p-adic special value formulae

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For an elliptic curve $E : y^2 = x^3 + ax + b$ over $\mathbb{Q}$,

- Arithmetic invariant: Mordell-Weil group $E(\mathbb{Q})$ and Shafarevich-Tate group $Ш(E/\mathbb{Q}) = \ker(H^1(\mathbb{Q}, E) \to \prod_v H^1(\mathbb{Q}_v, E))$
- Analytic invariant: Hasse-Weil L-function $L(E, s) = \prod_v L_v(E, s)$ when $\mathrm{Re}(s) \geq 3/2$,

$$L_p(E, s) = (1 - a_p p^{-s} + p^{1-2s})^{-1}, \quad a_p = 1 + p - \sharp E(\mathbb{F}_p) \quad \text{almost all} \quad p$$

Theorem (Mordell-Weil)

$E(\mathbb{Q})$ is a finitely generated abelian group.

Theorem (Wiles, Breuil-Conrad-Diamond-Taylor)

$L(E, s)$ extends to the entire $\mathbb{C}$-plane and satisfies $L(E, s) = \epsilon(E, s)L(E, 2 - s)$.

Conjecture (Birch-Swinnerton-Dyer,Tate)

$r_{an}(E) := \mathrm{ord}_{s=1} L(E, s) = \mathrm{rank}_{\mathbb{Z}} E(\mathbb{Q}) =: r(E)$ and $Ш(E/\mathbb{Q})$ is finite.

Theorem (Gross-Zagier, Kolyvagin)

When $r_{an}(E) \leq 1$, $r_{an}(E) = r(E)$ and $Ш(E/\mathbb{Q})$ is finite.

Crucial object and key ingredient: Take an imaginary quadratic field K.

Theorem (Heegner point)

Assume that $\epsilon(E_K, 1) = -1$. Then there exists a rational point $y_K \in E(\mathbb{Q})$.

Theorem (Gross-Zagier formula)

The following holds up to explicit factors:

$$\frac{L'(E_K, 1)}{\Omega_{E_K}} \doteq \langle y_K, y_K \rangle_{NT}$$

Here $\langle -, - \rangle_{NT}$ is Neron-Tate height pairing on $E(\mathbb{Q})$ and Ω_{E_K} is the Neron period.

- Gross-Zagier 1980s: under Heegner Hypothesis $\ell \mid \mathrm{Cond}(E) \implies \ell$ splits in K;
- Yuan-Zhang-Zhang 2010s: totally real field.

Conjecture (p-converse theorem)

$\mathbb{Z}_p$-Corank of $\mathrm{Sel}_{p^\infty}(E/\mathbb{Q}) = r$ implies that $\mathrm{ord}_{s=1}L(E,s) = r$.

Conjecture (p-part of BSD formula)

Let c_v be Tamagawa numbers and R_E be the discriminant of $\langle -,- \rangle_{NT}$.

$$\frac{L^*(E,1)}{\Omega_E R_E} \sim \frac{\sharp Ш(E/\mathbb{Q}) \prod_v c_v}{\sharp E(\mathbb{Q})^2_{\mathrm{tor}}} \quad \forall \textit{ prime } p$$

General strategy when $r_{an}(E) \leq 1$:

p-adic special value formula plus Iwasawa main conjecture.

Fact

For any finite order Hecke character χ on K, $\frac{L(1,E_K\times\chi)}{\Omega_{E_K}} \in \bar{\mathbb{Q}}$.

Let $\Gamma = \mathrm{Gal}(K_\infty/K)$ with K_∞/K the maximal $\mathbb{Z}_p$-extension unramfied outside p.

Question (Perrin-Riou, Hida, Disegni, F.-Pan-Tian. etc)

Assume E is crystabelian at p and choose a Frobenius eigenvalue α with $v_p(\alpha) < 1$.Does there exist a tempered distribution $\mathcal{L}_{p,\alpha}(E_K, -)$ on Γ such that

$$\mathcal{L}_{p,\alpha}(E_K,\chi) \doteq \frac{L(1, E_K \times \chi)}{\Omega_{E_K}} \quad \forall\chi :\ \Gamma \to \bar{\mathbb{Q}}^\times \subset \mathbb{C}_p^\times \ \textit{finite order?}$$

Assume $\epsilon(E_K, 1) = -1$ and let $\langle -, - \rangle_{p,\alpha}$ be the p-adic height pairing on $E(\mathbb{Q})$ w.r.t α.

$$\mathcal{L}'_{p,\alpha}(E_K, 1) \doteq \langle y_K, y_K\rangle_{p,\alpha}?$$

- Perrin-Riou: $v_p(\alpha) = 0$ plus p splits in K plus HH;
- Disegni: $v_p(\alpha) = 0$ plus totally real field; partial when p non-split;
- Kobayashi: p splits in K plus E supersingular ($\implies$ $v_p(\alpha) > 0$) plus HH
- F.-Pan-Tian: p splits in Kplus E supersingular.

Fact

Let Ω_K be the cm period of K. Then for any anticyclotomic Hecke character χ on K of infinite type $(k \geq 1, -k)$, $\frac{L(1, E_K \times \chi)}{\Omega_K^k} \in \bar{\mathbb{Q}}$.

Let $\Gamma^{ac} = \mathrm{Gal}(K_\infty^{ac}/K)$ with $K_\infty^{ac} \subset K_\infty$ the maximal anticyclotomic $\mathbb{Z}_p$-extension.

Question

Assume $\epsilon(E_K, 1) = -1$. Does there exist a continuous function $\mathcal{L}_p(E_K, -)$ on the space of continuous characters $\Gamma^{ac} \to \mathbb{C}_p^\times$ such that

$$\mathcal{L}_p(E_K, \chi) \doteq \frac{L(1, E_K \times \chi)}{\Omega_K^k} \quad \forall \chi : \ \Gamma^{ac} \to \mathbb{C}_p^\times \text{ algebraic of infinite type (k,-k)?}$$

Moreover let $\log_p$ be the p-adic logarithm on $E(\mathbb{Q})$. Then

$$\mathcal{L}_p(E_K, 1) \doteq \log_p^2 y_K?$$

- Bertolini-Darmon-Prasanna: p splits in K plus HH;
- Liu-Zhang-Zhang: p splits in K and totally real field;
- Andreatta-Iovita: p non-split in K plus HH plus $p \geq 3$ plus E crystalline at p;
- F.-Wan: p non-split plus E crystabelian at p.

Thanks for your attention!