Recent progress concerning local-global principle for zero-cycles on algebraic varieties

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Notation

- k : number field
- Ω : set of places of k
- k_v : completion of k at the place $v \in \Omega$
- X : smooth variety, geometrically integral over k, suppose that it is proper by taking a smooth compactification.
- $\mathrm{Br}(X) = \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$: cohomological Brauer group
- $X_v = X \otimes_k k_v$
- $M/n = \mathrm{Coker}(M \xrightarrow{n} M)$ for any abelian group M

Introduction

Local-global principle and Brauer–Manin obstruction

- Diagonal embedding $X(k) \to \prod_{v\in\Omega} X(k_v)$
- Hasse principle if $X(k_v) \neq \emptyset\ (\forall v \in \Omega) \Rightarrow X(k) \neq \emptyset$
- Weak approximation if $X(k)$ is dense in $\prod_{v\in\Omega} X(k_v)$
- (1970's) Manin pairing :

$$\begin{array}{ccccc} \prod_{v\in\Omega} X(k_v) & \times & \mathrm{Br}(X) & \to & \mathbb{Q}/\mathbb{Z} \\ (\ \{x_v\}_{v\in\Omega} & , & b\ ) & \mapsto & \sum_{v\in\Omega} \mathrm{inv}_v(b(x_v)) \end{array}$$

- Brauer–Manin set $[\prod X(k_v)]^{\mathrm{Br}}$: left “kernel” of the pairing
- Brauer–Manin obstruction

$$X(k) \subset \overline{X(k)} \subset \left[\prod X(k_v)\right]^{\mathrm{Br}} \subset \prod X(k_v)$$

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Zero-cycles

- Similarly, the Manin pairing can be defined on 0-cycles and it factorizes through the modified Chow groups

$$\prod_{v\in\Omega} \mathrm{CH}'_0(X_v) \times \mathrm{Br}(X) \to \mathbb{Q}/\mathbb{Z}$$

$$\mathrm{CH}'_0(X_v) = \begin{cases} \mathrm{CH}_0(X_v) & , v \text{ non-arch.} \\ \mathrm{Coker}\left[\mathrm{CH}_0(X_{\mathbb{C}}) \stackrel{N_{\mathbb{C}|\mathbb{R}}}{\to} \mathrm{CH}_0(X_{\mathbb{R}})\right] & , v \text{ real} \\ 0 & , v \text{ complex} \end{cases}$$

- Sequences [with $A_0 = \mathrm{Ker}(\deg : CH_0 \to \mathbb{Z})$ and $-^* = \mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$]

$$(\mathrm{E}) \qquad \varprojlim_n \mathrm{CH}_0(X)/n \to \prod_{v\in\Omega} \varprojlim_n \mathrm{CH}'_0(X_v)/n \to \mathrm{Br}(X)^*$$

$$(\mathrm{E}_0) \qquad \varprojlim_n A_0(X)/n \to \prod_{v\in\Omega} \varprojlim_n A_0(X_v)/n \to (\mathrm{Br}(X)/\mathrm{Br}(k))^*$$

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Main conjecture

Conjecture (Colliot-Thélène–Sansuc, Kato–Saito)

The sequence is exact for all proper smooth varieties.

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- the exactness of (E) implies :
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Classical Results – dimension 0 and 1

Dimension 0

- $X = \mathrm{Spec}(k)$
- Global class field theory : exact sequence

$$0 \to \mathrm{Br}(k) \to \bigoplus_{v\in\Omega} \mathrm{Br}(k_v) \to \mathbb{Q}/\mathbb{Z} \to 0$$

- Taking dual $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z}) \Rightarrow$ exactness of (E) for $\mathrm{Spec}(k)$

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- $X = C$ a smooth projective curve, assume the finiteness of the Tate–Shafarevich group $Ш(k, Jac(C))$

Theorem (Saito 89, Colliot-Thélène 99)

The sequence (E) *is exact for* C.

