HASSE PRINCIPLE FOR ZERO-CYCLES OF DEGREE ONE ON CERTAIN FIBRATIONS

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ABSTRACT. Let X be a projective smooth variety over a number field k, fibered over a curve with geometrically integral fibers. We will introduce the notion of Brauer-Manin obstruction for zero-cycles of degree 1 on X, and we will present some results on the Hasse principle for those zero-cycles.

Notations.

k = number field
$\Omega = \Omega_k = \{$ places of $k\}$
k_v = v-adic completion of k for any $v \in \Omega$
X = smooth projective geometrically integral k-variety
(variety=separated scheme, of finite type over k)
$X_v = X \times_k k_v$
$Z_0(X) = \{\sum n_P P\}$ = the group of zero-cycles of X, where $n_P \in \mathbb{Z}$ and P : closed point of X,
$deg : Z_0(X) \longrightarrow \mathbb{Z}$ the degree map
$Z_0^1(X)$ = the set of zero-cycles of degree 1
$BrX = H^2_{\text{ét}}(X, \mathbb{G}_m)$ the Brauer group of X
C = curve over k
(curve= smooth projective geometrically integral variety of dimension 1)
$K = k(C)$ the function field of C
$\eta = Spec(K)$ the generic point of C
$\pi : X \longrightarrow C$ "a fibration (over k)" means
(1)C is a curve,
(2)X is a variety,
(3)π is a non-constant morphism (hence flat),
(4)the generic fiber X_η is a geometrically integral K-variety.

Hasse principle.

Given a variety X over a number field k, we consider the k-rational points on X.
We know that $X(k) \subset \prod_{v\in\Omega} X(k_v)$, $X(k) \neq \emptyset \Rightarrow \prod_v X(k_v) \neq \emptyset$.
Question: Does Hasse principle hold for X?
(HP) $\prod_v X(k_v) \neq \emptyset \Rightarrow X(k) \neq \emptyset$?

Example 0.0.1. Let $X \subset \mathbb{P}^2$ be defined by the equation $a_1x_1^2 + a_2x_2^2 + a_3x_3^2 = 0$, where $a_i \in k^*(i = 1, 2, 3)$. (HP) holds for all curves of genus 0.

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Counter-example 0.0.2. Let $X \subset \mathbb{P}^2$ be defined by the equation $3x_1^3+4x_2^3+5x_3^3=0$, X is a curve of genus 1. (HP) does not hold for X.

Brauer-Manin obstruction. For $v \in \Omega$, a rational point $P_v \in X(k_v)$ induce a homomorphism $BrX \longrightarrow Brk_v$, we denote the image of $b \in BrX$ in Brk_v by $b(P_v)$.

When k is a number field, we define the **Brauer-Manin pairing** :

$$\begin{array}{lcccc} <\cdot,\cdot>_{BM}: \prod_{v\in\Omega} X(k_v) & \times & BrX & \to & \mathbb{Q}/\mathbb{Z}, \\ (\ \{P_v\}_{v\in\Omega} & , & b\ ) & \mapsto & \sum_{v\in\Omega_k} inv_v(b(P_v)), \end{array}$$

where $inv_v : Br(k_v) \hookrightarrow \mathbb{Q}/\mathbb{Z}$ is the local invariant map for $v \in \Omega$.

We denote (called the *Brauer-Manin subset*)

$$\begin{array}{rcl} BM & = & \{\{P_v\}_{v\in\Omega} \in \prod_{v\in\Omega} X(k_v); \{P_v\}_{v\in\Omega} \perp b \text{ for any } b \in BrX\} \\ & \subset & \prod_{v\in\Omega_k} X(k_v) \end{array}$$

Fact:

$X(k) \subset BM$ (class field theory: $0 \to Brk \to \bigoplus_v Brk_v \xrightarrow{\sum_v inv_v} \mathbb{Q}/\mathbb{Z} \to 0$)

If $BM = \emptyset$ then $X(k) = \emptyset$, in such case we say that there is a **Brauer-Manin obstruction** to the Hasse principle.

Question: "Brauer-Manin obstruction is the only obstruction" to the Hasse principle?

(BM only)$BM \neq \emptyset \Rightarrow X(k) \neq \emptyset$?

NO! Counter-examples have been given by Skorobogatov (1999, in dimension 2) and by Poonen (2008, in dimension 3).

Conjecture 0.0.3. ***(BM only)*** *holds for all curves.*

Remark 0.0.4. The conjecture is proved for curves of genus 1 by Y.I.Manin(1970s).

Zero-cycles of degree 1. We denote $Z_0(X)$ the group of zero-cycles of X, $Z_0^1(X)$ the subset of zero-cycles of degree 1, then

$$\begin{array}{ccc} X(k) & \hookrightarrow & \prod_v X(k_v) \\ \downarrow & & \downarrow \\ Z_0^1(X) & \hookrightarrow & \prod_v Z_0^1(X_v). \end{array}$$

If $g(C) \geqslant 2$, the theorem of Faltings says that there are not so many k-rational point on X, while there may be many zero-cycles of degree 1 on X.

Similarly we can define the Brauer-Manin pairing for the zero-cycles of degree 1,

$$\prod_v Z_0^1(X_v) \times BrX \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

We denote BM' the left kernel of this pairing.($BM = BM' \cap \prod_v X(k_v)$)

We know that $Z_0^1(X) \subset BM' \subset \prod_v Z_0^1(X_v)$.

Question: (BM' only)$BM' \neq \emptyset \Rightarrow Z_0^1(X) \neq \emptyset$?

