从 log-concavity 谈起

李平

同济大学

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Definition

Given a sequence $\{a_0, a_1, \ldots, a_n\}$ of real numbers.

- It is called unimodal if for some $0 \le j \le n$ we have

$$a_0 \le a_1 \le \cdots \le a_j \ge a_{j+1} \ge \cdots \ge a_n.$$

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- strongly log-concave $\Longrightarrow$ log-concave $\overset{a_i>0}{\Longrightarrow}$ unimodal.

Example

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Let $\lambda_1, \ldots, \lambda_n > 0$ and

$$e_i := \sum_{1 \le j_1 < \cdots < j_i \le n} \lambda_{j_1} \cdots \lambda_{j_i}, \qquad \widetilde{e}_i := \frac{e_i}{\binom{n}{i}}, \qquad 1 \le i \le n.$$

Then

1. $\widetilde{e}_i^2 \ge \widetilde{e}_{i-1}\widetilde{e}_{i+1}$; (Newton)
2. $\widetilde{e}_1 \ge \cdots \ge \widetilde{e}_i^{\frac{1}{i}} \ge \cdots \ge \widetilde{e}_n^{\frac{1}{n}}$. (Maclaurin)

Moreover, the equality in each case holds iff all the λ_i are equal.

Proposition

Let $P(x) = \sum_{i=0}^{n} a_i \cdot x^i$ be a real-coefficient polynomial with all the roots real, then the sequence $\{a_0, \ldots, a_n\}$ is strongly log-concave.

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Proof.

Set $b_i := a_i / \binom{n}{i}$, $D := d/dx$, $Q(x) := D^{j-1}P(x)$,

$R(x) := x^{n-j+1}Q(x^{-1})$, $D^{n-j-1}R(x) = (\cdots)(b_{j-1}x^2 + 2b_j x + b_{j+1})$.

Repeatedly applying Rolle's theorem yields the desired result. □

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Example

If A is an $n \times n$ Hermitian symmetric matrix. Then the coefficients of the characteristic polynomial $P(x) := \det(xI_n - A)$ are strongly log-concave.

Definition

The *q-binomial coefficient* $\binom{n}{k}_q$ $(n \geq k,\ n, k \in \mathbb{Z}_{\geq 0})$ is defined by

$$\binom{n}{k}_q := \frac{[n]!}{[k]![n-k]!}, \qquad [k]! := [1][2]\cdots[k], \qquad [j] := 1 - q^j.$$

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Remark/Exercise

- $\binom{n}{k}_q$ *is a polynomial in* q *with nonnegative integer coefficients.*
- $\binom{n}{k}_{q=1} = \binom{n}{k}$.
- *When* $q \geq 0$, $\binom{n}{0}_q, \binom{n}{1}_q, \ldots, \binom{n}{n}_q$ *is log-concave.*

Exercise

Let q be a prime power and $\mathbb{F}_q$ the q-element finite field. The number of k-dimensional linear subspace in $(\mathbb{F}_q)^n$ is $\binom{n}{k}_q$.

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Proposition

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$$\binom{n}{k}_q := \sum_{i=0}^{k(n-k)} a_i q^i.$$

Then the sequence a_i are unimodal and symmetric:

$$a_0 \leq a_1 \leq \cdots \leq a_{[\frac{k(n-k)}{2}]} \geq \cdots \geq a_{k(n-k)}, \qquad a_i = a_{k(n-k)-i}.$$

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$$\binom{4}{2}_q = 1 + q + 2q^2 + q^3 + q^4, \qquad \text{not log-concave.}$$

Definition

A poset (partially ordered set) P is a set with a binary relation "$\leq$" satisfying

1. $x \leq x, \forall x \in P$;
2. $x \leq y$ and $y \leq x \Longrightarrow x = y$;
3. $x \leq y$ and $y \leq z \Longrightarrow x \leq z$.

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Example

- $$P = (\mathbb{Z}, \leq), \qquad P = (\{1, 2, \cdots, n\}, \leq);$$
- $$E\text{: some set}, \qquad P = (2^E, \subset).$$

Theorem (R. Stanley, 1981)

- *Let P be a finite poset with $|P| = n$.*
- *A bijective map $\sigma : P \longrightarrow \{1, \dots, n\}$ is called order-preserving if $x < y$ in P implies that $\sigma(x) < \sigma(y)$.*

Fix some $v \in P$.

$$N_i(v) := \Big|\{\text{order-preserving bijections } \sigma \text{ with } \sigma(v) = i\}\Big|.$$

Then the sequence $N_1(v), \dots, N_n(v)$ is log-concave.

