An introduction to p-adic geometry

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GAGA: complex case

(1) GAGA functor. Let X be a finite type $\mathbb{C}$-scheme.
We can define the analytification functor

$$(\cdot)^{\mathrm{an}} : (\text{finite type } \mathbb{C}\text{-schemes}) \longrightarrow (\text{complex analytic spaces}), \quad X \longmapsto X^{\mathrm{an}},$$

which sends a variety over $\mathbb{C}$ to its associated complex analytic space.

Theorem

If X is a projective variety over $\mathbb{C}$, then the analytification functor induces

- *an equivalence of categories*

$$\mathrm{Coh}(X) \simeq \mathrm{Coh}(X^{\mathrm{an}}),$$

- *and for every coherent sheaf $\mathcal{F}$ on X,*

$$H^i(X, \mathcal{F}) \cong H^i(X^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}}) \quad \textit{for all } i \geq 0.$$

GAGA: rigid analytic case

(1) Rigid analytification functor. Let K be a non-archimedean field and let X be a finite type K-scheme.
One can define the rigid analytification functor

$$(\cdot)^{\mathrm{rig}} : (\text{finite type } K\text{-schemes}) \longrightarrow (\text{rigid analytic spaces over } K), X \longmapsto X^{\mathrm{rig}},$$

which associates to X a rigid analytic space.

Theorem

If X is a projective K-scheme, then the rigid analytification functor induces

- *an equivalence of categories*

$$\mathrm{Coh}(X) \simeq \mathrm{Coh}(X^{\mathrm{rig}}),$$

- *and for every coherent sheaf $\mathcal{F}$ on X,*

$$H^i(X, \mathcal{F}) \cong H^i(X^{\mathrm{rig}}, \mathcal{F}^{\mathrm{rig}}) \quad \textit{for all } i \geq 0.$$

Rigid analytic spaces

Definition (Rigid analytic spaces). Let K be a non-archimedean field.

- A *K-affinoid algebra* is a Banach K-algebra which plays the role of a ring of analytic functions over a non-archimedean field.
- An *affinoid rigid space* is the spectrum
$$\mathrm{Sp}(A)$$
of maximal ideals of a K-affinoid algebra A, equipped with its Grothendieck topology and structure sheaf.
- A *rigid analytic space* over K is a *G-space* which is locally isomorphic to affinoid rigid spaces, i.e. it admits an admissible covering by spaces of the form $\mathrm{Sp}(A)$.

Berkovich spaces: topological improvement

To deal with the topological issues of rigid analytic spaces, Berkovich introduced a new notion of non-archimedean analytic spaces.

- A *Berkovich space* is defined by enlarging the set of points: points correspond to multiplicative seminorms on affinoid algebras, rather than maximal ideals.
- As a result, Berkovich spaces carry a genuine topology. For example, for a K-affinoid algebra A, the Berkovich spectrum $\mathcal{M}(A)$ is Hausdorff and compact.

Philosophical comparison.

- Rigid analytic spaces make the *function theory* work correctly.
- Berkovich spaces make the *topology* work correctly.

Formal schemes: infinitesimal geometry

Motivation (infinitesimal behavior). Formal schemes are introduced to systematically encode *infinitesimal* neighborhoods.

- Example: the quotient
$$k[x]/(x^n)$$
describes the local behavior of the affine line $\mathrm{Spec} k[x]$ near the point 0 up to order n.
- Passing to all orders suggests the formal limit
$$\varprojlim_n k[x]/(x^n),$$
which should be thought of as the ring of functions on the *formal neighborhood* of 0. (This is intuition rather than the definition.)

Formal Schemes

Definition (affine formal scheme). Let A be a ring and let $I \subset A$ be an ideal. Assume that A is *I-adically complete and separated*, i.e.

$$A \cong \varprojlim_n A/I^n.$$

Then the *affine formal scheme* associated to (A, I) is

$$\mathrm{Spf}(A),$$

whose underlying topological space is $\mathrm{Spec}(A/I)$ and whose structure sheaf is obtained by I-adic completion on basic opens.

Formal GAGA

Theorem

Let X be a projective scheme over a noetherian ring A, and let $I \subset A$ be an ideal. Denote by $\widehat{X}$ the formal completion of X along I.
Then the completion functor induces an equivalence of categories

$$\mathrm{Coh}(X) \simeq \mathrm{Coh}(\widehat{X}),$$

and for every coherent sheaf $\mathcal{F}$ on X, the natural map

$$H^i(X, \mathcal{F}) \cong H^i(\widehat{X}, \widehat{\mathcal{F}}) \quad \textit{for all } i \geq 0$$

is an isomorphism.

