乘积空间~~[illegible]~~零维周[illegible]群的算术

武大. 2020.10.22.

梁永祺

~~§1 Introduction~~

Notation.

k: number field

$X_{/k}$ proper, smooth, geometrically ~~integral~~ connected variety $/k$

Ω: set of places of k.

k_v: completion of k at $v \in \Omega$.

Question $X(k) \overset{?}{\neq} \phi$

$$X(k) \xrightarrow[\text{diagonally}]{} \prod_{v \in \Omega} X(k_v)$$

Hasse principle: $\forall v,\ X(k_v) \neq \phi \Rightarrow X(k) \neq \phi$.

Weak approximation: $\overline{X(k)} = \prod_{v \in \Omega} X(k_v)$

1970s. Manin:

$$\operatorname{Br} X \times \prod_{v \in \Omega} X(k_v) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

$$b\ ,\ (x_v) \longmapsto \sum_{v \in \Omega} \operatorname{inv}_v(b(x_v))$$

(X proper $\Rightarrow$ finite $\sum$)

where $\operatorname{inv}_v : \operatorname{Br} k_v \hookrightarrow \mathbb{Q}/\mathbb{Z}$

$x_v : \operatorname{Spec} k_v \to X$ $\qquad x_v^* : \operatorname{Br} X \longrightarrow \operatorname{Br} k_v$

$$b \longmapsto b(x_v).$$

global class field theory: $0 \to \operatorname{Br} k \longrightarrow \bigoplus_{v \in \Omega} \operatorname{Br} k_v \xrightarrow{\sum \operatorname{inv}_v(-)} \mathbb{Q}/\mathbb{Z} \to 0$

$$X(k) \subseteq \overline{X(k)} \subseteq \Big[\prod_{v\in\Omega} X(k_v)\Big]^{Br} \subseteq \prod_{v\in\Omega} X(k_v)$$

$$\Big[\prod_{v\in\Omega} X(k_v)\Big]^{Br} = \Big\{(x_v)\in\prod X(k_v) \;\Big|\; (x_v)\perp b \;\; \forall b\in \mathrm{Br}X\Big\} \text{ closed subset.}$$

Def The Brauer-Manin obstruction is the only obstruction to

$\begin{cases} \text{Hasse principle} & \text{if } \Big[\prod_{v\in\Omega} X(k_v)\Big]^{Br} \neq \phi \Rightarrow X(k)\neq\phi \quad \text{(BMHP)} \\ \text{weak approximation} & \text{if } \overline{X(k)} = \Big[\prod X(k_v)\Big]^{Br} \quad \text{(BMWA)} \end{cases}$

Many varieties are proved satisfy BMHP / BMWA:

* abelian varieties (if Ш is finite)
* Smooth compactifications of linear algebraic groups
* several varieties with a fibration structure.

Prop (easy). X, Y smooth, proper, geom. connected.

X, Y BMWA $\Rightarrow$ $X\times Y$ BMWA.

proof: Take any $z_v = (x_v, y_v) \in (X\times Y)(k_v)$ st $(x_v, y_v) \perp \mathrm{Br}(X\times Y)$

~~Then~~ $\Rightarrow \begin{cases} (x_v)\perp \mathrm{Br}X \\ (y_v)\perp \mathrm{Br}Y \end{cases}$ BMWA for X and Y $\Rightarrow$ $\exists$ $x\in X(k)$, $y\in Y(k)$ very close to (x_v) and (y_v)

Then $z = (x, y) \in (X\times Y)(k)$ is very close to (x_v, y_v) #.

One can say much more:

Thm (Skorobogatov - Zarhin 2014)

$$\Big[\prod_{v\in\Omega}(X\times Y)(k_v)\Big]^{\mathrm{Br}(X\times Y)} = \Big[\prod_{v\in\Omega} X(k_v)\Big]^{\mathrm{Br}X} \times \Big[\prod_{v\in\Omega} Y(k_v)\Big]^{\mathrm{Br}Y} \subset \prod_{v\in\Omega}(X\times Y)(k_v).$$

0-cycles.

$$Z_0(X) := \bigoplus_{\substack{P \in X \\ \text{closed point}}} P\cdot\mathbb{Z}$$ a very big free abelian group of 0-cycles.

