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Zero-cycles on varieties fibered over Curves by Châtelet surfaces

Notation

k, Ω_k. k_v $(v \in \Omega)$

$X/_k$ algebraic variety (seperated scheme of finite type)

<u>Supposed</u> proper smooth. geometrically integral.

$X_v = X \otimes_k k_v$ $\qquad\qquad \mathrm{Br}X = H^2_{et}(X, \mathbb{G}_m)$.

$CH_0(X)$: Chow group of 0-cycles

<u>Question</u> Does there exist a 0-cycle of degree 1 on X?

Local-global principle on X?

The Brauer group $\mathrm{Br}(X)$ gives an obstruction to Hasse principle (HP) and Weak Approximation (WA) for 0-cycles.

Manin pairing

$$\prod_{v \in \Omega} Z_0(X_v) \times \mathrm{Br}X \longrightarrow \mathbb{Q}/\mathbb{Z}$$

$$\{z_v\} \;,\; b \longmapsto \langle \{z_v\}, b \rangle = \cancel{\sum_{v \in \Omega} \mathrm{inv}_v\Big(\sum_{P_v} \mathrm{cores}\Big)}$$

$$\{z_v\} = \sum n_{P_v} \cdot P_v$$

$$\langle \{z_v\}, b \rangle = \sum_{v \in \Omega} \mathrm{inv}_v\Big(\sum_{P_v} n_{P_v} \mathrm{cores}_{k(P_v)/k_v}(b(P_v))\Big)$$

$\mathrm{inv}_v : \mathrm{Br}k_v \hookrightarrow \mathbb{Q}/\mathbb{Z}$.

$$\prod_{v\in\Omega} CH_0(X_v) \times Br X \longrightarrow \mathbb{Q}/\mathbb{Z}$$

$$\prod_{v\in\Omega} CH_0'(X_v) \times Br X \longrightarrow \mathbb{Q}/\mathbb{Z}$$

$$CH_0'(X_v) = \begin{cases} CH_0(X_v) & v \text{ non-archimedean} \\ 0 & v \ \mathbb{C}. \\ \mathrm{Coker}[N_{\mathbb{C}/\mathbb{R}}: CH_0(X_{\mathbb{C}}) \to CH_0(X_{\mathbb{R}})] & v \ \mathbb{R} \end{cases}$$

Fact $\quad 0 \to Br k \to \bigoplus Br k_v \to \mathbb{Q}/\mathbb{Z} \to 0$

$\leadsto$

$$CH_0(X) \to \prod CH_0'(X_v) \to \mathrm{Hom}(Br X, \mathbb{Q}/\mathbb{Z})$$

is a complex.

$$CH_0(X)/m \to \prod_v CH_0'(X_v)/m \to \mathrm{Hom}(Br X[m], \mathbb{Q}/\mathbb{Z})$$

(E) $\quad \varprojlim_m CH_0(X)/m \to \prod \varprojlim_m CH_0'(X_v)/m \to \mathrm{Hom}(Br X, \mathbb{Q}/\mathbb{Z})$ complex

(E₀) $\quad \varprojlim A_0(X)/m \to \prod \varprojlim_m A_0(X_v)/m \to \mathrm{Hom}\left(\frac{Br X}{Br k}, \mathbb{Q}/\mathbb{Z}\right)$

where $\quad A_0 = \ker(CH_0 \xrightarrow{\deg} \mathbb{Z})$.

$\boxed{\text{Conjecture}}$ (Colliot-Thélène–Sansuc, Kato–Saito)

(E) is exact for all proper smooth (geo. int.) varieties.

Remark

(E) exact $\Rightarrow$

- (E_0) is exact.

- BM-obs is the only obstruction to HP for 0-cycles of deg. 1.

i.e. $\exists \{z_v\} \perp \mathrm{Br} X \Rightarrow \exists\, z$ $\deg(z)=1$.
$\hookrightarrow \deg = 1$ $\quad$ ($z \in CH_0(X)$)

- WA. 0-cycles of deg. δ

i.e. $\forall m>0$, $\forall S$ finite $\subseteq \Omega$.

$\{z_v\} \perp \mathrm{Br} X \Rightarrow \exists z = z_{m,S}$ st $z = z_v \in CH_0(X_v)/m$.
$\hookrightarrow \deg = \delta$ $\quad$ $\hookrightarrow \deg = \delta$ $\quad$ $(\forall v \in S)$

Some known results:

- $\dim X = 0$. $X = \operatorname{Spec} k$

(E) is the dual of "$\mathrm{Br} k \to \bigoplus \mathrm{Br} k_v \to \mathbb{Q}/\mathbb{Z}$".

