Brauer-Manin obstruction for zero-cycles on rationally connected varieties defined over number fields

notation:

k: number field.

Ω_k: ~~set of~~ places of k.

k_v: ~~v-adic~~ local field at $v \in \Omega_k$

X_k: algebraic variety (seperated scheme, finite type$/k$)

supposed proper, smooth, geometrically integral.

$X_v := X \otimes_k k_v$

$\mathrm{Br}\, X = H^2_{\acute{e}t}(X, \mathbb{G}_m)$ Brauer group of X.

$CH_0(X)$: 周-group of 0-cycles

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§1. Local-global principle for rational points

diagonal embedding $\quad X(k) \hookrightarrow \prod_{v \in \Omega_k} X(k_v)$

Hasse principle: (HP.) $\quad X(k_v) \neq \phi \ \forall v \Rightarrow X(k) \neq \phi$.

Weak approximation: (WA) $\quad \overline{X(k)} = \prod_{v \in \Omega_k} X(k_v)$

Manin (1970s): $\quad \prod_{v \in \Omega_k} X(k_v) \times \mathrm{Br}\, X \longrightarrow \mathbb{Q}/\mathbb{Z}$

$$\{x_v\}, \ b \longmapsto \sum_{v \in \Omega_k} \mathrm{inv}_v(b(x_v))$$

$\mathrm{inv}_v : \mathrm{Br}\, k_v \hookrightarrow \mathbb{Q}/\mathbb{Z}$ local invariant.

$\left[\prod X(k_v)\right]^{Br}$:= left "kernel" of the pairing.

Fact: $X(k) \subset \overline{X(k)} \subseteq \left[\prod_{v\in\Omega_k} X(k_v)\right]^{Br} \subseteq \prod_{v\in\Omega_k} X(k_v)$

If $\left[\prod X(k_v)\right]^{Br} = \phi$. then $X(k) = \phi$.

If $\left[\prod X(k_v)\right]^{Br} \subsetneqq \prod X(k_v)$ then we don't have ~~WA~~ weak approximation.

i.e. the Brauer group gives an obstruction to the Hasse principle and to ~~the~~ weak approximation.

Def The Brauer-Manin obstruction is the only obstruction to $\begin{cases}\text{the Hasse principle} \\ \text{weak approximation}\end{cases}$ if $\begin{cases}\left[\prod X(k_v)\right]^{Br} \neq \phi \Rightarrow X(k) \neq \phi. \\ \overline{X(k)} = \left[\prod X(k_v)\right]^{Br}\end{cases}$

examples
- (CT-Sansuc-Swinnerton-Dyer 1987) ~~certain fibrations over $\mathbb{P}^1$~~ Châtelet surfaces. $x^2 - ay^2 = P(z)$ $a\in k^*$ $P(z)\in k[z]$. $\deg P = 4$.
- (Borovoi 96)
  G: connected linear algebraic group
  Y: homogeneous space of G, with connected stabilizer.
  X: Smooth compactification of Y.
  Then. BM obstruction is the only obstruction to $\begin{matrix}\text{HP} \\ \text{WA}\end{matrix}$.
  <u>for rational points</u> on X

$\boxed{\text{Conjecture-pt}}$ The BM obstruction is the only obstruction to Hasse principle and to weak approximation for rational points on

① (Colliot-Thélène 88) all rationally connected varieties
② (Skorobogatov 2001) all curves.

§2. local-global principle for 0-cycles.

Similarly we can define a pairing:

$$\prod_{v \in \Omega_k} CH_0(X_v)$$

$$\prod_{v \in \Omega_k} CH_0'(X_v) \times \mathrm{Br} X \to \mathbb{Q}/\mathbb{Z}.$$

$$CH_0'(X_v) = \begin{cases} CH_0(X_v) & v: \text{fnt place} \\ 0 & v: \text{complex place} \\ \mathrm{Coker}\left[N_{\mathbb{C}/\mathbb{R}} : CH_0(\overline{X_v}) \to CH_0(X_v)\right] & v: \text{real place.} \end{cases}$$

$\rightsquigarrow$ a complex

$$CH_0(X) \to \prod_{v \in \Omega_k} CH_0'(X_v) \to \mathrm{Hom}(\mathrm{Br} X, \mathbb{Q}/\mathbb{Z})$$