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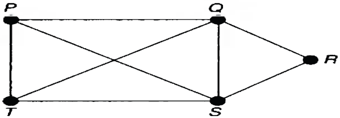
Remark

Its solution needs the Alexandrov-Fenchel inequalities in convex geometry!

Definition

- A connected graph G is a pair consisting of a vertex set $V(G)$ and an edge set $E(G)$ such that each pair of vertices can be joined by edges.
- A connected graph G is called simple if there are no loops or multiple edges.

simple, planar



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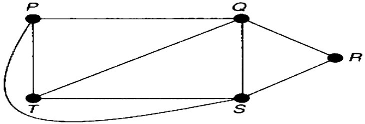
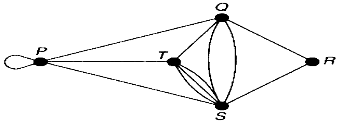


non-simple



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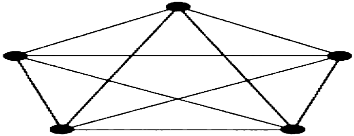
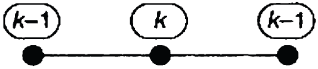


Definition (Birkhoff 1912, Whitney 1932)

Let G be a simple graph. For $k \in \mathbb{Z}_{\geq 0}$, let $\chi_G(k)$ be the number of ways of coloring the vertices of G with k colors so that no two adjacent vertices have the same color. $\chi_G(\cdot)$ is called the chromatic polynomial of G.

$\chi_G(k) = k(k-1)^2.$



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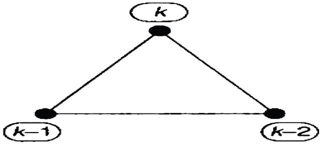

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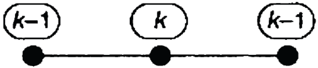

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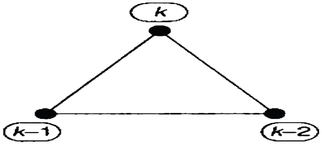
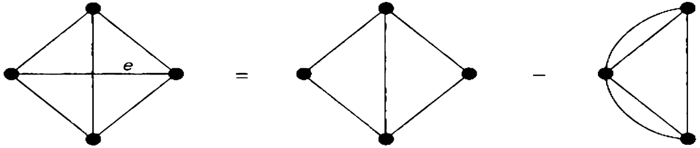


Theorem (Appel-Haken, 1976)

For every planar graph G we have $\chi_G(4) > 0$.

$$\chi_G(k) = \chi_{G-e}(k) - \chi_{G/e}(k) \qquad \text{(deletion-contraction relation)}$$



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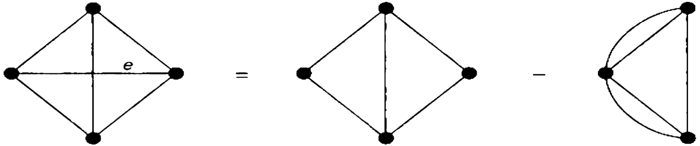

Proposition

- $\chi_G(k)$ *is a polynomial in* k *with degree* $|V(G)|$.
- $\chi_G(k) = k\left[a_0(G)k^d - a_1(G)k^{d-1} + \cdots + (-1)^d a_d(G)\right]$ *for some positive integers* $a_0(G) = 1, a_1(G), \ldots, a_d(G)$.

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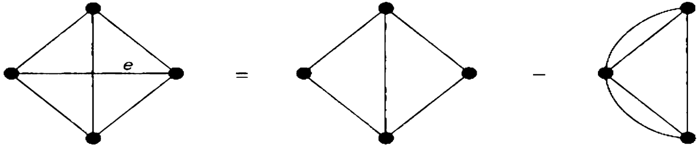

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Conjecture (Read 1968, Rota 1971)

The sequence $a_0(G), a_1(G), \ldots, a_d(G)$ is log-concave and hence unimodal.

