0$ otherwise.

Ruan's conjecture in the equivariant case

Now we have already described the structure of $\mathbb{C}^*$-equivariant Chen-Ruan cohomology of the orbifold $\mathcal{X} = [Y/G]$.
Naturally, we want to know what Ruan's conjecture will inspire us in the equivariant case. We may try to find a relation between the $\mathbb{C}^*$-equivariant Chen-Ruan cohomology of the orbifold $\mathcal{X}$ computed above and the equivariant quantum cohomology of the crepant resolution of the singular variety Y/G.

Figure 1: A photo with Zhenghang Du and Yufan Ge at Leiden

Thank you for your attention!