A REMARK ON AN ARTICLE OF BOROVOI

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ABSTRACT. We give some details of a proof (applying the method of Borovoi in [1]) of the following fact: Let X be a homogeneous space of an connected linear algebraic group G with connected stabilizer (or Abelian stabilizer if G is assumed simply connected) over a number field or a p-adic field. If there exists a zero-cycle of degree 1 on X, then X has a k-rational point.

We are going to prove the following two results. Other approachs (by C. Demarche, by J. Starr and M. Borovoi) are sketched in [2], Proposition 4.6.3, the following approach are also mentioned there without details, and we give some more details in this note.[1]

We keep all the notations in Borovoi [1].

Theorem 0.1. *Let X be a homogeneous space of an connected linear algebraic group G over a field k with geometric stabilizer $\bar{H}$. We assume one of the following conditions.*

(1) The stabilizer $\bar{H}$ is connected.

*(2) The stabilizer $\bar{H}$ is Abelian and G^{ssu} is simply connected (*i.e. *G^{ss} is semi-simple simply connected).*

Suppose that k is a local field of characteristic 0. If there exists a zero-cycle of degree 1 on X, then X has a k-rational point.

Theorem 0.2. *Les X be a homogeneous space as in Theorem 0.1 satisfying (1) or (2). Suppose that k is a number field. If there exists a zero-cycle of degree 1 on $X_v = X \times_k k_v$ for all $v \in \Omega_k$, and if there is no Brauer-Manin obstruction associated to $\text{Б}(X)$ (i.e. $m_H(X) = 0$), then X has a k-rational point.*

In particular, the existence of a zero-cycle of degree 1 on X implies the existence of a k-rational point on X.

Since the argument of Borovoi, [1] §5, is purely group theoretic and does not depend on the base field, we are reduced to show the theorems with the following assumptions in place of (1), (2):

(2.1.1) The group G^{ssu} is simply connected, and

(2.1.2) the quotient $\bar{H}/\bar{H}^{\text{ssu}}$ is Abelian, hence of multiplicative type.

Firstly, if k is either $\mathbb{R}$ or $\mathbb{C}$, the statement is evidently true. We suppose that k is a p-adic field or a number field.

We remark that, if k is a number field, we can also define the Brauer-Manin obstruction $m_H(X) \in \text{Б}(X)^D$ using any family of local zero-cycles of degree 1 (well-defined independent of the choice of a family of local zero-cycles).

We can copy the following two lemmas (for k local or global).

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[1] This note is not written very carefully, there may be some mistakes.

Lemma 0.3 ([1], Lem. 3.1)**.** *Let X be a homogeneous space of a linear group G and N a normal subgroup of G. Then there exists a quotient $\varphi : X \to Y = X/N$. In particular, φ is surjective and its geometric fibers are the orbits of N.*

Lemma 0.4 ([1], Lem. 3.2)**.** *Let X be a homogeneous space of a unipotent group U over a perfect field k. Then,*
(i) There exist a k-rational point (a fortiori a zero-cycle of degree 1) on X.

Proposition 0.5 ([1], Prop. 3.3)**.** *If G is a torus, then*
(i) the assertions of Theorem 0.1 and 0.2 is valid,
(iii) assuming k is a number field, $X(k)$ is dense in $X(k_\infty)$ if $X(k) \neq \emptyset$.

Proof. Since G is commutative, the classic restriction-corestriction argument shows the existence of a k-rational point if k is a p-adic field (resp. the existence of a k_v-rational point ($\forall v \in \Omega$) if k is a number field). Then the argument of [1], Proposition 3.3, proves the statement. □

We copy the following proposition, which will only be used when k is a number field. We *don't* need a zero-cycle-version of this statement.

Proposition 0.6 ([1], Prop. 3.4)**.** *Assume that k is a number field. If G is simply connected (*i.e. *G^{red} is semi-simple simply connected) and $\bar{H} = \bar{H}^{\mathrm{ssu}}$, then*
(i) the homogeneous space X has a k-rational point if $X(k_\infty) \neq \emptyset$.

For the case where k is a number field, the proof of the following proposition uses Proposition 0.5(i) for zero-cycles, Proposition 0.6 for rational points, and Proposition 0.5(iii) for rational points. For the case where k is a p-adic field, the proof of the following proposition uses only Proposition 0.5(i) for zero-cycles.

Proposition 0.7 ([1], Prop. 3.5)**.** *Assume that G^{ss} is (semi-simple) simply connected and*
(*) the homomorphism $\bar{H}^{\mathrm{ssu}} \to G^{\mathrm{tor}}_{bark}$ induced by $\bar{H} \subset G_{\bar{k}}$ is injective.
Then
(i) the assertions of Theorem 0.1 and 0.2 are valid.

