An Introduction to Tropical Intersection Theory

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(Regensburg - Bordeaux 2021 - 2022)

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· Heights and Monge-Ampère measure

Let K be $\mathbb{R}$ or $\mathbb{C}$ (archimedean places)

or a complete field with a non-trivial non-archimedean discrete absolute value $|\cdot|$.

Let X be a proper variety over K.

X^{an} is a complex analytic space in archimedean case;

an analytic space in Berkovich sense in na case.

Let $\bar{L} = (L, \|\cdot\|)$ be a semipositive metrized line bundle on X

For every d-dim cycle Y of X and sections $s_0, \cdots, s_d$ that meet Y properly,

$$h_{\bar{L}}(Y ; s_0, \cdots, s_d) = h_{\bar{L}}(Y \cdot \operatorname{div} s_d ; s_0, \cdots, s_{d-1})$$

$$- \int_{X^{an}} \log \|s_d\| \, c_1(\bar{L})^{\wedge d} \wedge \delta_Y$$

(In particular, for $p \in X(K) \setminus |\operatorname{div}(s)|$, $h_{\bar{L}}(p ; s) = -\log \|s(p)\|$)

Let Y be a subvariety of X.

Then $c_1(\bar{L})^{\wedge d} \wedge \delta_Y$ is called the Monge-Ampère measure

If K is na, then

we can choose a model $(\mathcal{X}, \mathcal{L})$ of (X, L) that induces the metric of L.

Let $\mathcal{Y} \subset \mathcal{X}$ be the closure of Y, $\tilde{\mathcal{Y}}$ be its normalization

$\tilde{\mathcal{Y}}_s$ = the special fibre of $\tilde{\mathcal{Y}}$

$\tilde{Y} :=$ the generic fibre of $\tilde{\mathcal{Y}}$

$\tilde{\mathcal{Y}}_s^{(0)} :=$ the set of irreducible components of $\tilde{\mathcal{Y}}_s$.

$\forall V \in \tilde{\mathcal{Y}}_s^{(0)}$, let η_V be the generic point of V.

$\exists$ a reduction map $\mathrm{red}: \tilde{Y}^{an} \longrightarrow \tilde{\mathcal{Y}}_s$.

$$\Rightarrow \underline{c_1(\bar{L})^{\wedge d} \wedge \delta_Y} = \underline{\sum_{V \in \tilde{\mathcal{Y}}_s^{(0)}}} \mathrm{ord}_V(\pi) \cdot \deg_{\mathcal{L}}(V) \cdot \underline{\delta_{\mathrm{red}^{-1}(\eta_V)}}$$

the Dirac delta measure on X^{an} supported on $\mathrm{red}^{-1}(\eta_V)$

- **Tropical intersection theory**

A weighted fan (X, ω_X) of dim k in $\mathbb{R}^n$ is a fan X in $\mathbb{R}^n$ of pure dimension k, together with a map $\omega_X : X^{(k)} \to \mathbb{Z}$.

A tropical fan (X, ω_X) of dim k in $\mathbb{R}^n$ is a weighted fan satisfying the following balancing condition for every $\tau \in X^{(k-1)}$:

$$\sum_{\sigma : \tau < \sigma} \omega_X(\sigma) \cdot n_{\sigma/\tau} = 0 ,$$

where $n_{\sigma/\tau}$ is the normal vector of σ relative to τ.

An affine tropical cycle $[(X, \omega_X)]$ is an equivalence class of tropical fans in $\mathbb{R}^n$.

If $\omega_X(\sigma) \geq 0$, $\forall \sigma \in X$,

we call $[(X, \omega_X)]$ a tropical variety.

- Let C be an affine tropical cycle

A rational function on C is a cts piecewise linear function $\varphi : |C| \to \mathbb{R}$,

i.e. $\exists$ a representative (X, ω_X) of C,

$= \varphi_\sigma + c$

s.t. $\varphi|_\sigma$ is an integral affine linear function, $\forall \sigma \in X$

- Associated Weil divisor

$$\operatorname{div}(\varphi) := \varphi \cdot C := \left[\left(\bigcup_{i=0}^{k-1} X^{(i)}, \omega_\varphi\right)\right] \text{ a } (k-1)\text{-cycle},$$

where $\omega_\varphi : X^{(k-1)} \longrightarrow \mathbb{Z}$

$$\tau \longmapsto \sum_{\substack{\sigma \in X^{(k)} \\ \tau < \sigma}} \varphi_\sigma\left(\omega(\sigma)\, n_{\sigma/\tau}\right) - \varphi_\tau\left(\sum_{\substack{\sigma \in X^{(k)} \\ \tau < \sigma}} \omega(\sigma)\, n_{\sigma/\tau}\right)$$