$C \xrightarrow{\varphi} X$ non-constant morphism
curve

$\forall f \in k(C)$ $\quad \varphi_*(\mathrm{div}(f)) \in Z_0(X)$

these divisors generate a subgroup of $Z_0(X)$

the quotient is $CH_0(X)$ Chow group of 0-cycles.

$\deg : CH_0(X) \to CH_0(\mathrm{Spec}\, k) = \mathbb{Z}$

* 0-cycles of degree 1 are generalizations of rational points.

* In general, $CH_0(X)$ is not easy to compute.

Manin's pairing:

$$\mathrm{Br}X \times \prod_{v\in\Omega} CH_0(X_{k_v}) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

can also be defined.

- P : closed pt of X_{k_v}

$$\begin{tikzcd} \mathrm{Br}X \arrow[r] \arrow[rrrr, bend right] & \mathrm{Br}X_{k_v} \arrow[r] & \mathrm{Br}\, k_v(P) \arrow[r, "\mathrm{cores}_{k_v(P)/k_v}"] & \mathrm{Br}k_v \arrow[r, hook, "\mathrm{inv}_v"] & \mathbb{Q}/\mathbb{Z} \end{tikzcd}$$

$$b \longmapsto b(P)$$

- get $\mathrm{Br}X \times \prod_{v\in\Omega} Z_0(X_{k_v}) \longrightarrow \mathbb{Q}/\mathbb{Z}$

factors through $\prod_{v\in\Omega} CH_0(X_{k_v})$

method of Skorobogatov-Zarhin $\Rightarrow$

$$\Big[\prod_{v\in\Omega} CH_0^1(X\times Y_{k_v})\Big]^{\mathrm{Br}(X\times Y)} \longrightarrow \Big[\prod_{v\in\Omega} CH_0^1(X_{k_v})\Big]^{\mathrm{Br}X} \times \Big[\prod_{v\in\Omega} CH_0^1(Y_{k_v})\Big]^{\mathrm{Br}Y}$$

where $CH_0^1 = \deg^{-1}(1)$.

injectivity ? not clear

<u>BMWA for 0-cycles</u> of degree 1 on $X \times Y$?

↳ Manin's pairing ⇒

$$CH_0(X) \longrightarrow \prod_{v\in\Omega} CH_0'(X_{k_v}) \longrightarrow \mathrm{Hom}(\mathrm{Br}X, \mathbb{Q}/\mathbb{Z}) \quad \text{is a complex}$$

where
$$CH_0'(X_{k_v}) = \begin{cases} CH_0(X_{k_v}) & \text{if } v \text{ is non-archimedean} \\ 0 & \text{if } v \text{ is complex} \\ CH_0(X_{\mathbb{R}}) / N_{\mathbb{C}|\mathbb{R}}\, CH_0(X_{\mathbb{C}}) & \text{if } v \text{ is real.} \end{cases}$$

$\forall n \in \mathbb{N}$

$$CH_0(X)/n \longrightarrow \prod_{v\in\Omega} CH_0'(X_{k_v})/n \longrightarrow \mathrm{Hom}\big((\mathrm{Br}X)[n], \mathbb{Q}/\mathbb{Z}\big)$$

where $A/n = A/nA$ $\qquad A[n] = \ker(n : A \to A)$

X: regular ⇒ $\mathrm{Br}X$ is a ~~torsion~~ group.

Take $\varprojlim_n$:

$$(E) \quad \varprojlim_n CH_0(X)/n \longrightarrow \varprojlim_n \prod_{v\in\Omega} CH_0'(X_{k_v})/n \longrightarrow \mathrm{Hom}(\mathrm{Br}X, \mathbb{Q}/\mathbb{Z})$$

<u>Conjecture</u> (Colliot-Thélène - Sansuc, Kato-Saito)

(E) is exact for all smooth proper geom. conn. variety X over a number field k.