- $\dim X = 1$ $\quad$ X = curve.

Saito 89, Colliot-Thélène 99:

(E) is exact if $Ш(\mathrm{Jac} X, k) < +\infty$

In particular, if $X = E$ an elliptic curve, we have

$$\overline{E(k)} \to \prod_{v \in \Omega} E(k_v)' \to H^1(k, E)^* = \left(\frac{\mathrm{Br} E}{\mathrm{Br} k}\right)^* \quad (\to Ш(k, E)^* \to 0)$$

$* = \mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$

Cassels-Tate exact sequence.

- $\dim X > 1$

2 types of results $\begin{cases} \text{fibrations} \\ \text{homogeneous spaces.} \end{cases}$

Thm (Wittenberg 2009)

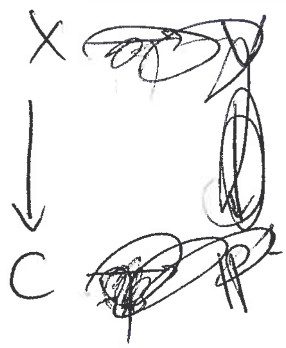
$X \to C$ fibration. C = curve. $Ш(\mathrm{Jac}\, C, k) < +\infty$

- all fibers are abelian-split (e.g. if the fibers are geometrically integral)
- almost all fibers satisfy WA (for rational points or for 0-cycles of deg. 1)

Then (E) is exact for X.

~~Example~~ Poonen's example (2010)

Consider:



- $Ш(\mathrm{Jac}(C), k) < +\infty$
- $\emptyset \neq C(k)$ but finite
- all fibers are geometrically integral.
- generic fiber = Châtelet surface

$$\left(\begin{array}{l} y^2 - a z^2 = P(x), \quad \deg P = 4, \\ a \in k^* \setminus k^{*2}, \quad P(x) \in k(C)[x]. \end{array} \right)$$

[P is irred/$k(C)$ in this example]

- $\forall\, P \in C(k)$ the fiber X_P : smooth.

$X_P(k_v) \neq \emptyset \quad \forall v \in \Omega_k$

but $X_P(k) = \emptyset$.

~~For Châtelet su~~

Poonen constructed such a X, $X(k_v) \neq \emptyset$, $X(k) = \emptyset$.

and moreover, $\exists\, x_v \in X(k_v)$ s.t. $\{x_v\} \perp \mathrm{Br}\, X$.

Thm (Colliot-thélène 2010)

$\exists\ z \in CH_0(X)$. $\deg z = 1$.

Attention : One can not apply Wittenberg's thm. for the ~~ex~~ exactness of (E).

Since for Châtelet Surfaces. BM-obs. is the only obstruction to ~~B~~ HP/WA for rational points (or 0-cycles of deg 1).

~~but~~ HP and WA often fail when the Brauer group is not trivial (modulo constant ~~$\frac{Br}{Brk}$~~ $\frac{Br}{Br\,k}$).

Def (~~gen~~ generalised hilbertian subset)

V : ~~geometrically~~ Geometrically integral variety $/k$.

Hil : subset of closed points of V is said to be a generalized Hilbertian subset. if

$\exists\ Z \xrightarrow[\substack{\text{étale}\\ \text{finite}}]{\rho} \underset{\ne \phi}{U} \xhookrightarrow[\text{open}]{} V$ where Z is integral.

st.

$Hil = \{\theta \in U \mid \text{closed pt. st } \rho^{-1}(\theta) \text{ is connected}\}$.

Δ For Poonen's 3-fold.

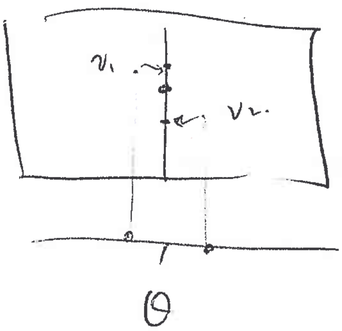
$Hil = \{\theta \in C \mid P_\theta(x) \in k(\theta)[x] \text{ is irred.} / k(\theta)\} \subseteq C$.

is a generalized Hilbertian subset.

