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某些~~双~~二次丛上的有理点与零-cycle.

Rational points and zero-cycles on certain biquadratic bundles.

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X : algebraic variety $/k$ number field
(Supposed proper smooth. geometrically integral.)

Ω_k, $k_v\ (v\in\Omega)$,

We consider local-global principle for rational points and for 0-cycles on varieties.

The Brauer group $\mathrm{Br}X = H^2_{et}(X,\mathbb{G}_m)$ gives an obstruction to Hasse Principle (HP) and to Weak Approximation (WA).

Manin pairing:

$$\prod_{v\in\Omega} X(k_v) \times \mathrm{Br}X \longrightarrow \mathbb{Q}/\mathbb{Z}$$

$$\{x_v\},\quad b \longmapsto \sum_{v\in\Omega_k} \mathrm{inv}_v(b(x_v))$$

$\mathrm{inv}_v : \mathrm{Br}\, k_v \hookrightarrow \mathbb{Q}/\mathbb{Z}$ local invariant.

$$\left[\prod X(k_v)\right]^{\mathrm{Br}} := \left\{ \{x_v\} \,\middle|\, \{x_v\}\perp \mathrm{Br}X \right\}$$

Fact

$$X(k) \subseteq \overline{X(k)} \subseteq \left[\prod X(k_v)\right]^{\mathrm{Br}} \subseteq \prod X(k_v)$$

Obstructions to HP
WA.

~~Similarly~~,

Def. Brauer-Manin obstruction is the only obstruction to $\begin{cases} \text{HP} \\ \text{WA} \end{cases}$ if $\begin{cases} \left[\prod X(k_v)\right]^{\mathrm{Br}} \neq \phi \Rightarrow X(k) \neq \phi. \\ \overline{X(k)} = \left[\prod X(k_v)\right]^{\mathrm{Br}}. \end{cases}$

Similarly, we have a pairing for Chow groups of 0-cycles

$$\prod_{v \in \Omega} CH_0'(X_v) \times \mathrm{Br}X \to \mathbb{Q}/\mathbb{Z}.$$

CH_0' modified Chow group.

$$CH_0'(X_v) = \begin{cases} CH_0(X_v) & , v \text{ non archimedean} \\ 0 & , v : \mathbb{C} \\ \mathrm{Coker}\left[N_{\mathbb{C}/\mathbb{R}} : CH_0(X_{\mathbb{C}}) \to CH_0(X_{\mathbb{R}})\right] & , v : \mathbb{R} \end{cases}$$

Fact $\rightsquigarrow$ complex

$$CH_0(X) \to \prod_{v \in \Omega} CH_0'(X_v) \to \mathrm{Hom}(\mathrm{Br}X, \mathbb{Q}/\mathbb{Z})$$

$$\to (E) : \varprojlim_m CH_0(X)/m \to \prod_{v \in \Omega} \varprojlim_m CH_0'(X_v)/m \to \mathrm{Hom}(\mathrm{Br}X, \mathbb{Q}/\mathbb{Z})$$

Conjecture (Colliot-Thélène et al.)

- BM-obs. is the only obstruction ~~for~~ to HP and WA for ~~uni-rational varieties~~ rational points on ~~the ra~~ (geometrically) uni-rational varieties. ($\mathbb{P}^n \dashrightarrow X$ dominant)

- (E) is exact for all proper smooth varieties
  i.e. BM-obs. is the only obstruction ~~for 0-cycles (of deg 1)~~ ~~on all~~ to HP and WA for 0-cycles on all varieties

Thm 0. (Colliot-Thélène - Skorobogatov - Swinnerton-Dyer 98 ($n=1$), Wittenberg 2007 ($\forall n$), Liang 2011 (for cycles))

$X \to \mathbb{P}^n$

- codim 1 fibers are abelian-split.
- almost all closed fibers satisfy $\begin{cases} HP \\ WA \end{cases}$.