Adic spaces

Huber rings. A *Huber ring* is a topological ring A admitting an open subring $A_0 \subset A$ such that

- the topology on A_0 is I-adic for some finitely generated ideal I,
- the topology on A is induced from that of A_0.

Affinoid pairs. An *affinoid pair* is a pair (A, A^+), where A is a Huber ring and $A^+ \subset A$ is an open, integrally closed subring.

The adic spectrum. To an affinoid pair (A, A^+) one associates the adic spectrum $\mathrm{Spa}(A, A^+)$, whose points are equivalence classes of continuous valuations on A bounded by 1 on A^+.

Adic Spaces

On $\mathrm{Spa}(A, A^+)$ one can define presheaves

$$\mathcal{O}_X \quad \text{and} \quad \mathcal{O}_X^+,$$

which in most well-behaved situations are in fact sheaves.

Adic spaces. An *affinoid adic space* is the space $\mathrm{Spa}(A, A^+)$ for affinoid pair (A, A^+) with $\mathcal{O}_X$ a sheaf.
An *adic* space is a space which is locally affinoid adic.[1]

[1] Here we should consider an adic space in a category (V) with object be a space together with a sheaf $\mathcal{O}_X$ and a family of valuation information $\{v_x\}$

Adic spaces

Theorem (Huber)

Adic spaces provide a common framework for rigid analytic geometry and formal geometry via fully faithful embeddings.

- *There exists a fully faithful functor*

$$r_K : (\textit{rigid analytic spaces over } K) \longrightarrow (\textit{adic spaces}),$$

  which sends $\mathrm{Sp}(A)$ *to* $\mathrm{Spa}(A, A^\circ)$.

- *There exists a fully faithful functor*

$$t : (\textit{locally noetherian formal schemes}) \longrightarrow (\textit{adic spaces}),$$

  which sends $\mathrm{Spf}(A)$ *to* $\mathrm{Spa}(A, A)$.

In this sense, adic spaces simultaneously generalize rigid analytic spaces and formal schemes.

Why adic spaces?

From rigid analytic geometry.

- The category of rigid analytic spaces does not admit well-behaved inverse limits.
- As a consequence, constructions such as perfectoid spaces can only live in adic spaces.

From formal geometry.

- A formal scheme describes an infinitesimal neighborhood of a point or a closed subscheme. However, they do not carry genuine analytic neighborhoods.
- Adic spaces provide a geometric realization of these formal neighborhoods, interpolating between formal and analytic geometry.

Perfectoid Spaces

Perfectoid rings. Let K be a perfectoid field. A Huber pair (R, R^+) is called a *perfectoid affinoid K-algebra* if

- R is a complete uniform Tate K-algebra,
- the Frobenius map
$$\varphi : R^\circ/p \longrightarrow R^\circ/p$$
is surjective.

Perfectoid spaces. A *perfectoid space* is an adic space which is locally isomorphic to $\mathrm{Spa}(R, R^+)$ for some perfectoid affinoid K-algebra (R, R^+).

Tilting

Definition (Tilt of a perfectoid ring). Let (R, R^+) be a perfectoid affinoid K-algebra. The *tilt* of R is defined as

$$R^\flat := \varprojlim_{x \mapsto x^p} R,$$

with ring structure induced from R. The subring

$$R^{\flat +} := \varprojlim_{x \mapsto x^p} R^+$$

defines a perfectoid affinoid algebra $(R^\flat, R^{\flat +})$ over the tilt $K^\flat$.

Tilting equivalence

Theorem

1. *The tilting construction induces an equivalence of categories* $\{K\text{-}Perfd\} \simeq \{K^\flat\text{-}Perfd\}$.
2. *Let* $X = \mathrm{Spa}(R, R^+)$ *be a perfectoid adic space. Then its tilt* $X^\flat := \mathrm{Spa}(R^\flat, R^{\flat+})$ *is a perfectoid space over* $K^\flat$*, and the assignment* $X \longmapsto X^\flat$ *induces an equivalence of categories between perfectoid spaces over* K *and over* $K^\flat$.
   Moreover, X *and* $X^\flat$ *are homeomorphic and have canonically identified rational subsets.*

An important example: the Fargues–Fontaine curve

Untilts. Let F be a perfectoid field of characteristic p. An *untilt* of F is a perfectoid field K of characteristic 0 together with an identification

$$K^\flat \cong F.$$

The ring A_{inf}. Let $\mathcal{O}_F$ be the ring of integers of F. Set

$$A_{\mathrm{inf}} := W(\mathcal{O}_F).$$

For suitable *distinguished elements* $\xi \in A_{\mathrm{inf}}$ (defined so that they cut out untilts), one obtains a precise bridge between untilts and quotients of A_{inf}.