• Associated Weil divisor

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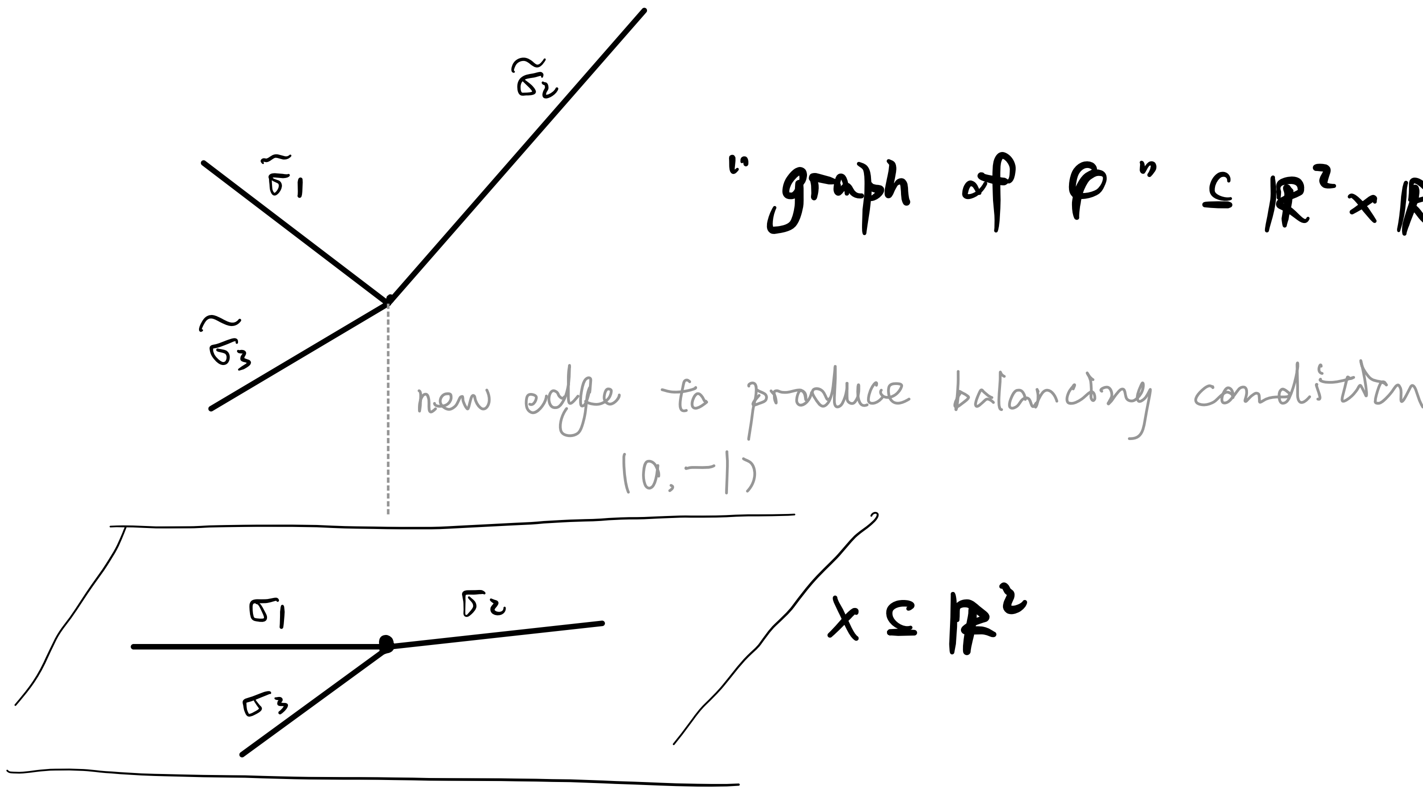


- Intersection product of cycles

Observe that

$[(\Delta, 1)] := [(\{(x,x) : x \in \mathbb{R}^n\}, 1)] \in Z_n(\mathbb{R}^n \times \mathbb{R}^n)$

Let $\psi_i := [(\mathbb{R}^n \times \mathbb{R}^n, \max\{0, x_i - y_i\})] \in \mathrm{Div}(\mathbb{R}^n \times \mathbb{R}^n)$.

Then $[(\Delta, 1)] = \psi_1 \cdots \psi_n \cdot (\mathbb{R}^n \times \mathbb{R}^n)$.