Then (1) BM-obs is the only obstruction to $\begin{cases} HP \\ WA \end{cases}$ for rational points

Assuming Schinzel's hypothesis.

(2) (E) is exact.

可看一看 Schinzel's hypothesis:

$f_1, \cdots, f_m \in \mathbb{Q}[X]$ irred. s.t. $f_i(\mathbb{Z}) \subseteq \mathbb{Z}$ & $\nexists p$ s.t. $\frac{f_i(\mathbb{Z})}{p} \subset \mathbb{Z}$ $\forall i$

then $\#\{n \in \mathbb{N} \mid f_1(n), f_2(n), \cdots, f_m(n) \text{ are all primes}\} = +\infty$.

$\Rightarrow$ Dirichlet's thm on primes in arithmetic progression

Twin prime conjecture

In particular, consider the compactification X of the affine variety

$$N_{K/k}(\vec{x}) = P(t_1, \cdots, t_n) \subseteq \mathbb{A}^d \times \mathbb{A}^n \qquad d = [K:k]$$

$P \in k[t_1, \cdots, t_n]$ polynomial.

if K/k is a cyclic extension.

Then the conjecture is true for X (for rational points, Schinzel assumed)

~~Question~~

We restrict to the equations

$$N_{K/k}(\vec{x}) = P(t_1, \dots, t_n).$$

Some recent results were obtained by:

* Heath-Brown - Skorobogatov
* CT - Harari - Sk.
* Browning - Heath-Brown.
* Wei Dasheng
* Derenthal - Smeets - Wei

~~[illegible]~~
~~[illegible] Q~~

However, a "simple" case is still open:

when K/k is biquadratic. i.e. $\mathrm{Gal}(K/k) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

$\boxed{\text{difficulty:}}$

a ~~the~~ general fiber is defined by

$$N_{K/k}(\vec{x}) = c.$$

. $c \in k^*$ can be a norm locally everywhere, but not a global norm, if K/k is not cyclic.

<u>Thm</u> (Cao-Liang) Let X be a smooth compactification of the affine variety defined by

$$\boxed{N_{K/k}(\vec{x}) \cancel{=Q(t_1, \dots, t_n)} = Q(t_1, \dots, t_n)^2}$$ in $\mathbb{A}^4 \times \mathbb{A}^n$ with K/k biquadratic.

Then the conjecture is true for X. (assuming Schinzel for rational points)

idea of proof.

reduce to the known cases.

$K = k(\sqrt{a}, \sqrt{b})$ $\quad \mathrm{Gal}(K/k) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

$k_a = k(\sqrt{a})$, $k_b = k(\sqrt{b})$, $k_{ab} = k(\sqrt{ab})$ three distinct subfields of K. ~~the only~~

Consider E/F biquadratic

$E = F(\sqrt{a}, \sqrt{b})$ $\quad \mathrm{Gal}(E/F) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

the only non-trivial subfields are

$F_a = F(\sqrt{a})$, $F_b = F(\sqrt{b})$, $F_{ab} = F(\sqrt{ab})$.

$$1 \to T \to \underbrace{R_{E/F}}_{\text{Weil restriction}} \mathbb{G}_m \xrightarrow{N_{E/F}} \mathbb{G}_m \to 1$$

T: torus: $N_{E/F}(\vec{x}) = 1$

$$\begin{tikzcd}
1 \arrow[r] & S \arrow[r] \arrow[d, "\alpha"] & R_{F_a/F}\mathbb{G}_m \times R_{F_b/F}\mathbb{G}_m \times R_{F_{ab}/F}\mathbb{G}_m \arrow[r, "\lambda"] \arrow[d] & \mathbb{G}_m \arrow[r] \arrow[d, "2"] & 1 \\
1 \arrow[r] & T \arrow[r] & R_{E/F}\mathbb{G}_m \arrow[r, "\mu"] & \mathbb{G}_m \arrow[r] & 1
\end{tikzcd}$$

product of 3 norms.