Proof (Sketch)

- recall of the fibration method

X
$\downarrow$
C



difficulties :

1. pass from 0-cycles to rational points (of high degree)

2. find $\theta \in C(k)$, instead of $\theta \in C$.

$\triangle$ difference : rat. pt $\leftrightarrow$ 0-cycles

$\langle$ not so much rational points on C.
lots of 0-cycles on C.

(1) "good places"

$X \downarrow C$ fibers are all geometrically integral

estimate of Lang-Weil (in family) + Hensel lemma

$\Longrightarrow$ $\exists\, S \underset{\text{fini}}{\subset} \Omega_k$ s.t. $\forall\, \theta \in C$ closed pt.

$\forall\, w \in \Omega_{k(\theta)} \setminus S$ $\quad X_\theta(k(\theta)_w) \neq \phi$.

(2) "bad places" $v \in S$

- moving lemma for 0-cycles

0-cycles $\rightsquigarrow$ effective 0-cycles in good position ($\simeq$ rational pts of high degree)

+ implicit function theorem

~~To conclude~~

To conclude, it remains to answer:

How to approximate effective 0-cycles by a closed point $\theta \in \underline{\mathrm{Hil}}$?

<u>Key Lemma</u> (Hilbert - Ekedahl - Liang)
$(d=1, g=0)$

Let C be a curve (projective, smooth, geo. int) over a number field k. $g = g(C)$.

$\mathrm{Hil} \subset C$ generalized Hilbertian subset.

$y \in Z_0(C)$. effective 0-cycle. $\deg y = d > 2g$.

$\underset{\text{finite}}{S} \subset \Omega_k$.

for $\forall v \in S$. $z_v \in Z_0(C_v)$ effective separable 0-cycle of degree d

st. $\mathrm{Supp}(z_v) \cap \mathrm{Supp}(y) = \phi$. and

$z_v \sim y$ on C_v

Then, $\exists$ closed point $\theta \in C$ st.

(1) $\theta \in \mathrm{Hil}$

(2) $\theta \sim y$ on C

(3) θ is sufficiently closed to z_v for all $v \in S$.

Proof:

$v \in S$ $\qquad z_v - y = \mathrm{div}_{C_v}(f_v)$, $\quad f_v \in k_v(C_v)^*/k_v^*$

$\deg y = d > 2g$. $\overset{\text{Riemann-Roch}}{\Longrightarrow}$ $\Gamma(C, \mathcal{O}_C(y))$ is a vector space of dim $r = d+1-g > g+1 \geq 1$

WA for $\mathbb{P}^{r-1}$ $\Rightarrow$ $\exists\, f \in k(C)^*/k^*$ st.

(i) f sufficiently close to f_v $(v \in S)$

(ii) $\mathrm{div}_C(f) = y' - y$ where y' is an effective 0-cycle st $\mathrm{Supp}(y') \cap \mathrm{Supp}(y) = \phi$

Then $\underset{\text{close enough}}{y' \simeq z_v}$ $\quad (v \in S)$

f defines a k-morphism $\varphi: C \to \mathbb{P}^1$ st

$\varphi^*(\infty) = y$

$\varphi^*(0) = y'$

$\mathrm{Hil} \subset C \quad \longleftrightarrow$

$$\begin{tikzcd} Z \arrow[r] \arrow[dr, "\varphi'"'] & U \subset C \arrow[d, "\varphi"] \\ & \mathbb{P}^1 \end{tikzcd}$$

$\exists\ \mathrm{Hil}' \subset \mathbb{P}^1 \quad \longleftrightarrow \quad \varphi'$

st $\forall\, \theta' \in \mathrm{Hil}' \Rightarrow \theta = \varphi^{-1}(\theta') \in \mathrm{Hil}$.

Hilbert's ~~irreducibl~~ irreducibility thm. (effective version by Ekedahl)

$\Rightarrow$ $\mathrm{Hil}' \cap \mathbb{P}^1(k)$ is dense in $\prod_{v \in S} \mathbb{P}^1(k_v)$

$\qquad\qquad \Downarrow$

hence $\exists\, \theta' \simeq 0 \in \mathbb{P}^1(k_v)$ $(\forall v \in S)$

$\Rightarrow$ $\theta := \varphi^{-1}(\theta') \in \mathrm{Hil}$. $\qquad \theta \simeq \varphi^*(0) = y' \simeq z_v$ $(v \in S)$

Moreover, $\theta \sim y' \sim y$ on C. $\qquad$ #