Conjecture 0.0.5 (Colliot-Thélène)**.** ***(BM' only)*** *holds for all varieties.*

Theorem 0.0.6. ***(BM' only)*** *holds for all curve C assuming finiteness of* $Ш(Jac(C))$.

(Manin (1970s, $g(C) = 1$), S.Saito (1989, first proof for $g(C)$ arbitrary), Colliot-Thélène (1999, simple proof), Eriksson/Scharaschkin (2008, more precise result))

For the higher dimensional varieties, not too many results are known. Only for the case of a fibration $X \longrightarrow C$ over a curve, there are some results.

From now on, we always assume that $Ш(Jac(C))$ is a finite group. In the following cases, **(BM' only)** holds for X.

(1)(Colliot-Thélène/Swinnerton-Dyer 1994)$X \longrightarrow \mathbb{P}^1$ fibration in Severi-Brauer varieties.

(2)(Colliot-Thélène/Skorobogatov/Swinnerton-Dyer 1998)$X \longrightarrow \mathbb{P}^1$, assuming

$\diamond$(H fiber)[1] -a technical hypothesis on the fibers. (X_Q is geometrically integral over $k(Q)$ for all $Q \Rightarrow$ (H fiber))

$\diamond$ almost all fibers satisfy **(HP')**.

(3)(Colliot-Thélène 2000, E.Frossard 2003)$X \longrightarrow C$ fibration in Severi-Brauer varieties of square-free index. ($[X_\eta] \in Brk(C), index = gcd\{n; [L : k(C)] = n, [X_\eta \times_{k(C)} L] = 0 \in BrL\}$)

(4)(O.Wittenberg 2010)$X \longrightarrow C$, assuming

$\diamond$(H fiber)

$\diamond$ almost all fibers satisfy **(HP')**,

Generalized Hilbert subsets.

Definition 0.0.7. Let V be a geometrically integral variety over a field k (any field). A subset H of closed points of X is called a **generalized Hilbert subset**, if there exist a finite étale morphism $Z \to U \subset X$ with U a non-empty open subset of X and Z an integral variety, such that H is set of closed points P of U with connected fiber X_P.

Theorem 0.0.8. *Let k be a number field. Let $X \longrightarrow C$ be a k-fibration over a curve with $Ш(Jac(C))$ finite. Suppose that*

$\diamond$ *all the fibers X_P are geometrically integral over $k(P)$,*

$\diamond$ *there exist a generalized Hilbert subset H of C, such that for any $P \in \mathsf{H}$ the fiber X_P satisfies **(HP')** Hasse principle for zero-cycles of degree* 1.

*Then **(BM' only)** holds for X.*

Key point of proof:

- By using some kinds of moving lemma to reduce the question on zero-cycles to the question on effective zero-cycles/rational points,
- Apply the fibration method for rational points.
- Hilbert's irreducibility theorem (effective version by Ekedahl).

Remark 0.0.9. Applying the same method, we can also obtain some results on "weak approximation".

An application of the theorem. We consider the fibration in Châtelet surfaces constructed by Poonen.

Let $V_0 \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{A}^1 \times \mathbb{A}^1$ be a variety defined by the equation

$$y^2 - az^2 = u^2 \tilde{P}_\infty(r, w) + v^2 \tilde{P}_0(r, w),$$

where $\tilde{P}_\infty(r, w)$ and $\tilde{P}_0(r, w)$ are the homogenizations of the polynomials $P_\infty(x), P_0(x) \in k[x]$ of degree 4. Take V to be a smooth compactification of V_0. We set $Z_1 \subset \mathbb{P}^1 \times \mathbb{P}^1$ the curve defined by $0 = u^2 \tilde{P}_\infty(r, w) + v^2 \tilde{P}_0(r, w)$.

[1](For any closed point Q of C, there exist an irreducible component Z of X_Q of multiplicity 1 such that the algebraic closure of $k(Q)$ in $k(Z)$ is an Abelian extension of $k(Q)$.)

$$\begin{array}{ccc} V & & \\ \downarrow & \searrow & \\ \mathbb{P}^1 & \longleftarrow & \mathbb{P}^1\times\mathbb{P}^1 \supset Z_1 \end{array} \qquad\qquad \begin{array}{ccc} (u:v;r:w;y,z) & & \\ \downarrow & \searrow & \\ (u:v) & \longleftarrow\!\shortmid & (u:v;r:w) \end{array}$$

For any (non-constant) morphism $\psi : C \longrightarrow \mathbb{P}^1$, we pull back everything to get a fibration $X = V \times_{\mathbb{P}^1} C \longrightarrow C$ and a finite covering $Z = Z_1 \times_{\mathbb{P}^1} C \longrightarrow C$. The finite covering $Z \longrightarrow C$ defines a generalized Hilbert subset H of C. For any closed point θ of C, the fiber X_θ is defined by $y^2 - az^2 = P_\theta(x)$ with $P_\theta(x) = \psi(\theta)P_\infty(x) + P_0(x) \in k(\theta)[x]$. The condition $\theta \in \mathsf{H}$ means that $P_\theta(x)$ is irreducible over $k(\theta)$, in which case **(HP)** (and also **(HP')**) Hasse principle holds for X_θ.

Poonen prove that if

- $C(k)$ is finite
- $\psi(C(k)) = \infty \in \mathbb{P}^1(k)$

then $BM \neq \emptyset$ but $X(k) = \emptyset$, i.e. **(BM only)** does not hold for rational points on X.

While, the theorem implies that **(BM' only)** holds for zero-cycles of degree 1 on X.

References

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