Brauer-Manin obstruction : rational points *versus* zero-cycles

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Notations

- k : number field
- k_v, for $v \in \Omega_k$. Ω_k^f, Ω_k^∞, $\Omega_k^\mathbb{R}$, $\Omega_k^\mathbb{C}$
- X : projective variety (separated scheme of finite type, geometrically integral) over k
- $Br(X) := H^2_{\text{ét}}(X, \mathbb{G}_m)$ the cohomological Brauer group
- $X_v = X \otimes_k k_v$

Rational points

- $X(k) \subset \prod_{v\in\Omega} X(k_v)$
- Brauer-Manin pairing
$$[\prod_{v\in\Omega} X(k_v)] \times Br(X) \to \mathbb{Q}/\mathbb{Z}$$
$$(\{x_v\}_{v\in\Omega}, \beta) \mapsto \langle \{x_v\}_v, \beta\rangle := \sum_{v\in\Omega} inv_v(\beta(x_v))$$
- $[\prod_{v\in\Omega} X(k_v)]^{Br}$ = left kernel of the pairing
- Fact. $\overline{X(k)} \subseteq [\prod_{v\in\Omega} X(k_v)]^{Br}$ (by class field theory)
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Zero-cycles

- (Colliot-Thélène) Similarly, Brauer-Manin pairing

$$[\prod_{v\in\Omega} Z_0(X_v)] \times Br(X) \to \mathbb{Q}/\mathbb{Z}$$

$$[\prod_{v\in\Omega} CH_0(X_v)] \times Br(X) \to \mathbb{Q}/\mathbb{Z}$$

$$[\prod_{v\in\Omega} CH_0'(X_v)] \times Br(X) \to \mathbb{Q}/\mathbb{Z}$$

- The modified Chow group:

$$CH_0'(X_v) = \begin{cases} CH_0(X_v), & v \in \Omega^f \\ CH_0(X_v)/N_{\mathbb{C}|\mathbb{R}} CH_0(\overline{X}_v), & v \in \Omega^{\mathbb{R}} \\ 0, & v \in \Omega^{\mathbb{C}} \end{cases}$$

- complex $CH_0(X) \to \prod_{v\in\Omega} CH_0'(X_v) \to Hom(Br(X), \mathbb{Q}/\mathbb{Z})$

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- $M\hat{} := \varprojlim_n M/nM = M \otimes \widehat{\mathbb{Z}}$ for any abelian group M

  $A_0(X) := ker(CH_0(X) \xrightarrow{deg} \mathbb{Z})$

- complex (E)

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  similarly, complex (E_0)

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  Question: Are they exact?

Remark (Wittenberg)

Exactness of $(E) \Longrightarrow$

- Exactness of (E_0)

- (E_1) : Existence of $z \in CH_0(X)$ of degree 1 supposing the existence of a family of degree 1 zero-cycles $\{z_v\} \perp Br(X)$.

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Examples and a conjecture

- (Cassels-Tate) (E_0) is exact if $X = A$ is an abelian variety (with finiteness of $\text{Ш}(A)$ supposed).
- (Colliot-Thélène) (E) is exact if $X = C$ is a smooth curve (with finiteness of $\text{Ш}(Jac(C))$ supposed).

Conjecture (Colliot-Thélène/Sansuc, Kato/Saito, Colliot-Thélène)

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Rationally connectedness

Definition

$X_{/k}$ is called *rationally connected*,
if for any $P, Q \in X(\mathbb{C})$, there exists a $\mathbb{C}$-morphism $f : \mathbb{P}^1_{\mathbb{C}} \to X_{\mathbb{C}}$ such that $f(0) = P$ and $f(\infty) = Q$.

- Counter-examples:
  - An abelian variety is *never* rationally connected.
  - A smooth curve of genus > 0 is *never* rationally connected.
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Main result

Theorem (Liang 2011)

Let X be a smooth (projective) rationally connected variety defined over a number field k.

Assume that the Brauer-Manin obstruction is the only obstruction to weak approximation for rational points on $X \otimes_k K$, for any finite extension K/k.

Then, the complex (E), hence (E_0), is exact for X.

(Outline of) Proof.

- BM obstruction is the only obs. to weak approx. for rational points on X_K, $\forall K/k$ finite.

$\Longrightarrow$ (Liang 2010)

- BM obstruction is the only obs. to "weak approx." for zero-cycles of degree 1 on X_K, $\forall K/k$ finite.

$\Longrightarrow$ (key: fibration method applied to $X \times \mathbb{P}^1 \to \mathbb{P}^1$, generalized Hilbertian subset)

- $\forall d \in \mathbb{Z}$, BM obstruction is the only obs. to "weak approx." for zero-cycles of degree d on $(X \times \mathbb{P}^1)_K$, $\forall K/k$ finite.

$\Longrightarrow$ (key: Theorem of Kollár-Szabó (X is RC), an argument of Wittenberg)

- Exactness of (E) for $X \times \mathbb{P}^1$.

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- Exactness of (E) for X.

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An application

- *Recall* : a result of Borovoi (1996).
  $G_{/k}$: connected linear algebraic group.
  Y : homogeneous space of G with connected stabilizer (or with abelian stabilizer if G is simply connected).
  X : smooth compactification of Y.
  Then the Brauer-Manin obstruction is the only obstruction to weak approximation for rational points on X.

Corollary

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Thank you for your attention !

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