Uniform Bogomolov Result

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December 2025

Canonical height function

Theorem

Let K be a number field. Let (X, f, L) be a triple called algebraic dynamical system, where:

- *X is a projective variety over K,*
- *$f : X \to X$ is a morphism over K,*
- *L is a line bundle on X such that $f^*L \simeq qL$ for some integer $q > 1$.*

*We have the **canonical height function***

$$\hat{h}_L : X(K) \longrightarrow \mathbb{R}$$

defined by

$$\hat{h}_L(x) = \lim_{n\to\infty} \frac{1}{q^n} h_L(f^n(x)),$$

where $h_L : X(K) \to \mathbb{R}$ is any Weil height associated to L.

Néron-Tate Height on Abelian Varieties

Definition

On an abelian variety A over a number field K, we have a canonical algebraic dynamical system. Then we can define the associated canonical height.
Suppose L is a line bundle on A:
If L is symmetric, we define

$$\hat{h}_L(x) = \lim_{n\to\infty} \frac{1}{4^n} h_L(2^n x), \quad x \in A(K)$$

If L is anti-symmetric, we define

$$\hat{h}_L(x) = \lim_{n\to\infty} \frac{1}{2^n} h_L(2^n x), \quad x \in A(K)$$

Néron-Tate Height on Abelian Varieties

In general, we define

$$\hat{h}_L = \frac{1}{2}\left(\hat{h}_{L+[-1]^*L} + \hat{h}_{L-[-1]^*L}\right)$$

For projective curves over number field, we have the Abel–Jacobi morphism $j_\alpha\colon X_K \longrightarrow J_K, x \mapsto x - \alpha$, where α is a line bundle of degree 1. Then we can define the θ divisor θ_α associate to α, and the ample symmetric line bundle $\Theta := \mathcal{O}\left(\theta_\alpha + [-1]^*\theta_\alpha\right)$
Let denote the canonical height of Θ by $\hat{h}_\Theta$.

Definition

The Néron–Tate height $\hat{h}\colon \mathrm{Pic}^0(X_K) \to \mathbb{R}$ is defined as

$$\frac{1}{2}\hat{h}_\Theta\colon J(K) \to \mathbb{R}$$

Bogomolov Conjecture for Curves

Theorem (Ullmo 1998)

Let C be a smooth projective curve of genus $g > 1$ over $\overline{\mathbb{Q}}$. Then for any divisor α on C of degree 1, there is a constant $c > 0$ such that

$$\#\{x \in C(\overline{\mathbb{Q}}) : \widehat{h}(x - \alpha) \leq c\} < \infty$$

Bogomolov Conjecture in General

Theorem (Zhang 1998)

Let A be an abelian variety over a number field K, and let L be a symmetric and ample line bundle on A. Let X be a closed subvariety of $A_{\bar{K}}$. Then the following statements are equivalent:

1. *X is the translation of an abelian subvariety of $A_{\bar{K}}$ by a torsion point;*
2. *For any $\epsilon > 0$, the set $\left\{x \in X(\bar{K}) : \hat{h}_L(x) < \epsilon\right\}$ is Zariski dense in X.*

Uniform Bogomolov Result

Theorem (Dimitrov–Gao– Habegger 2021 ; Yuan 2021)

Let $g > 1$, then there are constants $c_1, c_2 > 0$ depending only on g satisfying the following properties.
Let K be a number field. Then for any geometrically integral, smooth and projective curve C of genus g over K, and for any line bundle $\alpha \in \mathrm{Pic}(C_K)$ of degree 1, one has:

$$\#\{x \in C(K) : \hat{h}(x - \alpha) \leq c_1 \max\{h_{Fal}(C), 1\} + \hat{h}((2g-2)\alpha - \omega_{C/K})\} \leq c_2.$$

Here, $h_{Fal}(C)$ is the Faltings height of curves, which is defined by $h^*_{\mathrm{Fal}}(A) := \frac{1}{[K:\mathbb{Q}]}\deg(\underline{\hat{\omega_A}})$, where $\underline{\hat{\omega_A}}$ is Hodge line bundle with Faltings metric.

Uniformal Mordell Conjecture

Theorem (Kühne 2021)

There is a constant c depending only on $g \geq 2$ such that for every smooth projective curve C of genus g over a number field K, the number of K-rational points in C is bounded by c^{r+1}, where r is the rank of $J(K)$ for the Jacobian J of C.

Remark that the number of large points is bounded by Mumford's gap principle and Vojta's theorem. For points with small height, we apply uniformal Bogomolov theory.

Positive property

Definition

Let X be a projective variety over a field k. A line bundle L on X is **nef** (or *numerically effective*) if $L \cdot C \geq 0$ for any closed integral curve $C \subset X$.

