COMPATIBILITY OF THE BRAUER–MANIN PAIRING FOR WEIL RESTRICTIONS

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In [CL22, Section 2.4], we state and outline a proof of the compatibility of the Brauer–Manin pairing for Weil restrictions by citing Lemma 2.1(1) of the arXiv version of [CV21]. But from the published version (and the newest arXiv version) of the cited paper, the lemma has been removed. So we write down the details before they are forgotten. We use a better notation other than that in [CL22].

Let $k|k'$ be an extension of number fields and X be a quasi-projective algebraic variety defined over k. Then the Weil restriction $X' = \mathrm{R}_{k|k'}X$ exists as a k'-variety. It is a k'-variety such that $\mathrm{Mor}_{k'}(S', X') = \mathrm{Mor}_k(S'_k, X)$ for any k'-scheme S'. In particular, we have a canonical identification

$$\Phi : X(\mathbf{A}_k) \xrightarrow{=} X'(\mathbf{A}_{k'})$$

under which $X(k)$ and $X'(k')$ are identified. Moreover, the identity morphism $\mathrm{id}_{X'} : X' \to X'$ corresponds to a morphism $\alpha : X' \times_{k'} k \to X$ of k-varieties. This allows us to define a homomorphism $\phi = \mathrm{cores}_{k|k'} \circ \alpha^*$

$$\phi : \mathrm{Br}(X) \xrightarrow{\alpha^*} \mathrm{Br}(X' \times_{k'} k) \xrightarrow{\mathrm{cores}_{k|k'}} \mathrm{Br}(X')$$

between Brauer groups.

Proposition 1. *The identification Φ is compatible with the Brauer–Manin pairing under the homomorphism ϕ. More precisely,*

$$((x_w)_{w\in\Omega_k}, b)_{\mathrm{BM}_k} = (\Phi((x_w)_{w\in\Omega_k}), \phi(b))_{\mathrm{BM}_{k'}}$$

for any $(x_w) \in X(\boldsymbol{A}_k)$ and for any $b \in \mathrm{Br}(X)$.

Proof. As indicated in the first arXiv version of [CV21], the following proof is suggested by Olivier Wittenberg.

The bijection Φ^{-1} given by Weil restriction is a composition

$$\Phi^{-1} : X'(\mathbf{A}_{k'}) \xrightarrow{\mathrm{res}_{k|k'}} X'(\mathbf{A}_k) = X'_k(\mathbf{A}_k) \xrightarrow{\alpha} X(\mathbf{A}_k).$$

Over a place v of k', it is a bijection on the v-component

$$\Phi_v^{-1} : X'(k'_v) \xrightarrow{\mathrm{res}} X'(k_v) = X'_k(\prod_{w|v} k_w) = \prod_{w|v} X'_k(k_w) \xrightarrow{\alpha} \prod_{w|v} X(k_w),$$

where the étale algebra $k_v = k'_v \otimes_{k'} k = \prod_{w|v} k_w$.

For any $v \in \Omega_{k'}$ and any $(x'_v) \in X'(k'_v)$, we denote $(x'_w)_{w|v} = \mathrm{res}_{k|k'}(x'_v)$ and $(x_w)_{w|v} = \alpha((x'_w)_{w|v}) = \Phi_v^{-1}(x'_v)$. We are going to check that

$$\sum_{w|v} \mathrm{inv}_w(b(x_w)) = \mathrm{inv}_v(\phi(b)(x'_v))$$

for any $b \in \mathrm{Br}(X)$. Then the compatibility of the Brauer–Manin pairing follows by taking the sum $\sum_{v\in\Omega_{k'}}$.

According to the functoriality applied to α, we have

$$b((x_w)_{w|v}) = b(\alpha(\mathrm{res}_{k|k'}(x'_v))) = b(\alpha((x'_w)_{w|v})) = \alpha^*(b)((x'_w)_{w|v}) \in \bigoplus_{w|v} \mathrm{Br}(k_w).$$

By taking $\mathrm{cores}_{k|k'}(-) = \mathrm{cores}_{k'_v\otimes_{k'}k|k'_v}(-)$, we get

$$\sum_{w|v} \mathrm{cores}_{k_w|k'_v}(b(x_w)) = \mathrm{cores}_{k|k'}((b(x_w))_{w|v})$$

$$= \mathrm{cores}_{k|k'}(\alpha^*(b)((x'_w)_{w|v})) = \mathrm{cores}_{k|k'}(\alpha^*(b)(\mathrm{res}_{k|k'}(x'_v))) \in \mathrm{Br}(k'_v)$$

Thanks to [CTS21, Prop. 3.8.1], the evaluation of a Brauer element at a point commutes with the corestriction induced by finite étale morphism, hence we have

$$\mathrm{cores}_{k|k'}(\alpha^*(b)(\mathrm{res}_{k|k'}(x'_v)) = \mathrm{cores}_{k|k'}(\alpha^*(b))(x'_v) = \phi(b)(x'_v) \in \mathrm{Br}(k'_v).$$

Therefore

$$\sum_{w|v} \mathrm{cores}_{k_w|k'_v}(b(x_w)) = \phi(b)(x'_v) \in \mathrm{Br}(k'_v).$$

As $\mathrm{inv}_w = \mathrm{inv}_v \circ \mathrm{cores}_{k_w|k'_v}$, we obtain

$$\sum_{w|v} \mathrm{inv}_w(b(x_w)) = \mathrm{inv}_v(\phi(b)(x'_v))$$

as desired. □

References

[CL22] Yang Cao and Yongqi Liang, *Etale Brauer-Manin obstruction for Weil restrictions*, Adv. Math. **410** (2022), Paper No. 108718, 20. ↑(document)

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[CV21] B. Creutz and B. Viray, *Quadratic points on intersections of two quadrics*, 2021. Preprint available at arXiv:2106.08560. ↑(document)

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