<u>Rk</u> Exactness = Brauer-Manin obstruction is the only obstruction to $\left\{\begin{array}{l}\text{Hasse principle} \\ \text{weak approximation}\end{array}\right.$ for 0-cycles (of any degree) on X.

⇒ : BMHP : If $\exists z_v \in CH_0(X_{k_v})$ $(z_v) \perp \mathrm{Br}X$, $\deg z_v = 1$ then $\exists z \in CH_0(X)$ $\deg z = 1$.

⇒ BMWA 下页

BMWA for 0-cycles: $\forall S \subset \Omega$ finite $\forall n \in \mathbb{N}$ $(z_v) \perp \mathrm{Br} X$ $\deg z_v = \delta \ (\forall v \in \Omega)$ then $\exists \overset{z_{n,S}}{\overset{\shortparallel}{z}} \in CH_0(X)$ $\deg z = \delta$

s.t. $\forall v \in S$ $z = z_v \in CH_0(X_{k_v})/n$.

Exactness of (E) is known:

① $X = \mathbb{P}^n$ ~~exact~~ (E) is the dual of $\mathrm{Br}\, k \to \bigoplus_{v \in \Omega} \mathrm{Br}\, k_v \to \mathbb{Q}/\mathbb{Z}$

② $X = E$ elliptic curve (E) is the ~~Cass~~ Cassels-Tate $\underbrace{}_{\text{exact}}$ sequence.
(if Ш is finite)
(But unknown for general ab. var)

③ (Shuji Saito 1989) $X = C$ smooth projective curve.
(if $Ш(\mathrm{Jac}(C))$ is finite)

④ (Harpaz - Wittenberg 2014)
$X =$ many $\underbrace{\text{fibration}}_{\text{rationally connected}}$ over $\mathbb{P}^n$ or C

⑤ (Harpaz - Wittenberg 2020)
$X =$ Smooth compactification of homogeneous spaces of linear algebraic groups.

<u>Question</u> (E) exact for X and Y $\overset{?}{\Rightarrow}$ exact for $X \times Y$?

* if $Y = \mathbb{P}^n$ ✓. $CH_0(X \times \mathbb{P}^n) \cong CH_0(X)$
$\mathrm{Br}(X \times \mathbb{P}^n) \cong \mathrm{Br}\, X$.

<u>Thm</u> (Consequence of H-W 2014)
$Y = C$ curve $Ш(\mathrm{Jac}(C)) < +\infty$
X: rationally connected.
(E) exact for X_K $\forall K/k$ finite extension.
Then (E) exact for $X \times Y$.

~~Thm (Liang 2014, 2020)~~

~~X, Y: rationally connected $/k$~~

~~(E) exact for X_K, Y_K $\forall K/k$ finite extension~~

~~Then (E) is exact for $X \times Y$~~

~~Rk~~

Thm (Liang 2014, 2020) $\quad k$: number field.

X_i: rationally connected

(E) exact for $X_{i,K}$ $\forall K/k$ finite extension.

Then (E) is exact for $\prod X_i$

Rk. (1) similar statements hold for K3 surfaces, Kummer varieties.
(not exactly the same)

(2) at most one factor of the product is allowed to be a curve (with $Ш(\mathrm{Jac}(C)) < +\infty$) (not rationally connected)

* fibration method for rational points

Idea of proof.

* Consider

$$\begin{array}{c} X \times Y \times \mathbb{P}^1 \\ \downarrow \pi \\ \mathbb{P}^1 \end{array} \quad \text{trivial fibration.}$$

$\mathrm{Br}(X \times Y \times \mathbb{P}^1) \cong \mathrm{Br}(X \times Y)$ $\qquad$ (E) ~~holds~~ exact for $X \times Y \times \mathbb{P}^1 \Leftrightarrow$ (E) exact for $X \times Y$.