Definition

A matroid M is a pair $(E, \mathfrak{I})$, where E is a finite set and $\mathfrak{I} \subset 2^E$, called independent sets, and satisfy

1. $\emptyset \in \mathfrak{I}$;
2. $I \in \mathfrak{I},\ I' \subset I \Longrightarrow I' \in \mathfrak{I}$;
3. $I_1, I_2 \in \mathfrak{I},\ |I_2| > |I_1| \Longrightarrow \exists\ e \in I_2 - I_1$ such that $I_1 \cup \{e\} \in \mathfrak{I}$.

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Example

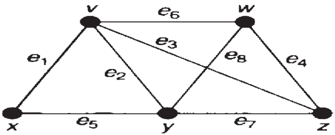
- V: vector space over some field;
- $E = \{v_1, \ldots, v_n\}$, $v_i \in V$;
- $$\mathfrak{I} := \{I \subset E \mid I \text{ is linearly-independent in } V\}.$$

Then $(E, \mathfrak{I})$ is a vector matroid.

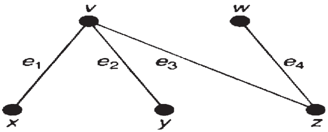
Example

- G: simple graph; $E = E(G)$;
- $\mathfrak{I} := \{I \subset E(G) \mid I$ does not contain any cycle of $G\}$.

Then $M(G) := (E, \mathfrak{I})$ is a graphic matroid.



a maximal independent set



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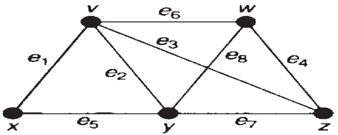

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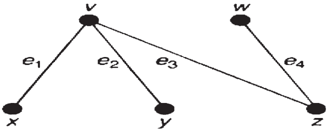

Exercise

Show that this $(E(G), \mathfrak{I})$ satisfies the three defining conditions.

Definition/Proposition

- $M = (E, \mathfrak{I})$: *matroid;* $\forall A \subset E$, $r(A) :=$ *rank of* A;
- *The characteristic polynomial of M is*

$$\chi_M(t) := \sum_{A \subset E} (-1)^{|A|} t^{r(E)-r(A)}$$
$$=: a_0(M)t^d - a_1(M)t^{d-1} + \cdots + (-1)^d a_d(M).$$

- $a_0(M) = 1, a_1(M), \ldots, a_d(M)$ *are positive integers.*
- *When G is a simple graph,* $\chi_G(t)/t = \chi_{M(G)}(t)$.

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Conjecture (Rota-Heron-Welsh, 1970s)

For a matroid M, *the sequence* $a_0(M), a_1(M), \ldots, a_d(M)$ *is log-concave.*

Theorem (Huh 2012, Huh-Katz 2012, Adiprasito-Huh-Katz 2018)

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Remark

- *(JAMS., 2012) Huh first proved it for vector matroids over fields of characteristic zero, which include graphic matroids.*
- *(Math. Ann., 2012) Huh and Katz extended it for all vector matroids.*
- *(Ann. Math., 2018) Adiprasito-Huh-Katz proved the general case by establishing a combinatorial Hodge theory for matroids.*

Definition/Proposition

- *In $\mathbb{R}^n$ a convex body is a compact convex subset with nonempty interior. Denote the set of all convex bodies by $\mathcal{K}^n$.*
- $V_n(K) :=$ *n-dim volume of K in $\mathbb{R}^n$, $\forall K \in \mathcal{K}^n$.*
- $\forall K_1, \ldots, K_m \in \mathcal{K}^n$, $\forall \lambda_1, \ldots, \lambda_m \geq 0$,

$$\lambda_1 K_1 + \cdots + \lambda_m K_m \overset{\text{Minkowski sum}}{:=} \{\lambda_1 k_1 + \cdots + \lambda_m k_m \mid k_i \in K_i\} \in \mathcal{K}^n.$$

- *(Minkowski) $V_n(\lambda_1 K_1 \cdots + \lambda_m K_m)$ is a degree n homogeneous polynomial of $\lambda_1, \ldots, \lambda_m$ with nonnegative coefficients.*