Proof. We define the quotient $\varphi : X \to Y = X/G^{\mathrm{ssu}}$ by the lemma 0.3. The base Y is a homogeneous space of a torus $G^{\mathrm{tor}} = G/G^{\mathrm{ssu}}$, and the fibers are principal homogeneous spaces of G^{ssu}.

If k is a p-adic field, Y has a k-rational point y by Proposition 0.5(i). The fiber X_y gives a class in $H^1(k, G^{\mathrm{ssu}}) = 0$ (by assumption G^{ss} is semi-simple simply connected, hence $H^1(k, G^{\mathrm{ss}}) = 0$, and G^{ssu} is an extension of G^{ss} by G^{u}.)

If k is a number field, we suppose that for all $v \in \Omega_k$, there exists a zero-cycle of degree 1 on X_v, so does Y_v. We know that $m_H(Y) = \varphi_*(m_H(X)) = 0$. By Proposition 0.5(i) Y has a k-rational point.

For any *infinite* places v, there is a zero-cycle of degree 1 on X_v, hence a k_v-rational point, *i.e.* $X(k_\infty) \neq \emptyset$. As φ is smooth, $\varphi(X(k_\infty))$ is open (non-empty) in $Y(k_\infty)$. There exists a k-rational point $y \in Y(k) \cap \varphi(X(k_\infty))$ (Proposition 0.5(iii)).

Consider the fiber X_y, the same argument as in [1] shows that $X_y(k_\infty)$ is not empty. By 0.6(i) X_y has a k-rational point, hence X has a k-rational point. □

We have to remove the assumption (*).

First, we define a k-form H^m of $\bar{H}^{\mathrm{mult}}$ as in [1]. We inject H^m into a quasi-trivial torus T, $j : H^m \hookrightarrow T$. We set $F = G \times T$, $H \to F = G \times T$. We define a

$F_{\bar{k}}$-equivariant map $\bar{\pi} : \bar{Y} = \bar{H}\backslash F_{\bar{k}} \to X_{\bar{k}}$, which is a torsor under $T_{\bar{k}}$. We verify that $\bar{H}^{\mathrm{mult}} \to F_{\bar{k}}^{\mathrm{tor}}$ is injective (*i.e.* satisfies (*)). Let k' be a finite extension of k, $\bar{\pi}$ descends to k' as soon as X has a k'-rational point.

The following lemma (will be proved later) works also for zero-cycles.

Lemma 0.8 ([1], Lem. 4.3)**.** *If k is a p-adic field, and if X has a zero-cycle of degree 1, then there exists a k-form (Y, π) of $(\bar{Y}, \bar{\pi})$.*

If k is a number field, and if X_v has a zero-cycle of degree 1 for any v, then there exists a k-form (Y, π) of $(\bar{Y}, \bar{\pi})$.

We copy the following lemma, which is used only when k is a number field.

Lemma 0.9 ([1], Lem. 4.4)**.** *If k is a number field, and assume the existence of a k-form (Y, π) of $(\bar{Y}, \bar{\pi})$. Then $Б(X) \xrightarrow{\simeq} Б(Y)$ is an isomorphism.*

Proof of Theorems 0.1 and 0.2 modulo Lemme 0.8. We only prove the case where k is a number field, if k is a p-adic field, the proof is similar without consideration of $m_H(\cdot)$.

We suppose that X_v has a zero-cycle of degree 1 for all v and $m_H(X) = 0$. For any *closed* point P (its residue field K is a finite extension of k_v) of X_v, the fiber Y_{vP} of $\pi_v : Y_v \to X_v$ has a K-rational point because $T_v \times_{k_v} K$ is a quasi-trivial torus $H^1(K, T) = 0$, then Y_v has a K-rational point. Hence "X_v has a zero-cycle of degree 1" implies that Y_v has a zero-cycle of degree 1. By Lemma 0.9 $m_H(Y) = 0$. The proposition 0.7(i) says that Y has a k-rational point. □

Finally we prove the lemma 0.8.

We construct a cohomological class $\eta \in H^2(k, T)$ from $(\bar{Y}, \bar{\pi})$ as in [1].

Lemma 0.10 ([1], Lem. 4.8)**.** *The class η equals to 0 if and only if there exists a k-form (Y, π) of $(\bar{Y}, \bar{\pi})$.*

Proof. If k is a local field, the restriction-corestriction argument on $H^2(k, T)$ shows that $\eta = 0$, the lemma 0.10 completes the proof.

If k is a number field, we've seen that $loc_v(\eta) \in H^2(k_v, T)$ is 0 for all v. As T is quasi-trivial, $\eta \in Ш^2(k, T) = 0$, the lemma 0.10 completes the proof. □

Actually, we don't change much in Borovoi's argument.

References

[1] M. Borovoi. The Brauer-Manin obstructions for homogeneous spaces with connected or abelian stabilizer. *J. reine angew. Math.*, 473:181–194, 1996.

[2] Y. Liang. Arithmetic of 0-cycles on varieties defined over number fields. Preprint, arXiv:1107.1634.

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