Let $\pi : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$; $(x, y) \longmapsto x$.

Then we define the intersection of products of cycles in $\mathbb{R}^n$ by

$$Z_{n-k}(\mathbb{R}^n) \times Z_{n-l}(\mathbb{R}^n) \longrightarrow Z_{n-k-l}(\mathbb{R}^n)$$

$$(C, D) \longmapsto C \cdot D := \pi_*(\psi_1 \cdots \psi_n \cdot (C \times D))$$

• [Antoine Chambert-Loir, Antoine Ducros 2012]:

MA measure $c_1(\bar{L})^{\wedge d} \wedge \delta_Y$ can be written as a "non-archimedean differential form" called Lagberg form

• $\left.\begin{array}{l}\text{Lagerberg forms} \\ \text{tropical cycles}\end{array}\right\}$ δ-forms [Gubler-Künnemann, 2016]

e.g. balancing condition $\leftrightsquigarrow$ Stoke's formula

Weil divisor $\leftrightsquigarrow$ $d'd''\varphi$ (analogue: $d' \leftrightarrow \frac{\partial}{\partial z}$; $d'' \leftrightarrow \frac{\partial}{\partial \bar{z}}$)

$\rightsquigarrow$ we have a tropical intersection theory for δ-forms on X^{an}.

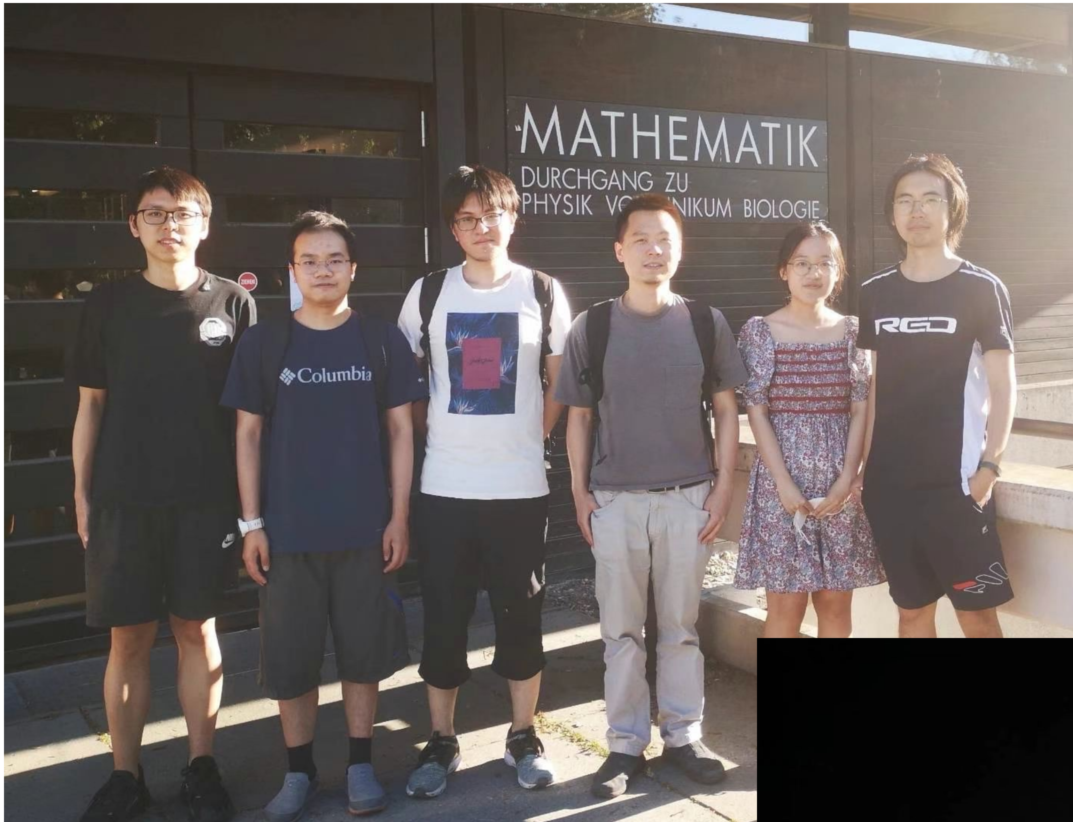
• **Applications**

[Burgos Gil-Gubler-Künnemann, 2025]

1. A tropical formula for non-archimedean local heights

[Garcia-Sankaran, 2019]

2. An application to arith. intersection problems on Shimura vars.
3. δ-forms on Lubin-Tate space [Mihatsch, 2025]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Thanks !