$(u, v, w) \mapsto u \cdot v \cdot w$ (in the field E.)

where S: torus: $N_{F_a/F}(\vec{u}) \cdot N_{F_b/F}(\vec{v}) \cdot N_{F_{ab}/F}(\vec{w}) = 1$.

<u>Lemma</u> (1) α is an epimorphism

(2) $\ker \alpha = (\mathbb{G}_m)^2$.

Take $F = \cancel{k(t)}\ k(t_1,\dots,t_n) \qquad a, b \in k^* \subset F^*$

$\cancel{K = k(\sqrt{a},\sqrt{b})}$

- Consider $N_{K/k}(\vec{x}) = Q(t_1,\dots,t_n)^2$

  it is the fiber W of μ over the point $Q(t_1,\dots,t_n)^2$ of $\mathbb{G}_{m,k(t_1,\dots,t_n)}$.

- the fiber V of λ over the point $Q(t_1,\dots,t_m)$ of $\mathbb{G}_{m,k(t_1,\dots,t_m)}$

  is defined by:

$$N_{k(\sqrt{a})/k}(\vec{u})\cdot N_{k(\sqrt{b})/k}(\vec{v})\cdot N_{k(\sqrt{ab})/k}(\vec{w}) = Q(t_1,\dots,t_m)$$

$(u,v,w) \longmapsto x = u\cdot v\cdot w$ (product in $F(\sqrt{a},\sqrt{b}) = E$)

defines a $k(t_1,\dots,t_n)$-morphism:

$$\begin{tikzcd} V \arrow[r, "\Phi"] \arrow[dr] & W \arrow[d] \\ & \operatorname{Spec} k(t_1,\dots,t_n) \end{tikzcd}$$

& it extends to a k-morphism

$$\begin{tikzcd} V_0 \arrow[r, "\Phi"] \arrow[dr] & W_0 \arrow[d] \\ & U \subseteq_{\text{open}} \mathbb{P}^n. \end{tikzcd}$$

One shows that: $V_0 \xrightarrow{\Phi} W_0$ is a $\underbrace{\mathbb{G}_m^2}_{\boxed{\ker\alpha}}$ – torsor

$$[V_0] \in H^1(W_0, \mathbb{G}_m^2) \longrightarrow H^1(k(W_0), \mathbb{G}_m^2) = 0$$

Hilbert 90.

$[V_0] \longmapsto [V_{0\eta}] = 0$

$\eta =$ generic pt of W_0

$\Rightarrow$ birationally one has

$$V_0 \underset{bir}{\sim} W_0 \times \mathbb{P}^2.$$

$$N_{k(\sqrt{a},\sqrt{b})}(\vec{x}) = Q(t_1,\dots,t_n)^2.$$

$\Updownarrow$

Then $\underline{\text{Statement for } V_0}$ $\Longrightarrow$ Statement for W_0

$\Updownarrow$

$$N_a(\vec{u}) \cdot N_b(\vec{v}) \cdot N_{ab}(\vec{w}) = Q(t_1,\dots,t_n)$$

$\Updownarrow$ birationally

$$N_a(\vec{u}) = \frac{Q(t_1,\dots,t_n)}{N_b(\vec{v})\, N_{ab}(\vec{w})}$$ OK by thm. 0.

\#.

<u>Remark</u> One may also consider K, $Gal(K/k) = \mathbb{Z}/p \oplus \mathbb{Z}/p$ with p a prime other than 2.

$$N_{K/k}(\vec{x}) = Q(t_1,\dots,t_n)^p.$$

However,
$$(\ker \alpha)_{\bar{k}} \cong \mathbb{G}_m^p \times (\mathbb{Z}/p\mathbb{Z})^{p-2} \not\cong \mathbb{G}_m^p \quad (\text{if } p \neq 2).$$
not connected.

our ~~proof~~ argument does not work anymore!