- Remark : The Brauer–Manin obstruction is conjectured, by Skorobogatov, to be the only obstruction to the Hasse principle and to weak approximation for rational points on curves. (Open even if $Ш < \infty$ supposed)

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Recent progress – higher dimensional varieties

- A general relation between the arithmetic of rational points and the arithmetic of 0-cycles
- Results on (smooth compactifications of) homogeneous varieties
- Results on fibrations
  - Fibrations over projective spaces
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General relation, rational points versus 0-cycles

Theorem (Liang 2011)

Let X be a geometrically rationally connected k-variety. Let L be a finite extension of k, denote by $\mathscr{K}_L$ the set of finite extensions of k which are linearly disjoint from L.
Assume that the Brauer–Manin obstruction is the only obstruction to weak approximation for K-rational points on X_K (i.e. $\overline{X(K)} = [\prod_{w\in\Omega_K} X(K_w)]^{\mathrm{Br}}$) for all $K \in \mathscr{K}_L$.
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Homogeneous varieties

The following result is deduced from the previous Theorem and Bovoroi's results (1996) on rational points on homogeneous varieties.

Theorem

Let G be a connected linear algebraic group. Let Y be a homogeneous space of G and X be one of its smooth compactifications. Suppose that the geometric stabilizer of Y is connected (or is abelian if G is semisimple simply connected). Then (E) *is exact for X.*

Fibrations over projective spaces (1)

Theorem (Liang 2010)

Let $X \to \mathbb{P}^n$ be a proper dominant morphism with geometrically rationally connected generic fibre. Suppose that

- *all codimension 1 fibres are geometrically integral;*
- *the Brauer–Manin obstruction is the only obstruction to weak approximation for rational points or 0-cycles of degree 1 on "almost all" closed fibres.*

Then the sequence (E) is exact for X.

Remark. Similar results for rational points were obtained by Harari (1994, 1997, 2007)

- Main ingredients in the proof :
  - induction on n
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Fibrations over projective spaces (2)

Theorem (Salberger 88, Colliot-Thélène–Swinnerton-Dyer 94, C-T–Skorobogatov–S-D 98, Wittenberg 07, Liang 2011)

Let $X \to \mathbb{P}^n$ be a proper dominant morphism with geometrically integral generic fibre. Suppose that

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Normic bundles

- For some varieties defined by explicit equations, the exactness of (E) has been proved by essentially the same fibration argument.
- Consider the equations of the form

$$N_{K|k}(\mathbf{x}) = P(t_1, \ldots, t_m)$$

where $K|k$ is a finite extension of degree d and P is a polynomial (or a rational function).
It defines a locally closed subvariety of $R_{K|k}\mathbb{A}^1 \times \mathbb{A}^m \simeq \mathbb{A}^{d+m}$, let X be a smooth compactification of the smooth locus of this variety.

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  - (Heath-Brown–Skorobogatov et al. 2002-2011) $m = 1$, $P(t) = ct^{n_1}(1-t)^{n_2}$ with $n_1, n_2 \in \mathbb{Z}_{\leq 0}$;
  - (Wei 2012) $K|k$ is of prime degree but not a cyclic extension;
  - (Derenthal–Smeets–Wei 2012) $m = 1$, $K|k$ is of degree 4, $P(t)$ is of degree 2, irreducible over k but split over K;
  - (Liang 2012) K is the compositum of extensions $K_1, \ldots, K_n$ of distinct prime degrees.
  - (Cao–Liang 2013) $K|k$ is a biquadratic extension, the rational function $P(t_1, \ldots, t_m) = Q(t_1, \ldots, t_m)^2$ is a complete square.
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Fibrations over curves

Theorem (C-T 2000, Frossard 2003, van Hamel 2003, Wittenberg 2012; C-T 2010, Liang 2011, Liang 2012)

Let $X \to C$ be a proper dominant morphism to a smooth projective curve with geometrically rationally connected generic fibre.
Suppose that

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- *"almost all" closed fibres satisfy weak approximation for rational points or for 0-cycles of degree 1.*

If $\text{Ш}(k, \text{Jac}(C))$ is finite, then the sequence (E) is exact for X.

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Thank you for your attention!