$(z_v) \in \mathrm{CH}_0((X \times Y \times \mathbb{P}^1)_{k_v})$

$(z_v) \perp \mathrm{Br}(X \times Y \times \mathbb{P}^1)$

take a closed point $Q \in X \times Y \times \mathbb{P}^1$

consider $(z_v + n\,Q)$

n large enough $z_v + n\,Q$ "very effective"

moving lemma $z_v + n\,Q \rightsquigarrow z'_v$

s.t. $\pi_*(z'_v) \in Z_0(\mathbb{P}^1)$ is "separable" (i.e. $= \sum n_P P$ with $n_P = 1$.)

$\longleftrightarrow$ separable polynomial $\in k_v[T]$.

find a closed point θ of $\mathbb{P}^1$ $\longleftrightarrow$ irred. polynomial $\in k[T]$

s.t. θ is sufficiently closed to $\pi_*(z'_v)$.

Furthermore choose $\theta \in \mathrm{Hil} \subseteq \mathbb{P}^1$ (Hilbertian subset of $\mathbb{P}^1$)

$$\begin{tikzcd} Z \arrow[r, "\text{finite étale}"'] \arrow[rr, bend left, "\rho"] & U \arrow[r, hook] & \mathbb{P}^1 \end{tikzcd}$$

integral variety.

$\mathrm{Hil} = \{P \text{ closed pt of } \mathbb{P}^1 \mid \rho^{-1}(P) \text{ is connected}\}$

(effective version of) Hilbert's irreducibility theorem:

Hil is dense under v-adic topology.

* $K = k(\theta)$. Once $\theta \in \mathrm{Hil}$. one can compare

$$\frac{\mathrm{Br}\,X}{\mathrm{Br}\,k} \cong \frac{\mathrm{Br}\,X_K}{\mathrm{Br}\,K} \qquad \frac{\mathrm{Br}\,Y}{\mathrm{Br}\,k} \cong \frac{\mathrm{Br}\,Y_K}{\mathrm{Br}\,K}$$

(we use: X, Y are rationally connected.)

$\cancel{(z_v) \perp \mathrm{Br}(X\times Y) \Rightarrow (z'_v) \perp \mathrm{Br}(X\times Y) \Rightarrow (z'_v) \perp \mathrm{Br}(X_K \times Y_K)}$

* implicite function theorem.

find local rat. points ~~effective 0-cycles~~ z''_v (≈ ~~rational points~~) on $\pi^{-1}(\theta) \cong X_K \times Y_K$ sufficiently closed to z'_v

(effective 0-cycles ≈ rat pt)

$(z_v) \perp \mathrm{Br}(X\times Y) \Rightarrow (z'_v) \perp \mathrm{Br}(X\times Y) \Rightarrow (z''_v) \perp \mathrm{Br}(X_K \times Y_K)$.

Consider $pr_X(z_v'')$ and $pr_Y(z_v'')$, apply exactness of (E)

to X_k and Y_k to conclude. #.

以下可不提

observation.

Lem. X, Y. proper smooth $/k$. $S \subset \Omega_k$ finite

$x \in CH_0(X)$ $y \in CH_0(Y)$

$(z_v)_{v \in S} \in \prod_{v \in S} CH_0((X \times Y)_{k_v})$ such that for a given $n \in \mathbb{N}$:

- $x = p_X(z_v) \in CH_0(X_{k_v})/n$ $\forall v \in S$
- $y = p_Y(z_v) \in CH_0(Y_{k_v})/n$ $\forall v \in S$

Suppose in addition the class z_v can be represented by a k_v-rational point on $X \times Y$ for any $v \in S$. (very restrictive assumption, but OK in our situation)

Then $p_X^*(x) \cap p_Y^*(y) = z \in CH_0((X \times Y)_{k_v})/n$. $\forall v \in S$