Strong approximation for Abelian varieties punctured at torsion points
去掉绕点后的某些阿贝尔簇的强逼近性质

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Notation

- k : number field
- $\Omega_k = \Omega_k^f \sqcup \infty_k$ set of places
- k_v for $v \in \Omega_k$
- $\mathcal{O}_v \subset k_v$ for $v \in \Omega_k^f$
- $\mathbf{A}_k$ ring of adèles
- $S \subset \Omega_k$ finite subset
  $\mathbf{A}_k^S$ adèles *without* S-components
  $pr^S : \mathbf{A}_k \to \mathbf{A}_k^S$ natural projection
- X : smooth variety over k (variety = separated scheme of finite type, geometrically integral)
- $\mathrm{Br}(X) = \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$ the cohomological Brauer group

Weak approximation

- $X(k) \hookrightarrow \prod_{v \in \Omega} X(k_v)$ diagonally
- Weak approximation holds if $X(k)$ is dense w.r.t. product topology
- $\emptyset \neq U \subset X$ Zariski open
- weak approximation on $X \implies$ weak approximation on U

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  - strong approximation on $X \not\Rightarrow$ strong approximation on U
- Example: $k = \mathbb{Q}$, $S \neq \emptyset$, $X = \mathbb{A}^1$, $U = \mathbb{A}^1 \setminus \{0\} = \mathbb{G}_m$
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  - purity for étale cohomology: $\mathrm{Br}(X) = \mathrm{Br}(U)$

First example: the affine space

- $X = \mathbb{A}^n$ satisfies strong approximation off $S \neq \emptyset$
- In this case, $\dfrac{\mathrm{Br}(X)}{\mathrm{Br}(k)} = \dfrac{\mathrm{Br}(X \setminus Z)}{\mathrm{Br}(k)} = 0$
- In general, should take into account the Brauer-Manin obstruction

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Theorem (D. Wei; Y. Cao & F. Xu)

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Brauer-Manin obstruction

- Manin's pairing:
- Fact: $X(k) \subseteq \overline{X(k)} \subseteq X(\mathbf{A}_k)^{\mathrm{Br}} \subseteq X(\mathbf{A}_k)$
- $S \subseteq \Omega_k$ finite subset
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- Similarly, we define Weak Approximation with Brauer-Manin obstruction using the product topology of $X(k_v)$ instead of the adélic topology.

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Question (We call it arithmetic purity)

Suppose that X satisfies approximation properties, what about $X \setminus Z$ for a closed subvariety Z of codimension ≥ 2?

- What about strong approximation with Brauer-Manin obstruction.

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Answers

- Positive answers
- Negative answers

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- Generalising examples $\mathbb{A}^n$ and $\mathbb{P}^n$

Positive answers

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  First result:

Theorem (D. Wei 2014)

Let X be a smooth toric variety such that $\bar{k}[X]^\times = \bar{k}^\times$. Then X verifies arithmetic purity for str. approx. with BM obs. off $S \neq \emptyset$.

Positive answers

- joint work with Y. Cao and F. Xu

Theorem (Cao-L.-Xu 2017)

Let G be a semi-simple simply connected linear algebraic group defined over a number field. Suppose that G is quasi-split (a Borel subgroup is defined over k). Then G verifies arithmetic purity for strong approximation off $S \neq \emptyset$.

- Example: SL_n
  For any Zariski closed subset Z such that $\mathrm{codim}(Z, SL_n) \geq 2$, $SL_n \setminus Z$ satisfies strong approximation off $S \neq \emptyset$.
- Open problem: remove the quasi-splitness condition.

Positive answers

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Theorem (Cao-L.-Xu 2017)

Let G be a semi-simple simply connected linear algebraic group defined over a number field. Suppose that G is quasi-split (a Borel subgroup is defined over k). Then G verifies arithmetic purity for strong approximation off $S \neq \emptyset$. (In this case $\mathrm{Br}(G)/\mathrm{Br}(k) = 0$.)

- Example: SL_n
  For any Zariski closed subset Z such that $\mathrm{codim}(Z, SL_n) \geq 2$, $SL_n \setminus Z$ satisfies strong approximation off $S \neq \emptyset$.
- Open problem: remove the quasi-splitness condition.

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Proposition

(1) GL_n verifies arithmetic purity (codim 2) for str. approx. with BM obs. off ∞_k if and only if the number field k is neither $\mathbb{Q}$ nor an imaginary quadratic field.

- The additional $1(= 3 - 2)$ dimension comes from $\mathbb{G}_m = GL_n/SL_n$ and Dirichlet's unit theorem.

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$GL_n \rightsquigarrow$ most general setting

- G connected linear algebraic group
- $G^{\mathrm{red}} = G/G^{\mathrm{u}}$, $G^{\mathrm{ss}} = [G^{\mathrm{red}}, G^{\mathrm{red}}]$, $G^{\mathrm{tor}} = G^{\mathrm{red}}/G^{\mathrm{ss}}$, $G^{\mathrm{sc}} \to G^{\mathrm{ss}}$

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Linear algebraic groups

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Theorem

Suppose that G^{sc} verifies arithmetic purity for str. approx. off ∞_k (in particular when it is quasi-split).
G verifies arithmetic purity of codimension $(2 + \dim G^{\mathrm{tor}})$ for str. approx. with BM obs. off ∞_k.

Negative answers

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- Example (Y. Cao & F. Xu 2013):
  $k = \mathbb{Q}$ or an imaginary quadratic field
  - $X = \mathbb{G}_m \times \mathbb{A}^1$ satisfies str. approx. with BM obs. off ∞_k
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Punctured Abelian varieties

Theorem (L. 2018)

E : elliptic curve, $rank(E(k)) > 0$
A : Abelian variety, $rank(A(k)) = 0$, $dim A \neq 0$
$T \subset E \times A$ a finite set of torsion points
$X = (E \times A) \setminus T$
If $pr_A(T)$ contains a k-rational point, then X does not satisfy Str. Approx. with BM obs. off ∞_k.
The converse is also true if $Ш(E \times A, k) < \infty$.

- Remark: the case where $T = \{O\}$ was known in [Cao-Liang-Xu 2017].

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Idea of proof

The proof use fibration methods with the following lemma.

Lemma

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- This lemma was known to [Harari-Voloch 2010] only in the very special case: $k = \mathbb{Q}$, $E : y^2 = x^3 + 3$, $rank(E(\mathbb{Q})) = 1$ and $T = \{O\}$.
- Final remark: As a consequence, $E \setminus O$ does not satisfy str. approx. with BM obs.off ∞_k

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Thank you for your attention !
谢谢大家!