Definition

Let L be a line bundle on X. The **volume** of L on X is defined to be

$$\mathrm{vol}(L) = \limsup_{n \to \infty} \frac{d!}{n^d} h^0(X, nL),$$

where $d = \dim X$. We say that L is **big** if $\mathrm{vol}(L) > 0$.

We can generalize these definition to adelic line bundle define on a quasi-projective scheme.In particular,

Definition

Let k be either $\mathbb{Z}$ or a field. Let U be a quasi-projective and flat integral scheme over k.Let χ be a projective model of U over k. The group of **model adelic divisors** is defined by the direct limit

$$\widehat{\mathrm{Div}}(U/k)_{\mathrm{mod}} = \varinjlim_{\chi} \widehat{\mathrm{Div}}(\chi, U),$$

where for each projective model χ of U, we set

$$\widehat{\mathrm{Div}}(\chi, U) = \widehat{\mathrm{Div}}(\chi)_{\mathbb{Q}} \times_{\mathrm{Div}(U)_{\mathbb{Q}}} \mathrm{Div}(U)$$

Choose the topologe on $\widehat{\mathrm{Div}}(U/k)_{\mathrm{mod}}$ induced by boundary normal, the we can consider complete space.

Definition

Let $\widehat{\mathrm{Div}}(U/k)$ be the completion of $\widehat{\mathrm{Div}}(U/k)_{\mathrm{mod}}$ with respect to the boundary topology. An element of $\widehat{\mathrm{Div}}(U/k)$ is called an **adelic divisor** on U/k.

Definition

Similarly, we can define the adelic line bundles for quasi-projective scheme. Let $\widehat{\mathrm{Pic}}(X)$ denote the category of hermitian line bundles on X, and let $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$ be the category of hermitian $\mathbb{Q}$-line bundles on X.

positive property

Definition

An adelic divisor is called effective if it is equal to a limit of effective arithmetic divisors. An adelic line bundle is called effective if it is the image of an effective adelic divisor.

Definition

For an adelic line bundle $\overline{L}$ (resp. adelic divisor $\overline{D}$):

- **Strongly nef**: $\overline{L} \cong \lim \overline{L}_n$ (resp. $\overline{D} = \lim \overline{D}_n$) with $\overline{L}_n$ (resp. $\overline{D}_n$) nef.
- **Nef**: $\exists$ strongly nef $\overline{M}$ such that $a\overline{L} + \overline{M}$ (resp. $a\overline{D} + \overline{M}$) is strongly nef $\forall a > 0$.
- **Integrable**: $\overline{L} \cong \overline{M}_1 - \overline{M}_2$ (resp. $\overline{D} = \overline{M}_1 - \overline{M}_2$) with $\overline{M}_1, \overline{M}_2$ strongly nef.

Arithmetic bigness

Definition (Volume)

For $\overline{L}$ an adelic line bundle on U, define:

$$\mathrm{vol}(\widehat{U},\overline{L}) = \lim_{m\to\infty} \frac{d!}{m^d}\widehat{h}^0(U, m\overline{L})$$

where $d = \dim U$.

Theorem (Yuan,Zhang 2021)

This limit always exists.

Definition (Big)

$\overline{L}$ is **big** if $\mathrm{vol}(\widehat{U},\overline{L}) > 0$.

Arithmetic bigness

Theorem (Hilbert–Samuel formula), [Yuan, Zhang 2021)

If $\overline{L}$ is nef, then:

$$\mathrm{vol}(\widehat{U}, \overline{L}) = \overline{L}^d$$

Corollary

For nef $\overline{L}$:

$$\overline{L} \text{ is big} \iff \overline{L}^d > 0$$

Admissable adelic line bundle

Let k be either $\mathbb{Z}$ or a field. Let S be a quasi-projective and flat normal integral scheme over k. Let $\pi : X \to S$ be a smooth relative curve over S of genus $g > 1$. Let $\omega_{X/S}$ be the relative dualizing sheaf, and $\Delta \subset X^2/S$ be the diagonal divisor.
Then Yuan proves that there exist a canonical admissible adelic line bundle satisfies some nice properties, which generalize Zhang's works on projective case.
Let denote the canonical admissible adelic line bundle on X extending $\omega_{X/S}$ by $\bar{\omega}_{X/S,a}$ and denote canonical admissible adelic line bundle on X^2/S extending $\mathcal{O}(\Delta)$ by $\bar{\mathcal{O}}(\Delta)_a$.

Main theory: arithmetic bigness

Theorem (Yuan 2021)

Let k be either $\mathbb{Z}$ or a field. Let S be a quasi-projective and flat normal integral scheme over k. Let $\pi: X \to S$ be a smooth relative curve over S of genus $g > 1$ with maximal variation. Then the admissible canonical bundle $\bar{\omega}_{X/S,a}$ is nef and big on X.

Recall that a relative curve $\pi: X \to S$ has maximal variation if the moduli morphism $S \to \mathcal{M}_{g,k}$ is generically finite.

Thank you !