Definition

The mixed volume $V(\cdot, \ldots, \cdot) : (\mathcal{K}^n)^n \longrightarrow \mathbb{R}_{\geq 0}$ is defined by

$$V(K_1, \ldots, K_n) = V(K_{\sigma(1)}, \ldots, K_{\sigma(n)}), \qquad \forall \sigma \in S_n$$

$$V_n(\lambda_1 K_1 + \cdots + \lambda_n K_n) = \sum_{i_1+\ldots+i_n=n} \frac{n!}{i_1! \cdots i_n!} \lambda_1^{i_1} \cdots \lambda_n^{i_n} V(K_1[i_1], \ldots, K_n[i_n]),$$

i.e., $n! V(K_1, \ldots, K_n)$ is the coefficient in front of $\lambda_1 \cdots \lambda_n$.

Theorem (Alexandrov-Fenchel, 1936)

For any $K_1, \ldots, K_n \in \mathcal{K}^n$*, we have*

$$V(K_1, K_2, K_3, \ldots, K_n)^2 \geq V(K_1, K_1, K_3, \ldots, K_n)V(K_2, K_2, K_3, \ldots, K_n).$$

The equality case holds if K_1 *and* K_2 *are homothetic, i.e.,* $K_2 = \lambda K_1 + t$ *for some* $t \in \mathbb{R}^n$ *and* $\lambda \in \mathbb{R}_{>0}$.

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Corollary

Let $K, L \in \mathcal{K}^n$ *and*

$$a_i := V(\underbrace{K, \ldots, K}_{i}, \underbrace{L, \ldots, L}_{n-i}), \qquad 0 \leq i \leq n.$$

The the sequence $a_0, a_1, \ldots, a_n$ *is log-concave.*

Theorem (R. Stanley, 1981)

Let P be a finite poset with $|P| = n$. Fix some $v \in P$.

$$N_i(v) := \Big|\{\text{order-preserving bijections } \sigma \text{ with } \sigma(v) = i\}\Big|.$$

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$P := \{v_1, \dots, v_{n-1}, v\}$. $\forall \sigma \in N_i(v)$, let $\pi \in S_{n-1}$ by

$$\sigma(v_{\pi(1)}) < \cdots < \sigma(v_{\pi(i-1)}) < \sigma(v_{\pi(i)}) < \cdots < \sigma(v_{\pi(n-1)}).$$

$$K := \{(t_1, \dots, t_{n-1}) \in \mathbb{R}^{n-1} \mid 0 \le t_i \le 1; t_i \le t_j \text{ if } v_i \le v_j; \\ t_i = 1 \text{ if } v < v_i.\}$$

$$L := \{(t_1, \dots, t_{n-1}) \in \mathbb{R}^{n-1} \mid 0 \le t_i \le 1; t_i \le t_j \text{ if } v_i \le v_j; \\ t_i = 0 \text{ if } v > v_i.\}$$

$\forall x, y \in \mathbb{R}_{\geq 0}$, we define

$$\Delta_\sigma := \{(t_1, \ldots, t_{n-1}) \in \mathbb{R}^{n-1} \mid 0 \leq t_{\pi(1)} \leq \cdots \leq t_{\pi(i-1)} \leq x \leq t_{\pi(i)} \leq \cdots \leq t_{\pi(n)} \leq x + y\}.$$

Then $\Delta_\sigma \subset xK + yL$ and

$$V(xK + yL) = \sum_\sigma V(\Delta_\sigma) = \sum_{i=1}^{n} N_i(v) \frac{x^{i-1}}{(i-1)!} \frac{y^{n-i}}{(n-i)!}.$$

$$N_i(v) = (n-1)! \cdot V(\underbrace{K, \ldots, K}_{i-1}, \underbrace{L, \ldots, L}_{n-i}).$$

Theorem (Classical Hodge theory)

X^n : *compact Kähler manifold*; $\mathcal{K}(X) \subset H^{1,1}(X;\mathbb{R})$: *Kähler cone.*
$\forall \omega \in \mathcal{K}(X)$ *and* $p+q \leq n$,

$$P^{p,q}(X) := \{\alpha \in H^{p,q}(X) \mid \alpha \wedge \omega^{n-p-q+1} = 0\} : \textit{ primitive subspace.}$$

- *(HL)* $H^{p,q}(X) \xrightarrow[\cong]{\wedge \omega^{n-p-q}} H^{n-q,n-p}$.
- *(LD)* $H^{p,q}(X) = P^{p,q}(X) \oplus \left[\omega \wedge H^{p-1,q-1}(X)\right]$;
- *(HR)*

$$Q(\alpha,\beta) := (\sqrt{-1})^{q-p}(-1)^{\frac{(p+q)(p+q+1)}{2}} \int_X \alpha \wedge \bar{\beta} \wedge \omega^{n-p-q}$$

is positive-definite on $P^{p,q}(X)$.