$\forall n \in \mathbb{Z}_{>0}.$

$$CH_0(X)/n \to \prod_{v \in \Omega_k} CH_0'(X_v)/n \to \mathrm{Hom}(\mathrm{Br}(X)[n], \mathbb{Q}/\mathbb{Z})$$

(where $/n$: coker $(n\cdot)$)

(E) $\varprojlim_n CH_0(X)/n \to \prod_v \varprojlim_n CH_0'(X_v)/n \to \mathrm{Hom}(\mathrm{Br} X, \mathbb{Q}/\mathbb{Z})$

(E$_0$) $\varprojlim_n A_0(X)/n \to \prod_v \varprojlim_n A_0(X_v)/n \to \mathrm{Hom}(\mathrm{Br} X, \mathbb{Q}/\mathbb{Z})$

where $A_0(X) = \ker(\deg : CH_0(X) \to \mathbb{Z})$

$\boxed{\text{Conjecture - 0cye}}$. (Colliot-Thélène - Sansuc, Kato - Saito)

(E) is exact for all varieties (smooth, proper).

Remark

(1) (Wittenberg) $\quad (E)$ exact $\Rightarrow$ (E_0) exact

(2) (Wittenberg) $\quad (E)$ exact $\Rightarrow$ The BM obstruction is the only ~~obs~~ obstruction to HP for 0-cycles of degree 1.

i.e. $\exists \{z_v\} \perp \operatorname{Br} X \Rightarrow \exists z \in CH_0(X)$
$\quad\ \deg 1 \qquad\qquad\qquad\qquad \deg z = 1$.

(3). (E) exact $\Rightarrow$ The BM obstruction is the only obstruction to WA for 0-cycles of degree δ. (for a fixed $\delta \in \mathbb{Z}$)

i.e. $\forall n \in \mathbb{Z}_{>0}$, $\forall S \underset{\text{finite}}{\subseteq} \Omega_k$.

If $\{z_v\} \perp \operatorname{Br} X \quad \deg z_v = \delta \ (\forall v \in \Omega_k)$

Then $\exists\ z = z_{n,S} \in CH_0(X)$ s.t. $z = z_v$ in $CH_0(X_v)/n$ ~~$\forall n$~~ for $\forall v \in S$.
$\quad \deg z = \delta$

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example

$X = C$ curve. supposing $Ш(\mathrm{Jac}(C), k) < +\infty$

$\Rightarrow (E)$ exact for C.

(Saito 89, Colliot-Thélène 99).

particular case: $X = E$ elliptic curve.

$$(E_0) \qquad \overline{E(k)} \longrightarrow \prod_{v \in \Omega_k} E(k_v)' \longrightarrow \left(\operatorname{Br} E/\operatorname{Br} k\right)^* = \left(H^1(k, E)\right)^*$$

$(\to Ш^1(k, E)^* \to 0)$

(Cassels-Tate)

† 修改于 Ω_k^∞.
~~由~~此处没取 $(\varprojlim_n \cdot/n)$.

Question

Is there a general relation between the arithmetic of rational points and the arithmetic of 0-cycles ?

Conjecture-pt $\overset{?}{\longleftrightarrow}$ Conjecture-0cyc.

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§3. ~~Q~~ rational points vs. 0-cycles

3.1. ~~Sur les cou~~
For curves, (E) is exact. ($Ш < +\infty$ supposed)

but Conjecture-pt is still open.

3.2

Poonen 2010 : $\exists X$. $\dim X = 3$.

$$X \downarrow C$$

fibration in Châtelet surfaces

(i.e. generic fiber X_η defined by
$x^2 - ay^2 = P(z) \in k(C)[z]$
$a \in k^*$. $\deg P = 4$)

$\boxed{\begin{array}{l}\cancel{g(C)>0}\\ \#C(k) < +\infty\end{array}}$

s.t $X(k) = \phi$

$[\prod X(k_v)]^{Br} \neq \phi$.

(a fortiori, $\exists \{z_v\}_{\deg 1} \perp Br X$)

Collioт-Thélène 2010 : ~~$\exists z \in CH_0(X)$ deg~~
There exists ~~a~~ 0-cycles of degree 1 on Poonen's 3-folds

Thm (L. 2010) (E) $_0$ is exact for Poonen's 3-folds.