Fargues-Fontaine Curves

Theorem

Let F be a perfectoid field of characteristic p. Then the assignment

$$\xi \longmapsto \mathrm{Frac}\big(A_{\mathsf{inf}}/(\xi)\big)$$

induces a bijection

$$(\{\textit{distinguished elements of } A_{\mathsf{inf}}\}/A_{\mathsf{inf}}^{\times}) \;\cong\; (\{\textit{untilts of } F\}/\cong).$$

Heuristic picture. For untilts C of F, $|p|_C$ varies from 0 to 1. We would like to use this to parametrize

$$Y := (\{\text{untilts of } F\}/\cong)$$

as the unit disk.

Fargues-Fontaine Curves

A holomorphic function on the punctured unit disk can be written as a Laurent series $\sum_{n \geq -k} c_n z^n$ satisfying the growth conditions

$$\limsup_{n\to\infty} |c_n|^{1/n} \leq 1, \qquad \limsup_{n\to\infty} |c_{-n}|^{1/n} = 0.$$

A p-adic analogue. Fix $\varpi \in \mathcal{O}_F$ with $0 < |\varpi| < 1$ and write $[\varpi] \in A_{\inf}$ for its Teichmüller lift. Inside

$$A_{\inf}\left[\frac{1}{p}, \frac{1}{[\varpi]}\right] = \left\{ \sum_{n \geq -k} [c_n]\, p^n \;\middle|\; (c_n) \subset F \text{ bounded} \right\},$$

one enlarges to a ring B such that whenever an element admits an expansion

$$\sum_{n \geq -k} [c_n]\, p^n,$$

the coefficients satisfy the same type of growth conditions:

$$\limsup_{n\to\infty} |c_n|^{1/n} \leq 1, \qquad \limsup_{n\to\infty} |c_{-n}|^{1/n} = 0.$$

We think of B as a *ring of holomorphic functions* on a space Y.

Fargues-Fontaine Curves

Definition (Fargues–Fontaine curve). Let φ denote Frobenius on B. Define

$$X_F := \mathrm{Proj}\left(\bigoplus_{n\geq 0} B^{\varphi=p^n}\right).$$

This is the (schematic) Fargues–Fontaine curve.

Fargues-Fontaine Curves

The Fargues–Fontaine curve behaves in many ways like a complete algebraic curve (of genus 0 in some sense). For example:

- The cohomology group $H^1(X_F, \mathcal{O}_{X_F})$ vanishes.
- Degree formula (no "missing points"): for any rational function f on X_F,

$$\sum_{x \in X_F} \deg_x(f) = 0.$$

- Vector bundles on X_F admit a canonical Harder–Narasimhan filtration (and are governed by slopes, as for bundles on curves), i.e.,

Theorem

Then every vector bundle $\mathcal{E}$ on X_F is isomorphic to a vector bundle of the form

$$\mathcal{E} \;\cong\; \bigoplus_{i=1}^{r} \mathcal{O}_{X_F}(\lambda_i),$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ are rational numbers.
Moreover, the multiset $\{\lambda_i\}$ is uniquely determined by $\mathcal{E}$.

The Fargues–Fontaine Curves

Let E be a finite extension over $\mathbb{Q}_p$ with uniformizer π.

The space $Y_{F,E}$. Let $W(\mathcal{O}_F)$ denote the ring of Witt vectors of $\mathcal{O}_F$, and set

$$W_{\mathcal{O}_E}(\mathcal{O}_F) := W(\mathcal{O}_F) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E.$$

One defines an adic space

$$Y_{F,E} := \mathrm{Spa}\big(W_{\mathcal{O}_E}(\mathcal{O}_F)\big) \setminus \{|\pi| = 0\},$$

which can be covered by suitable rational subsets. The Frobenius induces an automorphism

$$\varphi : Y_{F,E} \longrightarrow Y_{F,E}.$$

Definition The *adic Fargues–Fontaine curve* is defined as the quotient

$$X_{F,E} := Y_{F,E}/\varphi^{\mathbb{Z}}.$$

Fargues-Fontaine Curves

When $E = \mathbb{Q}_p$, the adic Fargues–Fontaine curve $X_{F,E}$ is canonically isomorphic to the adic analytification of the schematic Fargues–Fontaine curve X_F constructed earlier.

$\mathrm{Gal}(\mathbb{Q}_p)$ as a fundamental group

Let E be a p-adic field.

Theorem

One can construct an object Z_E in the pro-étale site of Perf_C, such that the category of finite étale covers of Z_E is equivalent to the category of finite étale E-algebras. Equivalently,

$$\pi_1^{\text{ét}}(Z_E) \cong \mathrm{Gal}(\overline{E}/E).$$

In this way, arithmetic Galois groups arise as fundamental groups of geometric objects in characteristic p-adic geometry.

Thank you.