Corollary

X^n : compact Kähler. Then its Betti numbers $b_i = b_i(X)$ satisfy

$$b_0 \le b_2 \le b_4 \le \cdots \le b_{[\frac{n}{2}]} \text{ or } b_{[\frac{n}{2}]} - 1, \qquad b_i = b_{2n-i}.$$

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Corollary

$G_k(\mathbb{C}^n)$: complex Grassmannian of k-dim subpaces in $\mathbb{C}^n$. Then

$$\sum_{i=0}^{k(n-k)} b_{2i}\big(G_k(\mathbb{C}^n)\big)\, q^i = \binom{n}{k}_q =: \sum_{i=0}^{k(n-k)} a_i q^i.$$

Consequently the sequence a_i are unimodal and symmetric:

$$a_0 \le a_1 \le \cdots \le a_{[\frac{k(n-k)}{2}]} \ge \cdots \ge a_{k(n-k)}, \qquad a_i = a_{k(n-k)-i}.$$

Theorem (mixed version, Gromov 1990, $\cdots$, Dinh-Nguyên 2006)

$\forall \omega, \omega_1, \ldots, \omega_{n-p-q} \in \mathcal{K}(X)$, $p+q \le n$ *and*
$\Omega := \omega_1 \wedge \cdots \wedge \omega_{n-p-q}$.

$$P^{p,q}(X) := \{\alpha \in H^{p,q}(X) \mid \alpha \wedge \omega \wedge \Omega = 0\}.$$

- *(HL)* $H^{p,q}(X) \xrightarrow[\cong]{\wedge\Omega} H^{n-q,n-p}$.
- *(LD)* $H^{p,q}(X) = P^{p,q}(X) \oplus \left[\omega \wedge H^{p-1,q-1}(X)\right]$;
- *(HR)*

$$Q(\alpha, \beta) := (\sqrt{-1})^{q-p}(-1)^{\frac{(p+q)(p+q+1)}{2}} \int_X \alpha \wedge \bar{\beta} \wedge \Omega$$

is positive-definite on $P^{p,q}(X)$.

Corollary

$\forall \omega_1, \ldots, \omega_n \in \mathcal{K}(X)$, we have the A-F type inequality

$$(\omega_1 \omega_2 \omega_3 \cdots \omega_n)^2 \geq (\omega_1^2 \omega_3 \cdots \omega_n) \cdot (\omega_2^2 \omega_3 \cdots \omega_n).$$

The equality holds iff ω_1 and ω_2 are proportional.

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The equality holds iff ω_1 *and* ω_2 *are proportional.*

Corollary

$\forall \gamma_1, \ldots, \gamma_n \in \overline{\mathcal{K}(X)}$ *are called nef classes, we still have*

$$(\gamma_1\gamma_2\gamma_3\cdots\gamma_n)^2 \geq (\gamma_1^2\gamma_3\cdots\gamma_n)\cdot(\gamma_2^2\gamma_3\cdots\gamma_n).$$

In particular, if $s_i := \gamma_1^i\gamma_2^{n-i}$, *we have the Khovanskii-Teissier inequalities (around 1978)*

$$s_i^2 \geq s_{i-1}s_{i+1}, \qquad 1 \leq i \leq n-1.$$

Definition

X: a mathematical object of "dimension" d. A Kähler package of X consist of a triple $\big(A(X), P(X), \mathcal{K}(X)\big)$:

- a real vector space with a graded algebra structure:

$$A(X) = \bigoplus_{q=0}^{d} A^q(X), \qquad A^0(X) \cong \mathbb{R};$$

- a symmetric bilinear pairing P on $A(X)$,
- a convex cone $\mathcal{K}(X) \subset A^1(X)$,

such that

Definition

- (PD) $P(\cdot,\cdot)$ is nondegenerate on $A^q(X) \times A^{d-q}(X)$.
- (HL)

$$A^q(X) \xrightarrow{\cong} A^{d-q}(X),\ \eta \longmapsto (\prod_{i=1}^{d-2q})\eta,\ \forall L_1,\dots,L_{d-2q} \in \mathcal{K}(X);$$

- (HR)For all $L_0, L_1, \dots, L_{d-2q} \in \mathcal{K}(X)$,

$$A^q(X)\times A^q(X) \longrightarrow \mathbb{R}, \qquad (\eta_1,\eta_2) \longmapsto (-1)^q P(\eta_1, (\prod_{i=1}^{d-2q} L_i)\eta_2)$$

is positive-definite on the kernel of the linear map $A^q(X) \longrightarrow A^{d-q+1}(X),\ \eta \longmapsto (\prod_{i=0}^{d-2q} L_i)\eta.$

Corollary

Let X be a mathematical object of "dimension" d with a Kähler package $(A(X), P(X), \mathcal{K}(X))$. Then we have the Alexandrov-Fenchel type inequalities

$$P(1, L_1 L_2 \cdots L_n)^2 \geq P(1, L_1^2 L_3 \cdots L_n) P(1, L_2^2 L_3 \cdots L_n), \ \forall L_i \in \mathcal{K}(X),$$

where the equality holds iff L_1 and L_2 are proportional.

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where the equality holds iff L_1 and L_2 are proportional.

- *In particular, if $\alpha, \beta \in \overline{\mathcal{K}(X)}$, the sequence*

$$s_i := P(1, \alpha^i \beta^{n-i}), \qquad 0 \leq i \leq n$$

is log-concave.

Theorem (Adiprasito-Huh-Katz, 2018)

- *Let M be a matroid of rank $d+1$. Then there exists a Kähler package of dimension d $(A(X), P(X), \mathcal{K}(X))$ on M.*
- *If the characteristic polynomial*

$$\chi_M(t) =: a_0 t^{d+1} - a_1 t^d + \cdots + (-1)^{d+1} a_{d+1}$$
$$=: (t-1)\left[\widetilde{a}_0 t^d - \widetilde{a}_1 t^{d-1} + \cdots + (-1)^d \widetilde{a}_d\right],$$

there exist $\alpha, \beta \in \overline{\mathcal{K}(X)}$ such that

$$\widetilde{a}_i = P(1, \alpha^i \beta^{d-i}), \qquad 0 \leq i \leq d.$$

Hence the sequence $\widetilde{a}_i$ as well as a_i is log-concave.

Question

How to completely characterize the equality

$$V(K_1, K_2, K_3, \ldots, K_n)^2 = V(K_1, K_1, K_3, \ldots, K_n)V(K_2, K_2, K_3, \ldots, K_n)$$

for convex bodies $K_1, \ldots, K_n$?

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Remark

- *The homothety of K_1 and K_2 is only a sufficient but not necessary condition.*
- *Only have some partial results!*

Theorem

X^n: compact Kähler, $\omega_1, \ldots, \omega_n \in \mathcal{K}(X)$. Then

$$(\omega_1\omega_2\omega_3\cdots\omega_n)^2 = (\omega_1^2\omega_3\cdots\omega_n)(\omega_2^2\omega_3\cdots\omega_n)$$

iff ω_1 and ω_2 are proportional.

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Question

How about the situation if these ω_i are only assumed to be nef and big?

Theorem (Boucksom-Favre-Jonsson 2009, Fu-Xiao 2014, Lehmann-Xiao 2016)

$\gamma_1, \ldots, \gamma_n$: *nef and big. Then*

$$(\gamma_1 \cdots \gamma_n)^n = \prod_{i=1}^{n} \gamma_i^n$$

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Theorem (L. 2017)

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iff $\gamma_1 \wedge \gamma_3 \wedge \cdots \wedge \gamma_n$ and $\gamma_2 \wedge \gamma_3 \wedge \cdots \wedge \gamma_n$ are proportional in $H^{n-1,n-1}(X;\mathbb{R})$.

谢谢大家!