(i.e. BM obstruction is the only obstruction also to WA for 0-cycles)

3.3

Recall.

Def. A variety $X/_k$ is rationally connected (RC) if for every pair of geometric points $Q_1, Q_2 \in X(\mathbb{C})$ there exist a ~~curve~~ $\mathbb{C}$-rational curve $f: \mathbb{P}^1_{\mathbb{C}} \to X_{\mathbb{C}}$ passing through Q_1 and Q_2. i.e. st. $f(0) = Q_1$, $f(\infty) = Q_2$.

(a purely geometric condition)

examples.

- ~~all $\bar{k}$-unirational~~
- all $\bar{k}$-unirational varieties are rationally connected. In particular, all homogeneous spaces of a connected linear algebraic group is rationally connected.
- No abelian variety is rationally connected.
- No curve of genus > 0 is rationally connected.

recall:

Conj. pt: BM only for rational points on RC. varieties.

Conj. 0cyc: BM only for 0cycles on all varieties.

3.4 General relation between two conjectures

We consider the following statements:

(pt-HP^{BM}): BM obstruction is the only obstruction to Hasse principle for rational points on X_K $\forall K/k$ finite ~~extension~~ extension.

(pt-WA^{BM}): BM obstruction is the only obstruction to Weak Approximation for rational points on X_K $\forall K/k$ finite extension.

i.e. $$\overline{X_K(K)} = \Big[\prod_{w \in \Omega_K} X_K(K_w)\Big]^{Br(X_K)}$$

$(\text{0cyc}^1\text{-PH}^{BM})$: BM obstruction is the only obstruction to Hasse principle for 0-cycles of degree 1 on X_K. $\forall K/k$ finite extension.

$(\text{0cyc}^1\text{-WA}^{BM})$: BM obstruction is the only obstruction to Weak Approx. for 0-cycles of degree 1 on X_K $\forall K/k$ finite extension.

Thm (L. 2011)
Let X/k be a (smooth proper) rationally connected variety.

Then, (1) $(\text{pt-HP}^{BM}) \Rightarrow (\text{0cyc}^1\text{-HP}^{BM})$

(2) $(\text{pt-WA}^{BM}) \Rightarrow (\text{0cyc}^1\text{-WA}^{BM}) \Rightarrow$ (E), (E_0) are exact for X_K, $\forall K/k$ fin.

设 X 是一个 k-簇，假设 X 是有理连通的.

则 (1) ____

(2) ____ (E), (E_0) 对 X_K 正合

X: rationally connected $\Rightarrow$ $\begin{cases} \mathrm{Br}\,\overline{X} : \text{finite} \\ \mathrm{Pic}\,\overline{X} : \text{torsion free} \\ \pi_1(\overline{X}) = 0 \end{cases}$

Consider field extension K/k.

$\circledast$ ~~$(\ast)$ $\circledast_k$ K.~~

$(\ast)/K$. $\rightsquigarrow$ compare $\dfrac{\mathrm{Br}\,X}{\mathrm{Br}\,k}$ with $\dfrac{\mathrm{Br}\,X_K}{\mathrm{Br}\,K}$

Brauer groups of the fibers of $\begin{array}{c} X \times \mathbb{P}^1 \\ \pi \downarrow \\ \mathbb{P}^1 \end{array}$.

~~Defn~~ for $\theta \in \mathrm{Hil}$, one can compare $\dfrac{\mathrm{Br}(\pi^{-1}(\theta))}{\mathrm{Br}\,k(\theta)}$

(d) $(\mathrm{Ocyc}^{S}\text{-}\mathrm{WA}^{\mathrm{BM}})$ for $X \Rightarrow$ (E) for X.

"weak approximation" considers the places in a certain finite set $S \subseteq \Omega_k$, but (E) concerns all places.

Thm (Kollár–Szabó)

$X/_k$ rationally connected.

Then for ~~almost all~~ all but finitely many $v \in \Omega_k$,

$\deg : \mathrm{CH}_0(X_v) \longrightarrow \mathbb{Z}$ is an isomorphism.

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